IDAF CURVES OF EXTREME STORM RAINFALL: A SCALING APPROACH

C. De Michele, N. T. Kottegoda and R. Rosso

DIIAR Politecnico di Milano, P.zza L. da Vinci, 32, Milano, I-20132 Italy

ABSTRACT

Intensity-duration-area frequency curves, IDAF, are determined for the evaluation of design storms using a scaling approach. The variability of maximum annual rainfall intensity in area and duration is represented through the scaling properties in time and space. Thus the scaling relationships of mean rainfall intensity with area and duration are derived using the concepts of dynamic scaling and statistical self-affinity. For a lognormal distribution of rainfall intensity a multiscaling lognormal model is obtained. This gives the IDAF curves of extreme storm rainfall. An application is made to the metropolitan area of Milano.

KEYWORDS

Extreme rainfall; dynamic scaling; IDF; IDAF; multiscaling; self-affinity

INTRODUCTION

Intensity-duration-area frequency curves, IDAF, that estimate the rainfall intensity over duration, $D$, area, $A$, and return period, $T$, provide important information for the evaluation of design storms. These curves are also one of the simplest methods of transposing storm data. The determination of IDAF curves is generally performed using diagrams (see, e.g., Raudkivi, 1979, p. 93, which shows the maximum depth-duration-area curves for a catchment) or by means of empirical formulae.

In Italy, IDAF curves are obtained by modifying the parameters of intensity-duration frequency curves, IDF, in order to consider the variability of intensity with area (see e.g. Supino, 1965, p. 119). In particular IDF curves are represented by a power law

$$I(D, T) = a_i(T)D^{\nu(T)}$$

where $a_i(T)$ is the intensity for unit duration and a return period $T$ and $\nu$ is a climatic exponent; the IDF parameters are modified as $a = a_i \left[1 - c_1 A + c_2 A^2\right]$ and $\nu' = \nu + c_3 A$, where the three coefficients $c_1$, $c_2$, and $c_3$ are dependent on physiographical characteristics and are generally determined by methods of regression.

More generally, the average areal intensity of specified duration and return period is obtained from the point rainfall for the same duration and return period by multiplying the point rainfall with a suitable factor termed "areal reduction factor", ARF, (see NERC, 1975, Vol. II, p. 38). For its estimation empirical formulae have been applied in the past (see Eagleson, 1970, and Raudkivi, 1979, for a review).

The concepts of scaling and multiscaling provide insights to the apparent complexities of hydrological phenomena, through simple mathematical formulations. Accordingly, simple models representing the intrinsic structure of this complexity can incorporate the spatial and temporal fluctuations of investigated processes. Previously scale invariance has been shown to explain the variability of a number of hydrological processes, such as river networks, surface and groundwater flows and precipitation. This indicates a unified framework to assess heuristic knowledge of these processes, sometimes providing a sound mathematical basis to empirical laws derived from field observations.

In this paper, we adopt a simple statistical approach to estimate the IDAF curves of extreme storm rainfall based on its scaling properties in space and time. In particular the concepts of dynamic scaling and statistical self-affinity are used to
model the variability of the rainfall random field in time and space. Because $IDAF$ curves reflect the variability of rainfall in space and time, it is necessary to make a joint analysis of scaling properties of the rainfall field in area and duration. Thus a scaling model is formulated to evaluate the $IDAF$ curves. The procedure is then applied to the metropolitan area of Milano, Italy.

**INTENSITY AND DURATION OF RAINFALL**

Let the random variable $I(D)$ denote the maximum annual value of local rainfall intensity over a duration $D$. It is defined as

$$I(D) = \max_{0 \leq d \leq D} \left( \frac{1}{D^{D/2}} \int_{d-D/2}^{d+D/2} X(\zeta) d\zeta \right)$$  \hspace{1cm} (2)$$

where $X(\zeta)$ is a time continuous stochastic process representing rainfall intensity and $d$ is a point in time.

Let $I(D, A)$ denote the maximum annual value of average rainfall intensity over a duration $D$, and area $A$ centered in $\mathbf{u}(u_1, u_2)$. It is defined as

$$I(D, A) = \max_{0 \leq d \leq D} \left( \frac{1}{DA} \int_{A-D/2}^{A+D/2} \int_{d-D/2}^{d+D/2} X(\zeta, \omega) d\zeta d\omega \right)$$  \hspace{1cm} (3)$$

where $X(\zeta, \omega)$ is the time-space continuous stochastic process representing rainfall intensity.

The variability of local intensity with duration and probability of non-occurrence, represented by the intensity-duration-frequency curves $IDF$, has been studied extensively in the past. Rosso and Burlando (1990) proposed a simple scaling formulation of $IDF$; Burlando and Rosso (1996) introduced a multifractal approach and Menabde et al. (1999) determined the scaling relationship of intensity with duration in terms of the parameters of the Gumbel probability distribution. However, the variability of average intensity with duration, area and probability of non-occurrence represented by the intensity-duration-area-frequency curves $IDAF$ has not yet been investigated. In the following sections, we consider the scaling properties of the rainfall random field in time and space and derive the relationships that link mean rainfall intensity with area and duration. This is followed by a multiscaling formulation of $IDAF$ curves.

**SPACE-TIME SCALING OF RAINFALL FIELD**

Let us consider a rainfall random field $\{I(.)\}$ represented by the variable $I$ parameterized in time (duration) and space (area) by the vector $(D, A)$. To study the variability of $I(.)$ in time and space we assume that the random field has *dynamic scaling* properties, i.e. for every pair $A_i$ and $A_j$, it is possible to rescale time using the relationship

$$D_i/D_j = \left( \frac{A_i}{A_j} \right)^z$$  \hspace{1cm} (4)$$

where $z$ is the dynamic scaling exponent. Denoting $\lambda = A_i/A_j$ and $\eta = D_i/D_j$, eq. (4) is written as $\eta = \lambda^z$. For $\eta > 0$ and $\lambda > 0$ the assumption of dynamic scaling allows the rescaled random field $\{I(\eta D, \lambda A)\}$ to be a function of the multiplier $\lambda$, that is, $\{I(\lambda D, \lambda A)\}$. Thus by invoking the concept of dynamic scaling in the space-time variability of the random field $\{I(.)\}$ one can effectively reduce a two dimensional problem with two ratios $\lambda$ (space) and $\eta$ (time) to a one-dimensional problem with a single ratio $\lambda$.

Then introducing the concept of scale invariance in the statistical sense and combining it with the concept of dynamic scaling, one can introduce the definition of *statistical self-affinity* in $(D, A)$ for the random field $\{I(.)\}$ (see also Schertzer and Lovejoy, 1989)

$$\{I(\lambda^a D, \lambda^b A)\} \overset{d}{=} \lambda^{-H} \{I(D, A)\}$$  \hspace{1cm} (5)$$

where the symbol $\overset{d}{=}$ denotes equality in probability distribution, $a$ and $b$ two positive constant scaling exponents, linked by the relation $a = b z$, and $H > 0$ is a scaling exponent. Note that if $H$ is independent of the level of probability then eq. (5) represents the simple case of *statistical self-affinity*; if $H$ varies with the level of probability then it gives the more general case of *statistical multiscaling*. For the problem considered, the determination of $IDAF$ curves, it is convenient to
center the random field with respect to the area $A_0$ intercepted by a raingauge, representing the minimum area of practical interest, thus eq. (5) is written as

$$\{I(H^a, H^b A^*)\} = \lambda^{-H} \{I(D, A')\}$$

(6)

where $A' = (A - A_0) \geq 0$ and $D \geq 0$.

From eq. (6) it is possible to derive a similar relation for the $q^{th}$ order moment as

$$E[I^q(H^a, H^b A^*)] = \lambda^{-qH} E[I^q(D, A')]$$

(7)

where $\xi(q)$ is a function of the order of the moment (a linear function, $\xi(q) = Hq$, in the case of self-affinity). The scale invariance solution of eq (6) has a form of the type (Theorem 2.2, p.453, Logan, 1987)

$$I(D, A') = D^{-\nu/a} f\left(\frac{A^{*a}}{D^b}\right)$$

(8)

where $f(.)$ is a random function. Thus eq. (8) defines the general form of the random field $\{I(.)\}$ under the transformation defined by eq. (6). The function, $f(A^{*a}/D^b)$, is obtained from its asymptotic properties. When $A' \to 0$, the rainfall intensity, $I(D, A')$, tends to the rainfall intensity at the raingauge $I(D, 0)$. Hence the random function $f(.)$ tends to $a_1$, say, the local intensity for unit duration. When $A' \to \infty$, the mean areal intensity of rainfall tends to zero and therefore $f(.) \to 0$. If $D \to \infty$, the rainfall intensity is uniform for all $A^*$ (as illustrated, for instance, by U.S. Weather Bureau, 1958) and the function $f(.) \to a_1$ (as for $A' \to 0$). Consequently the scaling function $f(.)$ is of the type

$$f\left(\frac{A^{*a}}{D^b}\right) = a_1 \left(1 + \frac{A^{*a}}{D^b}\right)^{-\beta}$$

(9)

where $\beta$ is a positive scaling exponent and $\varpi$ is a factor of normalization expressed in units of hrs$^b$km$^{-2a}$. The exponents $H$, of eqs. (7) and (8), and $\beta$ of eq. (9) are not independent but they are linked by the relationship

$$\beta = \frac{H}{ab}$$

(10)

This can be verified as follows: from eq. (8) and eq. (9), if $\varpi A^{*a}/D^b << 1$ one finds that $I(D, A') \propto D^{-H/a}$, or in other words, the variability of rainfall intensity is determined by $D$. On the contrary, if $\varpi A^{*a}/D^b >> 1$, one obtains, assuming eq. (10), that $I(D, A') \propto A^*^{-H/a}$; that is, the variability of rainfall intensity is determined by $A^*$. Thus by invoking eq. (10) one obtains these limiting relationships that are physically evident in the space-time variability of rainfall fields. Note that these relationships correspond to the approximation of the function, $f(.)$, by two straight lines. This is shown in Fig. 1, in which the random function, normalized with respect to $a_1$, $f(.)/a_1$, is plotted against the non-dimensional argument ($\varpi A^{*a}/D^b$) assuming $\beta = 0.9$. Thus eq. (8) becomes

$$I(D, A') = a_1 D^{-\nu/a} \left(1 + \frac{A^{*a}}{D^b}\right)^{\frac{H}{ab}} = a_1 D^{-\nu} \left[1 + \frac{A^{*a}}{D^b}\right]^{-\frac{H}{ab}}$$

(11)

where $\nu = H/a$ is the climatic exponent. From eq. (11), it is interesting to note that the rainfall intensity $I(.)$ is expressed as a product of two terms $a_1 D^{-\nu}$ representing the local intensity of rainfall, and $\left[1 + \frac{A^{*a}}{D^b}\right]^{-\frac{H}{ab}}$, that is, the reduction function of rainfall intensity with area. Thus, for $A^* = 0$, eq. (11) gives the well known formula

$$I(D, 0) = a_1 D^{-\nu}$$

(12)

that expresses the variability of local intensity with duration. We note that the empirical formula, represented by eq. (1), widely used in Italy to evaluate IDF curves, is similar to eq. (12) obtained theoretically from the hypothesis of statistical self-affinity of the random field $\{I(.)\}$. 


MULTISCALING FORMULATION OF IDAF CURVES

According to Schertzer and Lovejoy (1989), Gupta and Waymire (1990), Burlando and Rosso (1996), among others, the scaling exponent $H$ in eq. (6) is, generally, not constant but it is a function of the level of probability, or in other words of the return period. Consequently the exponent $\xi(q)$ in eq. (7) is not linear but a function of $q$, $\xi(q) = H_q q$.

Lovejoy and Schertzer (1990) proposed a general form for the variability of $H_q$ as

$$ H_q = H_1 - H_2 + q(H_2 - H_1) $$

where $H_1$ and $H_2$ are the values of the scaling exponent, $H$, for the first two statistical moments. Thus using the log-normal distribution to represent the variability of the rainfall intensity $I(.)$ with respect to the frequency $F$ or return period $T$, its cdf is

$$ \Pr[I(D, A^*) \leq I] = \frac{1}{\sqrt{2\pi \text{var}[\ln I]}} \exp\left[-\frac{1}{2} \left( \frac{\ln I - E[\ln I]}{\sqrt{\text{var}[\ln I]}} \right)^2 \right] \int_0^\zeta d\zeta $$

where $E[\ln I]$ and var[ln I] are the parameters of the transformed variable, lnI, related to the mean and variance of I as $E[\ln I] = \ln E[I] - \frac{1}{2} \ln \left( 1 + \frac{\text{var}[I]}{E^2[I]} \right)$ and var[ln I] = ln $\left( 1 + \frac{\text{var}[I]}{E^2[I]} \right)$. Remembering that var[I] = $E[I^2] - E^2[I]$, inverting eq. (14) the $T-th$ quantile of I becomes

$$ I_T(D, A^*) = \frac{E[I]}{E[I^2]} \left[ \Phi^{-1}_{1/T} \sqrt{E[I^2]} \right] $$. (15)

where $\Phi^{-1}_{1/T}$ is the standard normal variate with a probability of exceedance of 1/T. Writing eq. (11) in terms of statistical moments, according to eq. (13), we have

$$ E[I^* (D, A^*)] = E[q^*, D^{-\nu}] D^\nu \left[ 1 + \omega \left( \frac{A^*}{D} \right)^6 \right]^{\frac{\nu a}{T}}. $$

(16)
Substituting eq. (16) for the first two moments in eq. (15), it is possible to derive a second order log-normal multiscaling model that describes the variability of $I$ with duration, area and return period as follows:

$$I(D, A) = \frac{E[a_i^2]}{\sqrt{E[a_i^2]}} \exp \left[ \Phi^{-1} \left( \frac{1}{2} \right) \right] D^{-\frac{2}{2(\nu_1 - \nu_2)}} \left( 1 + \frac{A^\nu}{D^b} \right)^{2(\nu_1 - \nu_2)} \left( 1 + \frac{A^\nu}{D^b} \right)^{-2(\nu_1 - \nu_2)}.$$  \hspace{1cm} (17)

Eq. (17) gives the multiscaling formulation of IDA$F$ curves. It is a generalization of the second order log-normal multiscaling model proposed by Burlando and Rosso (1996) to describe the variability of $I$ with duration and return period.

**APPLICATION**

Here, the network of raingauges in the metropolitan area of Milano, Italy is considered. The database comprises 8-years (1973-1981) of continuous rain fall observed at 16 stations covering an area of about 300 km$^2$. The annual maximum of the average rainfall intensity $I(D, A)$ over a duration $D$ and area $A$ were obtained using the kriging method. The rainfall is aggregated over areas ranging from 0.25 km$^2$ to a maximum around 300 km$^2$, and duration from 20 minutes to 6 hours. Then annual maximum values of areal intensity are obtained. Fig. 2 gives the variability of expected values of local intensity with duration. From Fig. 2 it is evident that $E[I(D, 0)]$ has a linear behaviour in the plane $\ln D, \ln E[I(D, 0)]$. From eq. (16), written for $q = 1$ and $A^* = 0$, we can estimate $E[a_i]$ and the climatic exponent $\nu_1 = H_l/a$ using a linear regression. Thus we have $E[a_i] = 31.4$ mm/h$^{1-\nu}$ and $\nu_1 = 0.484$. Hence minimizing the differences between eq. (16) written hence $q = 1$ and the average values of areal intensity obtained from the data, we estimate the parameters $\nu, z$ and $b$ as $\nu = 0.0905$ h$^{-1}$/(km$^2$), $z = 1$ and $b = 0.540$. Fig. 3 shows the agreement between the model, represented by eq. (16), and the observed data. Note that the dynamic scaling exponent, $z$, is equal to 1 for the metropolitan area of Milano an isotropic behaviour of rainfall field in space and time. In a similar way, using eq. (16), written for $q = 2$ and $A^* = 0$, we estimate $E[a_i^2]$ and $\nu_2$ as $E[a_i^2] = 1269$ [mm/h$^{2-\nu}$] and $\nu_2 = 0.513$. Fig. 4 shows the observed and calculated second statistical moment of mean local rainfall intensity. In Fig. 5 we give the sample variability of the function $v_D(q)$ with the order of the moment $q$ (after estimating the values of $\nu_1$, $\nu_2$, $\nu_3$, $\nu_4$) compared to scaling theoretical behaviors: self-affinity and multiscaling. The first is represented by the linear function $v_D(q)$, the second is obtained using eq. (13) and estimated values of $\nu_1$ and $\nu_2$. From Fig. 5 it is evident that the multiscaling model gives a much closer representation of the sample variability of Milano rainfall data compared to the self-affine model. After estimating the parameters of the multiscaling model, we evaluate the $T$-year quantile of rainfall intensity for a fixed value of duration and area. In Fig. 6 the $T$-year quantiles (lines) for a duration of 1 hour and values of areas of 0, 0.20, 13 and 113 km$^2$, obtained using eq. (17), are compared to the Weibull plotting positions (dotted lines with points). In Fig. 7 we give the same graphs using a duration of 3 hours. These show the range of agreement between the model and the observed data. From Fig.6 and 7, it is also important to note the decreasing of rainfall intensity with increasing A, for fixed values of duration and return period.

**Figure 2.** Plot of observed (dots) and calculated values of mean local rainfall intensity in Milano against duration.

**Figure 3.** Plot of observed (dots) and calculated values of mean areal rainfall intensity in Milano against area, for different durations.
Figure 4. Plot of the second statistical moment of mean local rainfall intensity in Milano, observed (dots) and calculated, against duration.

Figure 5. Sample variability of the function $-\nu q$ with the order of the moment $q$ compared to the scaling behaviors: self-affinity and multiscaling.

Figure 6. The $T$-year quantiles (lines) of rainfall intensity for a duration of 1 hour and values of areas of 0, 0.20, 13 and 113 km$^2$, obtained using eq. (17), are compared to Weibull plotting positions (dotted lines with points) in Milano.
CONCLUSIONS

The focus of this paper is to derive the IDAF curves of extreme storm rainfall from the scaling properties of storm rainfall in time and space. On the basis of dynamic scaling and statistical self-affinity we obtained a scaling relation of average rainfall intensity in area and duration. Then introducing the lognormal probability law we derived a second order lognormal multiscaling model to describe the variability of $I$ with duration, area and return period and thus a multiscaling model of IDAF curves. The proposed model provides a close approximation to observations from the metropolitan area of Milano. The results of this study significantly support the conjecture that scaling holds for the rainfall intensity of extreme events in time and space. Hence the scaling approach provides assessment of design storm evaluation in hydrological practice.

ACKNOWLEDGEMENTS

This research has been carried out as a part of project “Framework, Flash-flood Risk Assessment under the iM pact of land use changes and river Engineering WORKs” granted by the European Commission, DGXII, contract ENV-CT97-0529.
REFERENCES


