Estimation of extreme regression risk measures

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A simple way to assess the (environmental) risk is to compute a measure linked to the value Y of the phenomenon of interest (rainfall height, wind speed, river flow, etc.):

- quantiles (= Value at Risk = return level),
- expectiles,
- conditional tail moments,
- spectral risk measures,
- distortion risk measures, etc.

Here, we focus on the first three measures: quantiles, expectiles and conditional tail moments.

We shall see how to estimate some extensions of such measures: L^p-quantiles and regression risk measures.

Outline

- Quantiles, expectiles and conditional tail moments
- L^p- quantiles
- Numerical experiments (simulated and real data examples)
- Conclusions

Quantiles

Let Y be a random variable with cumulative distribution function (cdf) F.

Definition (Quantile)

The quantile associated with Y is the function q defined on $(0, 1)$ by

$$q(\tau) = \inf\{t \in \mathbb{R} \mid F(t) \geq \tau\}.$$

In other words, the quantile $q(\tau)$ of level τ is the smallest real value exceeded by Y with probability less than $1 - \tau$.

For the sake of simplicity, we assume $Y \geq 0$ and F is continuous and strictly increasing.
Then, the quantile $q(\tau)$ is the unique real value such that

$$F(q(\tau)) = \tau.$$

Quantiles from an optimization point of view

From Koenker & Bassett (1978),

$$q(\tau) = \arg\min_{q \in \mathbb{R}} \mathbb{E}(\eta_\tau(Y - q, 1) - \eta_\tau(Y, 1)).$$

with the so-called check function

$$\eta_\tau(x, 1) = |\tau - \mathbb{1}_{\{x \leq 0\}}||x|.$$

- The initial motivation was to estimate quantiles in a linear regression framework, thanks to a minimization problem.
- The second term $\mathbb{E}(\eta_\tau(Y, 1))$ does not play any role in the minimization, but it ensures that the cost function exists even when $\mathbb{E}(Y) = \infty$.
- In particular, the median is the best L^1 predictor of Y:

$$q(1/2) = \arg\min_{q \in \mathbb{R}} \mathbb{E}|Y - q|.$$

Estimation of extreme quantiles

1) **Intermediate level.** Let $\{Y_1, \ldots, Y_n\}$ be a $n-$ sample from F. The empirical estimator of $q(\tau)$ is given by

$$\widehat{q}_n(\tau) := Y_{\lceil n\tau \rceil, n}$$

where $Y_{1,n} \leq \cdots \leq Y_{n,n}$ are the order statistics. It can be interpreted both as a minimizer of the empirical optimization problem:

$$\arg\min_q \frac{1}{n} \sum_{i=1}^{n} \eta_\tau(Y_i - q, 1)$$

and as a solution of the equation $F_n(q) = \tau$ where F_n is the empirical cdf.

The asymptotic properties of this estimator are well-known when τ is fixed. Here, we focus on the asymptotic behaviour of this estimator when $\tau = \tau_n \uparrow 1$ as $n \to \infty$. In such a case, $q(\tau) = q(\tau_n)$ is an extreme quantile. In the situation where, additionally, $n(1-\tau_n) \to \infty$, τ_n is referred to as an intermediate level.

In the following, we assume that Y has a heavy right tail.

Assumption (First order condition $\mathcal{C}_1(\gamma)$)

The survival function $\overline{F} := 1 - F$ is regularly varying at $+\infty$ with index $-1/\gamma < 0$:

$$\forall x > 0, \ \lim_{t \to +\infty} \frac{\overline{F}(tx)}{\overline{F}(t)} = x^{-1/\gamma}.$$

The next condition controls the rate of convergence in $\mathcal{C}_1(\gamma)$.

Assumption (Second order condition $\mathcal{C}_2(\gamma, \rho, A)$)

There exist $\gamma > 0$, $\rho < 0$ and a function A tending to zero at $+\infty$ with asymptotically constant sign such that:

$$\forall x > 0, \ \lim_{t \to \infty} \frac{1}{A(1/\overline{F}(t))} \left[\frac{\overline{F}(tx)}{\overline{F}(t)} - x^{-1/\gamma} \right] = x^{-1/\gamma} \frac{x^{\rho/\gamma} - 1}{\gamma\rho}.$$

It can be shown that $|A|$ is necessarily regularly varying with index ρ. The larger $|\rho|$ is, the smaller the approximation error $|A|$ is.

Under the second order condition, the estimator $\widehat{q}_n(\tau_n)$ is asymptotically Gaussian with (relative) rate of convergence $\sqrt{n(1-\tau_n)}$:

Theorem (Intermediate extreme quantiles, Theorem 2.4.1, de Haan & Ferreira, 2006)

Suppose $\mathcal{C}_2(\gamma,\rho,A)$ *holds. If* $\tau_n \uparrow 1$ *with* $n(1-\tau_n)\to\infty$ *and* $\sqrt{n(1-\tau_n)}A(1/(1-\tau_n))\to\lambda\in\mathbb{R}$, *then*

$$\sqrt{n(1-\tau_n)}\left(\frac{\widehat{q}_n(\tau_n)}{q(\tau_n)}-1\right)\xrightarrow{d}\mathcal{N}(0,\gamma^2)\quad \text{as}\quad n\to\infty.$$

2) **Arbitrary level.** Extreme quantiles of arbitrary large order τ_n' *i.e.* such that $n(1-\tau_n') \to c < \infty$ can be estimated thanks to Weissman's approximation deduced from condition $\mathcal{C}_1(\gamma)$:

$$q(\tau_n') \approx \left(\frac{1-\tau_n}{1-\tau_n'}\right)^{\gamma} q(\tau_n).$$

One chooses τ_n such that $n(1-\tau_n) \to \infty$ as well as an estimator $\widehat{\gamma}_n$ of γ (Hill estimator for instance) to compute Weissman estimator (1978):

$$\widehat{q}_n^W(\tau_n'|\tau_n) = \left(\frac{1-\tau_n}{1-\tau_n'}\right)^{\widehat{\gamma}_n} \widehat{q}_n(\tau_n).$$

$\widehat{q}_n^W(\tau_n'|\tau_n)$ inherits its asymptotic distribution from $\widehat{\gamma}_n$ with a slightly slower rate of convergence.

Extreme regression quantiles

When a d-dimensional covariate X is recorded simultaneously with Y, the regression quantile or conditional quantile is defined by

$$F(q(\tau|x)|x) = \tau,$$

where $F(.|x)$ denotes the cdf of Y conditional on $X = x$.

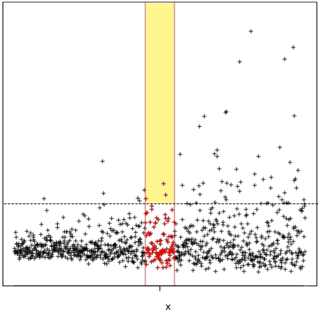
Kernel estimation. Let K be a kernel (a density on $\mathbb{R}^d$) and $h_n \to 0$ a bandwidth. Letting $K_{h_n}(\cdot) = h_n^{-d} K(h_n^{-1} \cdot)$, there are two equivalent methods for estimating $q(\tau|x)$.
Consider the locally weighted optimization problem:

$$\hat{q}_n(\tau|x) \in \arg\min_q \frac{1}{n} \sum_{i=1}^{n} K_{h_n}(x - X_i)\eta_\tau(Y_i - q, 1).$$

One can show that this entails that $\hat{F}_n(\hat{q}_n(\tau|x)|x) = \tau$ where $\hat{F}_n(.|x)$ is the kernel estimator of $F(.|x)$:

$$\hat{F}_n(y|x) = \frac{\sum_{i=1}^n K_{h_n}(x - X_i)\mathbb{1}_{\{Y_i \leq y\}}}{\sum_{i=1}^n K_{h_n}(x - X_i)}.$$

The asymptotic normality of $\hat{q}_n(\tau_n|x)$ has been established in the intermediate case *i.e.* when $n(1-\tau_n)h_n^d \to \infty$. An extrapolated version has also been developped to address arbitrary rates (Daouia *et al.* 2011).



Drawbacks of quantiles

The quantile of level τ does not provide information on the extreme values of Y lying beyond $q(\tau)$.

For instance, two distributions may have the same quantile of level 99% but different tail indices γ.

Similarly, the estimator $\widehat{q}_n(\tau_n)$ does not use the most extreme values of the sample (in the intermediate case).

$\Rightarrow$ Loss of tail information.

Our goal: "Adapt" the definition of quantiles to take into account the whole tail structure of the underlying distribution.

Regression Conditional Tail Moments

Let $a > 0$ and $x \in \mathbb{R}^d$ such that $\mathbb{E}(Y^a|X = x) < \infty$. The Regression Conditional Tail Moment of order a and level τ is defined as

$$\mathrm{RCTM}_a(\tau|x) = \mathbb{E}(Y^a|Y > q(\tau|X), X = x).$$

Two particular cases:

- Regression Expected Shortfall (mean of losses above the VaR): $\mathrm{RES}(\tau|x) = \mathrm{RCTM}_1(\tau|x)$
- Regression Conditional Tail Variance (variance of losses above the VaR): $\mathrm{RCTV}(\tau|x) = \mathrm{RCTM}_2(\tau|x) - \mathrm{RCTM}_1^2(\tau|x)$

Kernel estimation of RCTM

1) Intermediate level. A two-step procedure is used:

- First, the kernel estimator $\hat{q}_n(\tau_n|x)$ of the extreme regression quantile is computed.
- Second, a kernel estimator of the conditional expectation is implemented:

$$\mathrm{RC\hat{T}M}_a(\tau_n|x) = \frac{1}{1-\tau_n} \frac{\sum_{i=1}^n K_{h_n}(x - X_i) Y_i^a \mathbb{1}_{\{Y_i > \hat{q}_n(\tau_n|x)\}}}{\sum_{i=1}^n K_{h_n}(x - X_i)}.$$

2) Arbitrary level. A Weissman type estimator is constructed remarking that, under $\mathcal{C}_1(\gamma(x))$,

$$\frac{\mathrm{RCTM}_a(\tau|x)}{q^a(\tau|x)} \to \frac{1}{1-a\gamma(x)} \quad \text{as} \quad \tau \to 1.$$

Both estimators are asymptotically Gaussian, see Elmethni *et al.* (2014).

Expectiles

Let us assume that $\mathbb{E}(Y) < \infty$.

Definition (Expectiles, Newey & Powell, 1987)

The expectile associated with Y is the function ξ defined on $(0,1)$ by

$$\xi(\tau) = \underset{q\in\mathbb{R}}{\arg\min}\, \mathbb{E}(|\tau - \mathbb{1}_{\{Y\leq q\}}|(Y-q)^2 - |\tau - \mathbb{1}_{\{Y\leq 0\}}|Y^2).$$

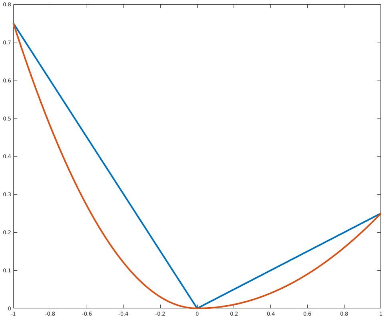
To define the expectile, the quantile check function

$$\eta_\tau(x,1) = |\tau - \mathbb{1}_{\{x\leq 0\}}||x|$$

introduced by Koenker & Bassett (1978) is replaced in the optimization problem by the function

$$\eta_\tau(x,2) = |\tau - \mathbb{1}_{\{x\leq 0\}}|x^2.$$

Comparison of cost functions



Red: expectiles $\eta_\tau(.,2)$, blue: quantiles $\eta_\tau(.,1)$ with $\tau = 1/3$.

The new cost function is continuously differentiable, the associated first order condition is:

$$(1-\tau)\mathbb{E}(|Y-\xi(\tau)|\mathbb{1}_{\{Y\leq\xi(\tau)\}}) = \tau\mathbb{E}(|Y-\xi(\tau)|\mathbb{1}_{\{Y>\xi(\tau)\}}).$$

In particular, $\xi(1/2)=\mathbb{E}(Y)$, and more generally:

$$\tau = \frac{\mathbb{E}(|Y-\xi(\tau)|\mathbb{1}_{\{Y\leq\xi(\tau)\}})}{\mathbb{E}(|Y-\xi(\tau)|)}.$$

An expectile is thus defined in terms of mean distance with respect to Y, and not only in terms of frequency.

Besides, the computation of an empirical expectile takes into account the whole tail information via $\mathbb{E}(|Y-\xi(\tau)|\mathbb{1}_{\{Y>\xi(\tau)\}})$.

Estimation of extreme expectiles

1) **Intermediate level.** The empirical estimator of $\xi(\tau)$ is

$$\widehat{\xi}_n(\tau) = \arg\min_{\xi \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} \eta_\tau(Y_i - \xi, 2).$$

Under the first order condition, the estimator $\widehat{\xi}_n(\tau_n)$ is asymptotically Gaussian with (relative) rate of convergence $\sqrt{n(1-\tau_n)}$:

Theorem (Intermediate extreme expectiles, Daouia *et al.* 2018)

Assume $\mathcal{C}_1(\gamma)$ holds with $0 < \gamma < 1/2$. If $\tau_n \uparrow 1$ such as $n(1-\tau_n) \to \infty$, then

$$\sqrt{n(1-\tau_n)} \left(\frac{\widehat{\xi}_n(\tau_n)}{\xi(\tau_n)} - 1 \right) \xrightarrow{d} \mathcal{N}\left(0, \gamma^2 \times \frac{2\gamma}{1-2\gamma}\right).$$

The proof is based on Geyer (1996): since the empirical criterion is convex, the asymptotic behaviour of the minimizer can be deduced from the asymptotic behaviour of the criterion itself.

2) **Arbitrary level.**

Proposition (Bellini *et al.* 2014 and Daouia *et al.* 2018)

Assume $\mathcal{C}_1(\gamma)$ holds with $0 < \gamma < 1$. Then,

$$\frac{\xi(\tau)}{q(\tau)} \to (\gamma^{-1} - 1)^{-\gamma} \ \text{ as } \ \tau \to 1.$$

Weissman's approximation thus still holds for expectiles:

$$\xi(\tau_n') \approx \left(\frac{1-\tau_n}{1-\tau_n'}\right)^{\gamma} \xi(\tau_n).$$

An extrapolated estimator can then be derived:

$$\widehat{\xi}_n^W(\tau_n'|\tau_n) = \left(\frac{1-\tau_n}{1-\tau_n'}\right)^{\widehat{\gamma}_n} \widehat{\xi}_n(\tau_n)$$

where $n(1-\tau_n) \to \infty$ and $n(1-\tau_n') \to c < \infty$.

Extreme regression expectiles

Extreme regression expectiles can be estimated following the same scheme as for extreme regression quantiles.

- Considering first the intermediate case, the following locally weighted optimization problem is introduced.

$$\hat{\xi}_n(\tau|x) = \arg\min_{\xi} \frac{1}{n}\sum_{i=1}^{n} K_{h_n}(x - X_i)\eta_\tau(Y_i - \xi, 2).$$

- Second, taking account of $\mathcal{C}_1(\gamma(x))$:

$$\frac{\xi(\tau|x)}{q(\tau|x)} \to (\gamma(x)^{-1} - 1)^{-\gamma(x)} \quad \text{as} \quad \tau \to 1,$$

a Weissman type estimator can also be introduced to estimate regression expectiles of arbitrary extreme levels.

Drawbacks of expectiles

Existence of expectiles requires $\mathbb{E}(Y) < \infty$ which amounts to supposing $\gamma < 1$.

In practice, to obtain reasonable estimates, even at the intermediate level, one needs $\gamma < 1/2$.

Similar problems occur when dealing with Expected Shortfall.

$\Rightarrow$ Restricts the range of potential fields of applications.

L^p-quantiles: Basic idea

Quantile of level τ: solution of the minimization problem

$$q(\tau) = \arg\min_{q\in\mathbb{R}} \mathbb{E}(|\tau - \mathbb{1}_{\{Y\leq q\}}||Y-q| - |\tau - \mathbb{1}_{\{Y\leq 0\}}||Y|).$$

Expectile of level τ: when $\mathbb{E}(Y) < \infty$, solution of the minimization problem

$$\xi(\tau) = \arg\min_{q\in\mathbb{R}} \mathbb{E}(|\tau - \mathbb{1}_{\{Y\leq q\}}||Y-q|^2 - |\tau - \mathbb{1}_{\{Y\leq 0\}}||Y|^2).$$

Definition (L^p-quantile, Chen 1996)

Assume $\mathbb{E}(Y^{p-1}) < \infty$. The L^p-quantile associated with Y is the function $q(\cdot, p)$ defined on $(0,1)$ as

$$q(\tau, p) = \arg\min_{q\in\mathbb{R}} \mathbb{E}(|\tau - \mathbb{1}_{\{Y\leq q\}}||Y-q|^p - |\tau - \mathbb{1}_{\{Y\leq 0\}}||Y|^p).$$

Advantages and drawbacks

- When $p > 1$, the L^p-quantile $q(\tau, p)$ exists, is unique and verifies

$$\tau = \frac{\mathbb{E}(|Y - q(\tau,p)|^{p-1} \mathbb{1}_{\{Y \leq q(\tau,p)\}})}{\mathbb{E}(|Y - q(\tau,p)|^{p-1})}.$$

  It can thus be interpreted in terms of (pseudo-)distance to Y in the space L^{p-1}.

- The condition for the existence of L^p-quantiles is $\mathbb{E}(Y^{p-1}) < \infty$. When $1 < p < 2$, it is a weaker condition than the existence condition for expectiles.

- When $p \neq 2$, the L^p-quantiles do not define a coherent risk measure (Bellini *et al.*, 2014) since they are not in general subadditive.

Estimation of extreme L^p-quantiles

1) **Intermediate levels.** Introducing

$$\eta_\tau(x,p) = |\tau - 1\!\!1_{\{x\le 0\}}||x|^p,$$

one has

$$q(\tau,p) = \underset{q\in\mathbb{R}}{\arg\min}\, \mathbb{E}(\eta_\tau(Y-q,p) - \eta_\tau(Y,p)).$$

The empirical estimator of $q(\tau,p)$ is obtained by minimizing the empirical counterpart of the previous criterion:

$$\widehat{q}_n(\tau,p) = \underset{q\in\mathbb{R}}{\arg\min}\, \frac{1}{n}\sum_{i=1}^{n} \eta_\tau(Y_i - q, p).$$

According to Geyer (1996), since the empirical criterion is convex, the asymptotic behaviour of the minimizer directly depends on the asymptotic behaviour of the criterion itself.

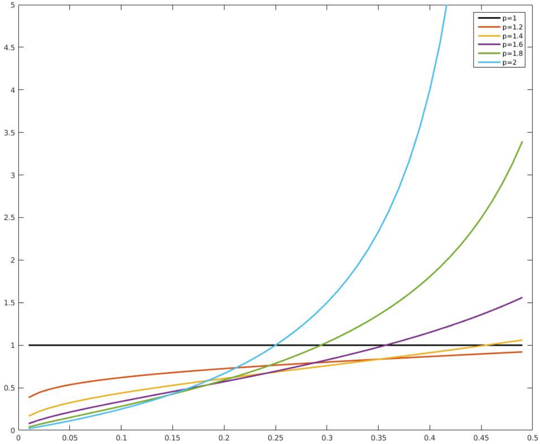
Theorem (Intermediate extreme L^p-quantiles, Daouia *et al.* 2018)

Let $p > 1$. *Assume* $\mathcal{C}_2(\gamma, \rho, A)$ *holds with* $0 < \gamma < [2(p-1)]^{-1}$.

If $\tau_n \uparrow 1$ *such that* $n(1-\tau_n) \to \infty$ *and* $\sqrt{n(1-\tau_n)}A(1/(1-\tau_n)) \to \lambda \in \mathbb{R}$ *then,*

$$\sqrt{n(1-\tau_n)}\left(\frac{\widehat{q}_n(\tau_n, p)}{q(\tau_n, p)} - 1\right) \xrightarrow{d} \mathcal{N}\left(0, \gamma^2 V(\gamma, p)\right)$$

where $V(\gamma, p) = \dfrac{\Gamma(2p-1)\Gamma(\gamma^{-1}-2p+2)}{\Gamma(p)\Gamma(\gamma^{-1}-p+1)}$ *and* $\Gamma(x)$ *is the Gamma function.*



Behaviour of variances $\gamma \in (0, 1/2) \mapsto V(\gamma, p)$ for some values of $p \in [1, 2]$.

2) **Arbitrary levels.** In order to remove the condition on τ_n, one has to show that Weissman's approximation is still valid for L^p-quantiles.

Proposition (Daouia *et al.* 2018)

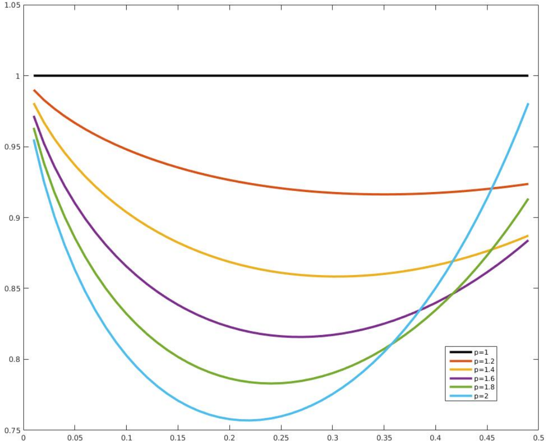
Let $p > 1$. *Assume* $\mathcal{C}_1(\gamma)$ *with* $\gamma < 1/(p-1)$. *Then,*

$$\lim_{\tau \uparrow 1} \frac{q(\tau, p)}{q(\tau, 1)} = C(\gamma, p),$$

where $C(\gamma, p) = \left[\frac{\gamma}{B(p, \gamma^{-1} - p + 1)}\right]^{-\gamma}$ *and* $B(x, y)$ *is the Beta function.*

Extreme L^p-quantiles are asymptotically proportional to extreme ordinary quantiles, for all $p > 1$.

For $p = 2$, one has $C(\gamma, 2) = (\gamma^{-1} - 1)^{-\gamma}$, which coincides with the previous result on expectiles.



Behaviour of constants $\gamma \in (0, 1/2) \mapsto C(\gamma, p)$ for some values of $p \in [1, 2]$.

Weissman's approximation is thus still valid for L^p-quantiles:

$$q(\tau_n', p) \approx \left(\frac{1-\tau_n}{1-\tau_n'}\right)^{\gamma} q(\tau_n, p).$$

An extrapolated estimator can be derived for L^p-quantiles:

$$\widehat{q}_n^W(\tau_n'|\tau_n, p) = \left(\frac{1-\tau_n}{1-\tau_n'}\right)^{\widehat{\gamma}_n} \widehat{q}_n(\tau_n, p)$$

where $n(1-\tau_n) \to \infty$ and $n(1-\tau_n') \to c < \infty$.

Establishing the asymptotic behaviour of this estimator requires to investigate the error term in Weissman's approximation.

Theorem (Arbitrary extreme L^p-quantiles, Daouia *et al.* 2018)

Suppose $\mathcal{C}_2(\gamma, \rho, A)$ *holds with* $\gamma < [2(p-1)]^{-1}$.
Let τ_n, $\tau_n' \uparrow 1$ *such that* $n(1-\tau_n) \to \infty$, $n(1-\tau_n') \to c < \infty$ *and*

$$\begin{aligned} \sqrt{n(1-\tau_n)} \max\left(\frac{1}{q(\tau_n,1)}, 1-\tau_n, A\left(\frac{1}{1-\tau_n}\right)\right) &= O(1) \\ \sqrt{n(1-\tau_n)}(\widehat{\gamma}_n - \gamma) &\xrightarrow{d} N. \end{aligned}$$

Then,

$$\frac{\sqrt{n(1-\tau_n)}}{\log([1-\tau_n]/[1-\tau_n'])}\left(\frac{\widehat{q}_n^W(\tau_n'|\tau_n,p)}{q(\tau_n',p)} - 1\right) \xrightarrow{d} N.$$

3) **Exploiting links between quantiles and L^p-quantiles**

– First, remark that from the property $q(\tau, p) \sim C(\gamma, p) q(\tau, 1)$, one can build other estimators of extreme L^p-quantiles:

- at intermediate levels: $\widetilde{q}_n(\tau_n, p) := C(\hat{\gamma}_n, p)\widehat{q}_n(\tau_n, 1)$ where $\hat{\gamma}_n$ is an estimator of the tail index and $\widehat{q}_n(\tau_n, 1) = Y_{\lceil n\tau_n \rceil, n}$.
- at arbitrary levels: $\widetilde{q}_n^W(\tau_n' | \tau_n, p) := C(\hat{\gamma}_n, p)\widehat{q}_n^W(\tau_n' | \tau_n, 1)$ where $\widehat{q}_n^W(\tau_n' | \tau_n, 1)$ is Weissman's estimator.

Asymptotic normality results have been established under the condition $\gamma < (p-1)^{-1}$ instead of $\gamma < [2(p-1)]^{-1}$ in the previous theorem. Such results are deduced from the (joint) asymptotic normality of $\hat{\gamma}_n$ and extreme quantile estimators.

– Second, recall that the L^p-quantile $q(\tau_n, p)$ exists, is unique and verifies

$$\tau_n = \frac{\mathbb{E}(|Y - q(\tau_n, p)|^{p-1} \mathbb{1}_{\{Y \leq q(\tau_n, p)\}})}{\mathbb{E}(|Y - q(\tau_n, p)|^{p-1})}.$$

It is thus possible to find levels α_n and τ_n such that $q(\tau_n, p) = q(\alpha_n, 1)$ by imposing

$$\tau_n = \frac{\mathbb{E}(|Y - q(\alpha_n, 1)|^{p-1} \mathbb{1}_{\{Y \leq q(\alpha_n, 1)\}})}{\mathbb{E}(|Y - q(\alpha_n, 1)|^{p-1})}.$$

Asymptotically, as $n \to \infty$, one can show that

$$\frac{1 - \tau_n}{1 - \alpha_n} \to \frac{1}{\gamma} B\left(p, \frac{1}{\gamma} - p + 1\right).$$

Starting from the two equations in red, one can estimate extreme quantiles from extreme L^p-quantiles.

Letting

$$\widehat{\tau}'_n(p,\alpha_n) := 1-(1-\alpha_n)\frac{1}{\widehat{\gamma}_n}B\left(p,\frac{1}{\widehat{\gamma}_n}-p+1\right),$$

the extreme quantile $q(\alpha_n,1)$ can be estimated by $q_n(\widehat{\tau}'_n(p,\alpha_n)|\tau_n,p)$ where $q_n(\cdot|\tau_n,p)$ is an estimator of the extreme L^p-quantile $q(\cdot,p)$ *i.e.* $q_n(\cdot|\tau_n,p)=\widehat{q}_n^W(\cdot|\tau_n,p)$ or $q_n(\cdot|\tau_n,p)=\widetilde{q}_n^W(\cdot|\tau_n,p)$.

Asymptotic normality results have been established for both estimators $\widehat{q}_n^W(\widehat{\tau}'_n(p,\alpha_n)|\tau_n,p)$ and $\widetilde{q}_n^W(\widehat{\tau}'_n(p,\alpha_n)|\tau_n,p)$.

Illustration on simulations

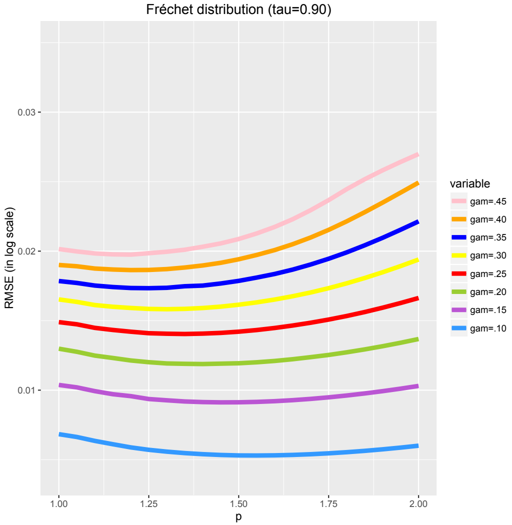
We focus on L^p-quantiles for $p \in (1, 2)$ (alternative risk measure to expectiles with a weaker existence condition).

The accuracy is assessed by computing the relative mean-squared error (MSE) on 3000 replications of samples of size $n = 200$ from a Fréchet distribution: $F(x) = e^{-x^{-1/\gamma}},\ x > 0$.

1) Intermediate extreme level

Which L^p-quantiles can be estimated accurately with $\widehat{q}_n(\tau_n, p)$?
We consider a L^p-quantile of level $\tau_n = 0.9$.
In the following, $\gamma \in \{0.1, 0.15, \ldots, 0.45\}$ and $p \in \{1, 1.05, \ldots, 2\}$.

Relative MSE - Fréchet distribution - Intermediate level



Horizontally: p, Vertically: relative MSE (in log scale) for different values of γ (small values: bottom curves, large values: top curves).

First conclusions:

- The estimation accuracy is getting lower when γ increases.
- When $\gamma \geq 0.2$, the estimation of expectiles ($p = 2$) is more difficult than the estimation of quantiles ($p = 1$).
- The value of p minimizing the relative MSE depends on the tail index γ. However, $p \in [1.2, 1.4]$ seems to be a good compromise.

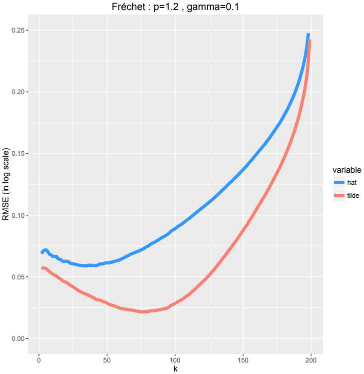
2) Arbitrary extreme level

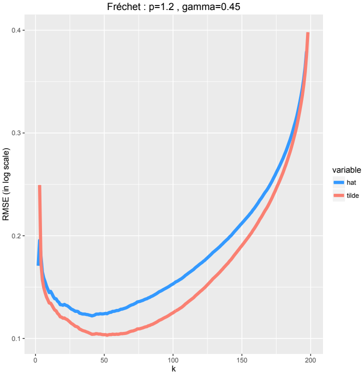
Comparison of:

- $\widehat{q}_n^W(\tau_n'|\tau_n, p)$ (based on empirical criterion + extrapolation),
- $\widetilde{q}_n^W(\tau_n'|\tau_n, p)$ (based on the extreme L^1- quantile).

We consider a L^p-quantile of level $\tau_n' = 1 - 1/n$.
In the following, $\tau_n = 1 - k/n$ where $k \in \{2, \ldots, n-1\}$, $\widehat{\gamma}_n$ is Hill's estimator, $\gamma \in \{0.1, 0.45\}$ and $p \in \{1.2, 1.5, 1.8\}$.

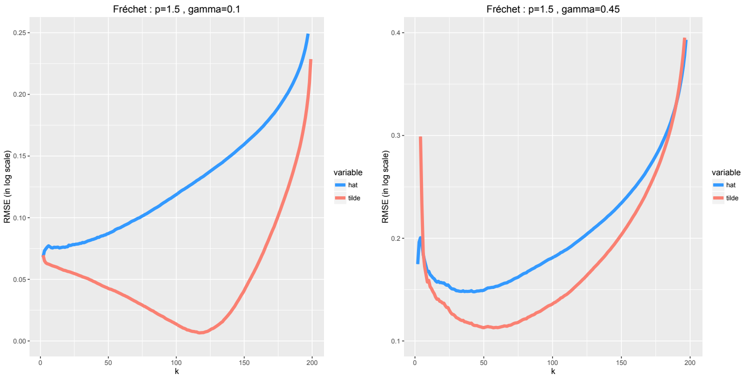
Relative MSE - Fréchet distribution - Extreme level





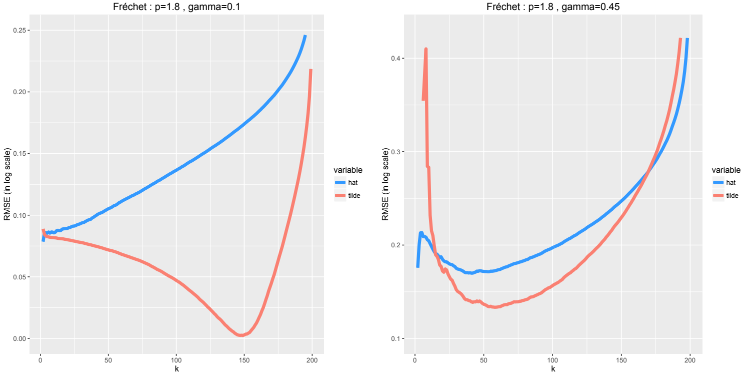
Horizontally: k, Vertically: relative MSE (in log scale) of $\widehat{q}_n^W(\tau_n'|\tau_n, p = 1.2)$ and $\widetilde{q}_n^W(\tau_n'|\tau_n, p = 1.2)$ as a function of $k \in \{2, \ldots, n-1\}$ (left: $\gamma = 0.1$, right: $\gamma = 0.45$).

Relative MSE - Fréchet distribution - Extreme level



Horizontally: k, Vertically: relative MSE (in log scale) of $\widehat{q}_n^W(\tau_n'|\tau_n, p = 1.5)$ and $\widetilde{q}_n^W(\tau_n'|\tau_n, p = 1.5)$ as a function of $k \in \{2, \ldots, n-1\}$ (left: $\gamma = 0.1$, right: $\gamma = 0.45$).

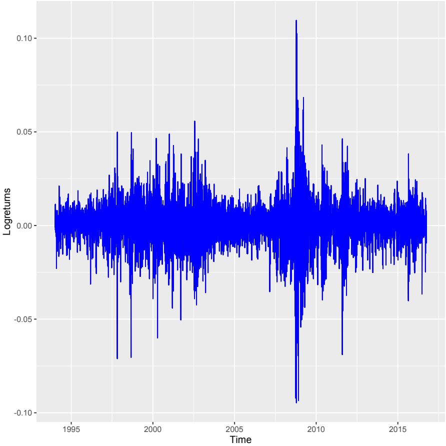
Relative MSE - Fréchet distribution - Extreme level



Horizontally: k, Vertically: relative MSE (in log scale) of $\widehat{q}_n^W(\tau_n'|\tau_n, p = 1.8)$ and $\widetilde{q}_n^W(\tau_n'|\tau_n, p = 1.8)$ as a function of $k \in \{2, \ldots, n-1\}$ (left: $\gamma = 0.1$, right: $\gamma = 0.45$).

Illustration on real data

S&P500 index from Jan, 4th, 1994 to Sep, 30th, 2016 (5727 trading days).



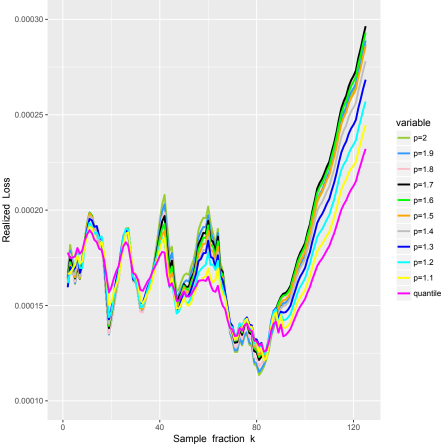
To reduce the potential serial dependence, we used lower frequency data by choosing weekly returns in the same sample period (Cai *et al.*, 2015). This results in a sample $\{Y_1, \ldots, Y_{1176}\}$ of size 1176.

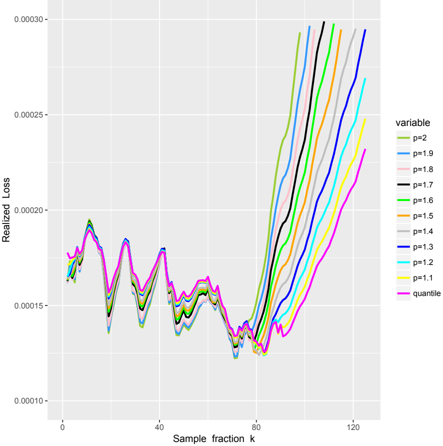
Computation of prediction errors

For $t = 1, \dots, 656$

- Starting from $\{Y_t, \dots, Y_{t+n-1}\}$ a training sample with $n = 520$,
- Our goal is to estimate $q(1/n, 1)$ which can be viewed as the weekly loss return for a once-per-decade financial crisis.
- Three estimators are computed:
  - $\widehat{q}_n^W(1/n|\tau_n, p = 1)$ (Weissman estimator for L^1- quantiles),
  - $\widehat{q}_n^W(\widehat{\tau}_n'(p, 1/n)|\tau_n, p)$ and $\widetilde{q}_n^W(\widehat{\tau}_n'(p, 1/n)|\tau_n, p)$ (based on estimators for L^p- quantiles).
- The associated prediction errors are computed with respect to Y_{t+n}.

Prediction errors - Weekly loss returns





Horizontally: k, Vertically: Prediction error for $\widehat{q}_n^W(\widehat{\tau}_n'(p,1/n)|\tau_n,p)$ (left), $\widetilde{q}_n^W(\widehat{\tau}_n'(p,1/n)|\tau_n,p)$ (right) and $\widehat{q}_n^W(1/n|\tau_n,p=1)$ (magenta) as a function of k.

Conclusion

- The extreme behaviour of L^p-quantiles has been established.
- Classical quantiles as well as expectiles are particular cases of L^p-quantiles.
- In contrast to quantiles, L^p-quantiles take into account the whole tail structure of the distribution.
- The condition for existence of L^p-quantiles is weaker than for expectiles.
- It is possible to extrapolate to arbitrarily large levels.
- The theory has been extended to a mixing dependence framework and to real-valued distributions.
- L^p-quantiles may be adapted to the regression framework (work in progress).

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