Estimation of tail risk based on extreme expectiles

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Outline

- Quantiles, expectiles & expected-shortfall.
- Tail behaviour, application to inference:
  - Intermediate vs extreme levels,
  - Asymptotic results,
  - Illustration on simulations.
- Application on a real data example.

Quantiles

If X is a real-valued random variable, its univariate τth quantile

$$q_\tau := \inf\{t \in \mathbb{R} \text{ s.t. } \mathbb{P}(X \leq t) \geq \tau\}$$

can be obtained by solving the optimisation problem (Koenker & Bassett, 1978)

$$q_\tau = \arg\min_{q \in \mathbb{R}} \mathbb{E}(\varphi_\tau(X - q) - \varphi_\tau(X))$$

where φ_τ is the "check function" defined by

$$\varphi_\tau(x) = (1-\tau)|x|\mathbb{I}\{x < 0\} + \tau|x|\mathbb{I}\{x \geq 0\}.$$

Remarks:

- Subtracting $\mathbb{E}(\varphi_\tau(X))$ makes the cost function well-defined even when $\mathbb{E}|X| = \infty$.
- In particular, the median $q_{1/2}$ of X is obtained by minimising $\mathbb{E}|X - q|$ with respect to q.
- q_τ is also referred to as the Value-at-Risk (VaR) of level τ.

Expectiles

If X is a real-valued random variable, its univariate τth expectile is defined by the optimisation problem (Newey & Powell, 1987)

$$\xi_\tau = \arg\min_{\theta \in \mathbb{R}} \mathbb{E}(\eta_\tau(X - \theta) - \eta_\tau(X))$$

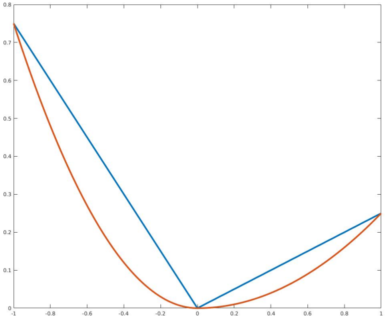
where η_τ is the function defined by

$$\eta_\tau(x) = (1 - \tau)x^2 \mathbb{I}\{x < 0\} + \tau x^2 \mathbb{I}\{x \geq 0\}.$$

Remarks:

- Subtracting $\mathbb{E}(\eta_\tau(X))$ makes the cost function well-defined provided that $\mathbb{E}|X| < \infty$.
- In particular, the mean $\xi_{1/2}$ of X is obtained by minimising $\mathbb{E}(X - \theta)^2$ with respect to θ.

Comparison of cost functions



Red: expectiles η_τ, blue: quantiles φ_τ with $\tau = 1/3$.

Expectiles vs quantiles

Theoretical point of view

- Both families of quantiles and expectiles are embedded in the more general class of M-quantiles (Breckling & Chambers (1988)) as the minimizers of an asymmetric convex loss function.
- The only M-quantiles that are coherent risk measures are the expectiles, for $\tau > 1/2$ (Bellini *et al.* (2014)).

Practical point of view

- Expectiles are more sensitive to the magnitude of extremes than quantiles are.
- Sample expectiles provide a class of smooth curves as functions of the level τ, which is not the case for sample quantiles.
- Expectiles do not have an intuitive interpretation as direct as quantiles.

Expected shortfall

- The (quantile-based) expected shortfall, also known under the names Conditional Value at Risk or Average Value at Risk, is defined as the average of the quantile function above a given confidence level τ:

$$\mathrm{QES}(\tau) := \frac{1}{1-\tau}\int_{\tau}^{1} q_\alpha \, d\alpha.$$

When X is continuous, $\mathrm{QES}(\tau) = \mathbb{E}(X|X > q_\tau)$.

- Similarly, one may define an alternative expectile-based expected-shortfall as

$$\mathrm{XES}(\tau) := \frac{1}{1-\tau}\int_{\tau}^{1} \xi_\alpha \, d\alpha.$$

Contributions

Let $X_1, \ldots, X_n$ be an i.i.d. sample from F. Our aim is to estimate expectiles ξ_{τ_n} and the associated expectile-based expected-shortfall $\mathrm{XES}(\tau_n)$ when $\tau_n \to 1$ as $n \to 1$ when F is an heavy-tailed distribution. Two situations are investigated:

- Intermediate levels, $n(1 - \tau_n) \to \infty$,
- Extreme levels, $n(1 - \tau_n) \to c \geq 0$ (extrapolation needed).

Inference (for intermediate levels)

We assume $\tau_n \to 1$ and $n(1-\tau_n) \to \infty$ as $n \to \infty$ (intermediate level). Let $k = [n(1-\tau_n)]$ be an intermediate sequence.

- Intermediate quantile (Thm 2.4.1, de Haan & Ferreira (2006)):

$$\hat{q}_{\tau_n} = X_{n-k,n},$$

- Intermediate quantile-based expected-shortfall (Elmethni *et al.*, 2014):

$$\widehat{\mathrm{QES}}(\tau_n) = \frac{1}{k}\sum_{i=1}^{n} X_i \mathbb{I}\left(X_i > \hat{q}_{\tau_n}\right),$$

- Intermediate expectile:

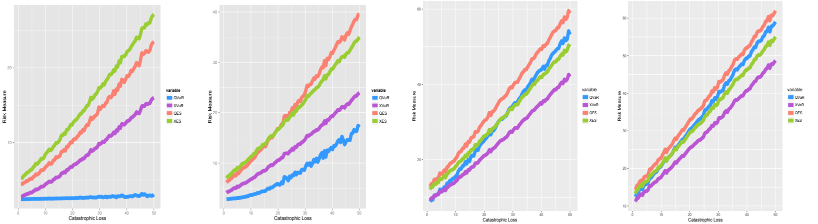
$$\tilde{\xi}_{\tau_n} = \arg\min_{u \in \mathbb{R}} \frac{1}{n}\sum_{i=1}^{n} \eta_{\tau_n}(X_i - u),$$

- Intermediate expectile-based expected-shortfall:

$$\widetilde{\mathrm{XES}}(\tau_n) = \frac{1}{k}\sum_{i=1}^{n} X_i \mathbb{I}(X_i > \tilde{\xi}_{\tau_n}).$$

Numerical illustration

Duffie & Pan (1997): X is simulated from the mixture model
$X \sim (1-p)\mathcal{N}(0, 1/(1-p)) + p\mathcal{N}(c, 1/p)$ where $p = 0.005$ and $c \in [1, 50]$.
The sample size is $n = 1000$.



Horizontally: c, vertically: Monte-Carlo averages (over 1000 replications) of the estimated risk measures. Blue: quantile $\hat{q}_\tau$, violet: expectile $\tilde{\xi}_\tau$, red: quantile-ES $\widehat{\mathrm{QES}}(\tau)$ and green: expectile-ES $\widetilde{\mathrm{XES}}(\tau)$ for $\tau \in \{0.99, 0.995, 0.999, 0.9995\}$.

Heavy-tailed distributions

Definition. The cumulative distribution function F is said to be heavy-tailed if it belongs to Fréchet Maximum Domain of Attraction *i.e.*

$$F(x) = 1 - x^{-1/\gamma}\ell(x),\ x > 0$$

where

- $\gamma > 0$ is the extreme-value index (or tail index),
- ℓ is a slowly-varying function *i.e.* such that $\ell(tx)/\ell(t) \to 1$ as $t \to \infty$ for all $x > 0$.

Consequences.

- $\gamma < 1$ implies $E|X| < \infty$ and thus the existence of expectiles.
- The survival function $\bar{F} := 1 - F$ is said to be regularly-varying with index $-1/\gamma$ *i.e.* $\bar{F}(tx)/\bar{F}(t) \to x^{-1/\gamma}$ as $t \to \infty$ for all $x > 0$.
- Equivalently, the tail quantile function $U := (1/\bar{F})^{\leftarrow}$ is regularly-varying with index γ.

Second order condition

- The regular-variation property is also referred to as a first order condition: $U(tx)/U(t) \to x^{\gamma}$ as $t \to \infty$ for all $x > 0$.
- The goal of the second order condition is to quantify the rate of convergence: there exist $\gamma > 0$, $\rho \leq 0$, and a function A converging to 0 at infinity such that for all $x > 0$,

$$\lim_{t\to\infty} \frac{1}{A(t)} \left[\frac{U(tx)}{U(t)} - x^{\gamma} \right] = x^{\gamma} \frac{x^{\rho} - 1}{\rho}.$$

This condition is denoted by $\mathcal{C}_2(\gamma, \rho, A)$. Note that $(x^{\rho} - 1)/\rho$ is to be understood as $\log x$ when $\rho = 0$.

Asymptotic distribution of $\tilde{\xi}_{\tau_n}$

From the continuity and the convexity of η_{τ_n} and a result of Geyer (1996):

Theorem 1

If F is heavy-tailed with $0 < \gamma < 1/2$ and $\tau_n \to 1$ is such that $n(1-\tau_n) \to \infty$, then

$$\sqrt{n(1-\tau_n)}\left(\frac{\tilde{\xi}_{\tau_n}}{\xi_{\tau_n}} - 1\right) \xrightarrow{d} \mathcal{N}\left(0, V_1(\gamma)\right) \quad \text{with} \quad V_1(\gamma) = \frac{2\gamma^3}{1-2\gamma}.$$

- No need for a second-order condition,
- Restriction on the extreme-value index.

First order expansions

Proposition 1

For all heavy-tailed distribution such that $0 < \gamma < 1$, *when* $\tau \to 1$, *one has*

$$\begin{aligned} \frac{XES(\tau)}{QES(\tau)} &\sim \frac{\xi_\tau}{q_\tau} &\sim (\gamma^{-1} - 1)^{-\gamma}, \\ \frac{XES(\tau)}{\xi_\tau} &\sim \frac{QES(\tau)}{q_\tau} &\sim \frac{1}{1-\gamma}. \end{aligned}$$

- Second order approximations have been established under $\mathcal{C}_2(\gamma, \rho, A)$ (Daouia *et al.*, 2016).
- If $\gamma < 1/2$ then, asymptotically, $\mathrm{XES}(\tau) < \mathrm{QES}(\tau)$ and $\xi_\tau < q_\tau$.

Inference for heavy-tailed distributions

The order statistics are denoted by $X_{1,n} \leq \cdots \leq X_{n,n}$.

- Hill estimator for the tail index (Hill, 1975)

$$\hat{\gamma}_H = \frac{1}{k} \sum_{i=1}^{k} \log \frac{X_{n-i+1,n}}{X_{n-k,n}},$$

- Weissman estimator for extreme quantiles (Weissman, 1978)

$$\hat{q}^{\star}_{\tau'_n} = \hat{q}_{\tau_n} \left(\frac{1-\tau_n}{1-\tau'_n} \right)^{\widehat{\gamma}_H},$$

- Estimator of the quantile-ES (Elmethni *et al.*, 2014)

$$\widehat{\mathrm{QES}}^{\star}(\tau'_n) = \widehat{\mathrm{QES}}(\tau_n) \left(\frac{1-\tau_n}{1-\tau'_n} \right)^{\widehat{\gamma}_H}.$$

An alternative estimator of the (intermediate) expectile

The property $\xi_\tau \sim q_\tau(\gamma^{-1} - 1)^{-\gamma}$ as $\tau \to 1$ suggests an estimator based on an intermediate quantile:

$$\hat{\hat{\xi}}_{\tau_n} = X_{n-k,n}(\hat{\gamma}_H^{-1} - 1)^{-\hat{\gamma}_H}$$

Theorem 2

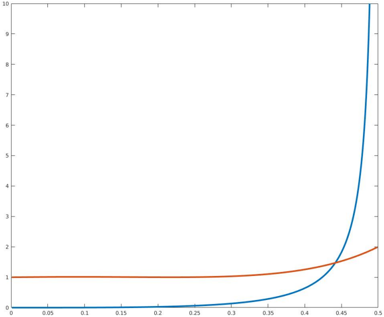
If F verifies $\mathcal{C}_2(\gamma, \rho, A)$ with $0 < \gamma < 1$ and $\tau_n \to 1$ is such that $n(1-\tau_n) \to \infty$, $\sqrt{n(1-\tau_n)}q_{\tau_n}^{-1} \to 0$ and $\sqrt{n(1-\tau_n)}A((1-\tau_n)^{-1}) \to 0$, then

$$\sqrt{n(1-\tau_n)}\left(\frac{\hat{\hat{\xi}}_{\tau_n}}{\xi_{\tau_n}} - 1\right) \xrightarrow{d} \mathcal{N}\left(0, V_2(\gamma)\right)$$

with $V_2(\gamma) = 1 + \left(\frac{\gamma}{1-\gamma} - \gamma \log\left(\frac{1}{\gamma} - 1\right)\right)^2$.

- Need for a second-order condition,
- Bias conditions on τ_n.

Comparison of asymptotic variances



Horizontally: $\gamma \in (0, 1/2)$, Vertically: asymptotic variances $V_1(\gamma)$ in blue and $V_2(\gamma)$ in red.

Estimation of extreme expectiles

Let $\tau'_n \to 1$ and $n(1-\tau'_n) \to c \geq 0$ as $n \to \infty$ (extreme level).

The property $\xi_\tau \sim q_\tau(\gamma^{-1}-1)^{-\gamma}$ as $\tau \to 1$ also entails $\xi_{\tau'}/\xi_\tau \sim q_{\tau'}/q_\tau$ as both $\tau \to 1$ and $\tau' \to 1$. Thus, the same extrapolation factor can be applied for expectiles and quantiles leading to two possible estimators for extreme expectiles:

$$\tilde{\xi}^\star_{\tau'_n} = \tilde{\xi}_{\tau_n}\left(\frac{1-\tau_n}{1-\tau'_n}\right)^{\widehat{\gamma}_H}$$

and

$$\hat{\xi}^\star_{\tau'_n} = \hat{\xi}_{\tau_n}\left(\frac{1-\tau_n}{1-\tau'_n}\right)^{\widehat{\gamma}_H} = X_{n-k,n}(\hat{\gamma}_H^{-1}-1)^{-\hat{\gamma}_H}\left(\frac{1-\tau_n}{1-\tau'_n}\right)^{\widehat{\gamma}_H} = \hat{q}^\star_{\tau'_n}(\hat{\gamma}_H^{-1}-1)^{-\hat{\gamma}_H}.$$

In the following slide, we focus on the first estimator.

Asymptotic distribution of $\tilde{\xi}^{\star}_{\tau'_n}$

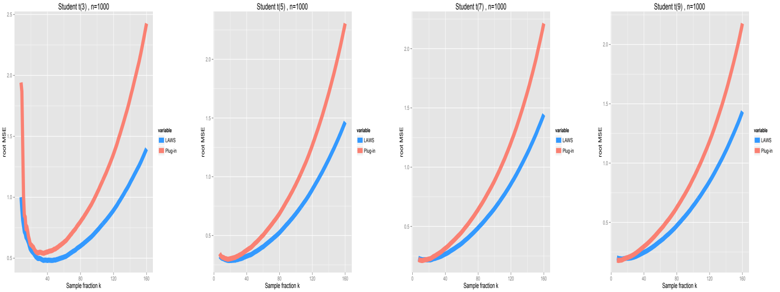
Theorem 3

If F *verifies* $\mathcal{C}_2(\gamma, \rho, A)$ *with* $0 < \gamma < 1/2$, $\rho < 0$ *and* $\tau_n \to 1$, $\tau'_n \to 1$ *are such that* $n(1-\tau_n) \to \infty$, $n(1-\tau'_n) \to c \geq 0$, $\sqrt{n(1-\tau_n)}q_{\tau_n}^{-1} \to 0$ *and* $\sqrt{n(1-\tau_n)}A((1-\tau_n)^{-1}) \to 0$, *then*

$$\frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\tau'_n)]}\left(\frac{\tilde{\xi}^{\star}_{\tau'_n}}{\xi_{\tau'_n}} - 1\right) \xrightarrow{d} \mathcal{N}\left(0, \gamma^2\right).$$

A similar asymptotic result is available for $\hat{\xi}^{\star}_{\tau'_n}$ (Daouia *et al.*, 2016).

Numerical illustration



Horizontally: k, vertically: root MSE estimates (over 10,000 replications) for the t_3, t_5, t_7 and t_9-distributions, with sample size $n = 1000$. Red: $\hat{\xi}^{\star}_{\tau'_n}$, blue: $\tilde{\xi}^{\star}_{\tau'_n}$.

Estimation of the extreme expectile-based expected-shortfall

The property $\mathrm{XES}(\tau) \sim \xi_\tau/(1-\gamma)$ as $\tau \to 1$ suggests two possible estimators for the extreme expectile-based expected-shortfall:

$$\widetilde{\mathrm{XES}}^\star(\tau_n') = \tilde{\xi}^\star_{\tau_n'}/(1-\hat{\gamma}_H) \quad \text{and} \quad \widehat{\mathrm{XES}}^\star(\tau_n') = \hat{\xi}^\star_{\tau_n'}/(1-\hat{\gamma}_H).$$

In the following theorem, we focus on the first estimator.

Theorem 4

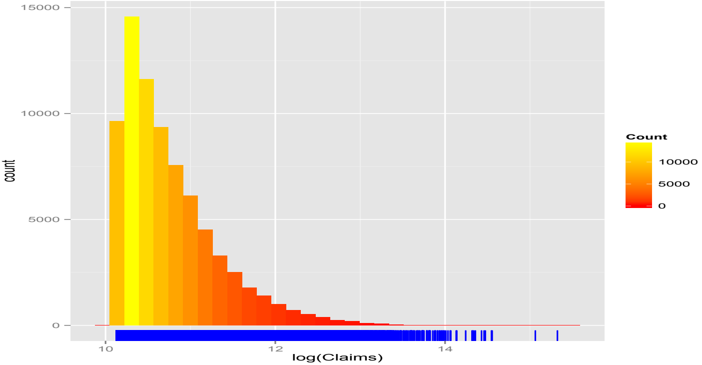
Under the assumptions of Theorem 3,

$$\frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\tau_n')]}\left(\frac{\widetilde{XES}^\star(\tau_n')}{XES(\tau_n')}-1\right) \xrightarrow{d} \mathcal{N}\left(0,\gamma^2\right).$$

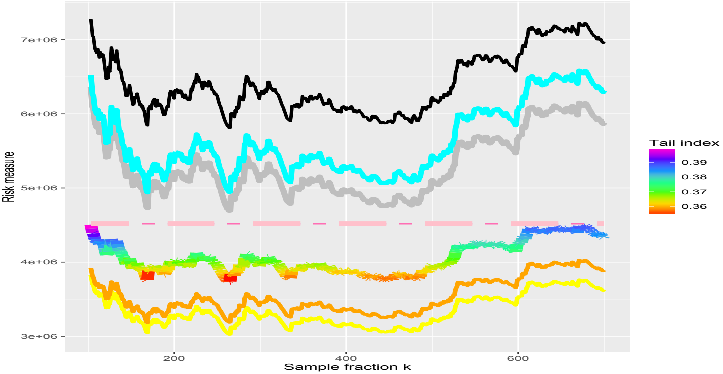
A similar asymptotic result is available for $\widehat{\mathrm{XES}}^\star(\tau_n')$ (Daouia *et al.*, 2016).

Illustration on real data

The Society of Actuaries Group Medical Insurance Large Claims Database records all the claim amounts exceeding 25,000 USD over the period 1991-92. As in Beirlant *et al.* (2004), we only deal here with the $n = 75,789$ claims for 1991. Moreover, we focus on the extreme level $\tau_n' = 1 - 10^{-5}$.



Results



Horizontally: k, vertically: expectiles $\hat{\xi}^{\star}_{\tau'_n}$ in yellow and $\tilde{\xi}^{\star}_{\tau'_n}$ in orange, expectile-based expected-shortfall $\widehat{\text{XES}}^{\star}(\tau'_n)$ in gray and $\widetilde{\text{XES}}^{\star}(\tau'_n)$ in cyan, quantile $\hat{q}^{\star}_{\tau'_n}$ as a rainbow curve, $\widehat{\text{QES}}^{\star}(\tau'_n)$ in black, sample maximum $Y_{n,n}$ as an horizontal pink line. The estimated sample fraction is $\hat{k} = 486$ (Beirlant *et al.* (2004)).

References

Beirlant, J., Goegebeur, Y., Segers, J. and Teugels, J. (2004). *Statistics of extremes: Theory and applications*, Wiley.

Bellini, F., Klar, B., Müller, A. and Gianina, E.R. (2014). Generalized quantiles as risk measures, *Insurance: Mathematics and Economics*, **54**, 41–48.

Breckling, J. and Chambers, R. (1988). M-quantiles, *Biometrika*, **75**, 761–772.

Daouia, A., Girard, S. and Stupfler, G. (2016). Estimation of tail risk based on extreme expectiles. `https://hal.archives-ouvertes.fr/hal-01142130`

Duffie, D. and Pan, J. (1997). An overview of value at risk, *Journal of Derivatives*, **4**, 7–49.

El Methni, J., Gardes, L. and Girard, S. (2014). Nonparametric estimation of extreme risks from conditional heavy-tailed distributions, *Scandinavian Journal of Statistics*, **41** (4), 988–1012.

Geyer, C.J. (1996). On the asymptotics of convex stochastic optimization, unpublished manuscript.

de Haan, L. and Ferreira, A. (2006). *Extreme Value Theory: An Introduction*. Springer-Verlag, New York.

Koenker, R. and Bassett, G. S. (1978). Regression Quantiles, *Econometrica*, **46**, 33–50.

Newey, W.K. and Powell, J.L. (1987). Asymmetric least squares estimation and testing, *Econometrica*, **55**, 819–847.