Some asymptotic equivalence results in the Le Cam sense

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Grenoble, 04 December 2014

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Comparison of statistical models

Example (Le Cam and Yang (1990))

A physicist decides to estimate the half life of Carbon 14. He assumes that the life of a C^{14} has an exponential distribution of parameter θ, unknown. He takes a batch of n atoms and he begins to think about how to run the experiment. He has in mind two different ways to proceed :

1. Let P_θ be the law of X_1 = numbers of atoms that decay before a certain time $\rightsquigarrow \mathscr{P}_1 : (P_\theta)_{\theta\in(0,\infty)}$.
2. Let Q_θ be the law of X_2 = the first moment when the number of disintegrated atoms reaches a certain threshold $\rightsquigarrow \mathscr{P}_2 : (Q_\theta)_{\theta\in(0,\infty)}$.

Can one compare the experiments $\mathscr{P}_1$ and $\mathscr{P}_2$? How different can their statistical properties be?

The Le Cam Δ-distance

Definition

The **deficiency** $\delta(\mathcal{E}_1, \mathcal{E}_2)$ of $\mathcal{E}_1$ with respect to $\mathcal{E}_2$ is defined as

$$\delta(\mathscr{P}_1, \mathscr{P}_2) = \sup_L \inf_{\pi_1} \sup_{\pi_2} \sup_\theta |R(\mathscr{P}_1, \pi_1, L, \theta) - R(\mathscr{P}_2, \pi_2, L, \theta)|.$$

The so called **Δ-distance** between $\mathcal{E}_1$ and $\mathcal{E}_2$ is defined as:

$$\Delta(\mathcal{E}_1, \mathcal{E}_2) = \max\left(\delta(\mathcal{E}_1, \mathcal{E}_2), \delta(\mathcal{E}_2, \mathcal{E}_1)\right).$$

Two sequences of models $(\mathcal{E}_1^n)_{n\in\mathbb{N}}$ and $(\mathcal{E}_2^n)_{n\in\mathbb{N}}$ are **asymptotically equivalent** if $\Delta(\mathcal{E}_1^n, \mathcal{E}_2^n) \to 0$ as $n \to \infty$.

- **Interpretation**: Asymptotic equivalence means that any statistical inference procedure can be transferred from one experiment to the other in such a way that the asymptotic risk remains the same, at least for bounded loss functions.

1. Introduction
2. Main results (Asymptotic equivalences)
   - Additive processes with unknown drift function
   - Compound Poisson processes with unknown Lévy density
   - Diffusion processes and their Euler scheme: small variance case
3. Work in progress

Additive processes

$$X_t = \int_0^t f(s)ds + \int_0^t \sigma(s)dW_s + \sum_{i=1}^{N_t} Y_i, \quad t \geq 0,$$

- $W = \{W_t\}_{t\geq 0}$ is a standard Brownian motion;
- $N = \{N_t\}_{t\geq 0}$ is an inhomogeneous Poisson process with intensity function $\lambda(\cdot)$, independent of W;
- $(Y_i)_{i\geq 1}$ is a sequence of i.i.d. real random variables with distribution G (either concentrated on $\mathbb{Z}$ or absolutely continuous with respect to Lebesgue), independent of W and N;
- $f(\cdot)$ belongs to a certain non-parametric class $\mathcal{F}$; $\lambda(\cdot)$ and G are unknown and $\sigma^2(\cdot)$ is known (to be discussed).

The problem we consider

We suppose to observe $\{X_t\}_{t\geq 0}$ at discrete times $0 = t_1 < \cdots < t_n = T_n$ such that

$$\Delta_n = \max_{1\leq i\leq n} \big\{|t_i - t_{i-1}|\big\} \downarrow 0 \text{ as } n \to \infty.$$

Problem: To estimate the drift function $f(\cdot)$ from the discrete data $(X_{t_i})_{i=1}^n$.

At least two natural questions arise:

1. How much information about the parameter $f(\cdot)$ do we lose by observing $(X_{t_i})_{i=1}^n$ instead of $\{X_t\}_{t\in[0,T_n]}$?
2. Can we construct an easier (read: mathematically more tractable), but equivalent, model from $(X_{t_i})_{i=1}^n$?

The case G concentrated on $\mathbb{Z}$

Theorem (M., 2014)

For n big enough:

$$\Delta\big(D_n, W\big) \leq \sqrt{2\sum_{i=1}^{n}\bigg(\frac{6}{\sigma_i}\varphi\Big(\frac{1}{6\sigma_i}\Big)+4\phi\Big(\frac{-1}{6\sigma_i}\Big)\bigg)}$$

$$+\sup_{f\in\mathcal{F}}\bigg(\sum_{i=1}^{n}\frac{(f(t_i)-f(\gamma_i))^2}{2\sigma^2(\eta_i)}\Delta_n+\int_0^{T_n}\frac{(f(t)-\bar{f}_n(t))^2}{\sigma^2(t)}dt\bigg)$$

$$+2\sqrt{\sum_{i=1}^{n}\lambda_i^2}.$$

Remark: If $\lambda \in L_\infty(\mathbb{R})$, then $\sqrt{\sum_{i=1}^{n}\lambda_i^2} = O(\sqrt{T_n\Delta_n})$.

Rate of convergence in the small variance case

$\mathcal{F}$: a class of α-Hölder, uniformly bounded functions on $\mathbb{R}$, $\lambda(\cdot) \in L_\infty(\mathbb{R})$ and let T be fixed.

Theorem (M., 2014)

$$\begin{aligned} \Delta\big(D_n, W\big) &\to 0 \\ \Delta\big(C, D_n\big) &\to 0 \end{aligned} \qquad \textit{as } n \to \infty,$$

under either of the following two sets of conditions:

1. Y_1 *is discrete with support on* $\mathbb{Z}$, $\alpha \geq \frac{1}{2}$; *in this case the rate of convergence is* $O(\sqrt{\Delta_n})$.
2. Y_1 *admits a density with respect to the Lebesgue measure on* $\mathbb{R}$, $\alpha \geq \frac{1}{4}$; *in this case the rate of convergence is* $O\Big(\Delta_n^{\frac{1}{4}}\Big)$.

Compound Poisson processes

$$L_t = \sum_{i=1}^{N_t} Y_i, \quad t \geq 0,$$

- $N = \{N_t\}_{t\geq 0}$ is a Poisson process with mean λ;
- $(Y_i)_{i\geq 1}$ is a sequence of i.i.d. real random variables with densities (with respect to Lebesgue) g, independent of N and concentrated on $[0,1]$;
- $f(\cdot) := \lambda g(\cdot)$ belongs to a certain non-parametric class $\mathcal{F}$;

We observe $\{L_t\}_{t\geq 0}$ at n equidistant discrete times $0 = t_1 < \cdots < t_n = T_n$, $t_i = T_n \frac{i}{n}$:

$$\Delta_n = \frac{T_n}{n} \downarrow 0 \text{ and } T_n \to \infty \text{ as } n \to \infty.$$

Problem: To estimate the Lévy density $f(\cdot)$ from the discrete data $(L_{t_i})_{i=1}^n$.

Definition of the experiments

- $P_{T_n}^f$: law of the process $\{L_t\}_{t\in[0,T_n]}$ on $(D,\mathcal{D})$,
- Q_i^f: law of the random variable $L_{t_i}-L_{t_{i-1}}$ on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$,
- $\mathbb{W}^f$: law induced on $(C,\mathscr{C})$ by

$$dY_t=\sqrt{f(t)}dt+\frac{dW_t}{2\sqrt{T_n}},\quad t\in[0,1].$$

The statistical models we consider are:

- $\mathscr{P}_n=\big(D,\mathcal{D}_{T_n},\{P_{T_n}^f:f\in\mathcal{F}\}\big)$,
- $\mathscr{Q}_n=\big(\mathbb{R}^n,\mathcal{B}(\mathbb{R}^n),\{\bigotimes_{i=1}^n Q_i^f:f\in\mathcal{F}\}\big)$,
- $\mathscr{W}=\big(C,\mathscr{C},\{\mathbb{W}^f:f\in\mathcal{F}\}\big)$.

Equivalence between discrete and continuous observations

Theorem (M. 2014)

For every choice of a sequence of integer $m = m_n$, we have an asymptotic equivalence with the following rates of convergence (provided that the stated sequences do converge to zero):

$$\Delta(\mathscr{Q}_n, \mathscr{W}) = O\Big(\sqrt{T_n\Delta_n} + \frac{m\ln m}{\sqrt{n}} + \sqrt{T_n}m^{-\gamma}\Big),$$

$$\Delta(\mathscr{P}_n, \mathscr{W}) = O\Big(T_n^{-\frac{\gamma}{4+2\gamma}}\Big),$$

$$\Delta(\mathscr{P}_n, \mathscr{Q}_n) = O\Big(T_n^{-\frac{\gamma}{4+2\gamma}} + \sqrt{T_n\Delta_n} + \frac{m\ln m}{\sqrt{n}} + \sqrt{T_n}m^{-\gamma}\Big).$$

Example: If $n = T_n^{\alpha}$ with $\alpha > 2$, then $m = T_n^{\frac{1+\alpha}{2(1+\gamma)}}$. In this case one finds that: $\Delta(\mathscr{Q}_n, \mathscr{W}) = O\Big(\sqrt{T_n^{2-\alpha}} + T_n^{\frac{1-\alpha\gamma}{2(1+\gamma)}}\ln(T_n)\Big)$.

Diffusion processes: small variance case

$$dy_t = f(y_t)dt + \varepsilon\sigma(y_t)dW_t, \quad t \in [0,T], \quad y_0 = 0.$$

We observe $\{y_t\}_{t\geq 0}$ at n equidistant discrete times $t_i = T\frac{i}{n}$.

Problem: To show that estimating $f(\cdot)$ from the discrete data $(y_{t_i})_{i=1}^n$ is equivalent to estimate $f(\cdot)$ from the Euler scheme:

$$Z_0 = 0, \quad Z_i = Z_{i-1} + \frac{f(Z_{i-1})}{n} + \varepsilon\sigma(Z_{i-1})\xi_i, \quad i = 1,\dots,n,$$

with independent standard normal variables ξ_i.

Motivation: Inference for discretely observed diffusion models is generally difficult (because of the intractability of the transitions densities). A way to bypass this problem is to consider statistical procedures based on the corresponding Euler scheme.

Main result

$$
\begin{aligned}
\mathscr{P}_y^{n,T} &= (C, \mathcal{C}_T, (P_f^{n,y}, f \in \mathcal{F})), \quad P_f^{n,y} = \mathcal{L}(y)\\
\mathscr{Q}_y^{n} &= (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), (Q_f^{n,y}, f \in \mathcal{F})), \quad Q_f^{n,y} = \mathcal{L}((y_{t_i})_i),\\
\mathscr{Q}_Z^{n} &= (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), (Q_f^{n,Z}, f \in \mathcal{F})), \quad Q_f^{n,Z} = \mathcal{L}((Z_i)_i).
\end{aligned}
$$

Theorem

If $\varepsilon n \to \infty$ as $n \to \infty$ and $\varepsilon \to 0$, the experiments $\mathscr{P}_y^T$ and $\mathscr{Q}_Z^n$ are asymptotically equivalent. More precisely we have

$$
\begin{aligned}
\Delta\big(\mathscr{P}_y^T, \mathscr{Q}_Z^n\big) &= O\Big(\frac{1}{\varepsilon n} + (n^{-1} + \varepsilon)^{1/4}\Big),\\
\Delta\big(\mathscr{P}_y^T, \mathscr{Q}_y^n\big) &= O\Big(\frac{1}{\varepsilon n}\Big).
\end{aligned}
$$

Perspectives

Extensions:

- A generalization for X_t taking values in $\mathbb{R}^d$.
- To generalize to the case ν infinite Lévy measure.

Future work:

- Asymptotic equivalence for stationary distribution.

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Thank you for your attention !