Measuring Spatial Dependence among Maxima

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Outline of the Talk

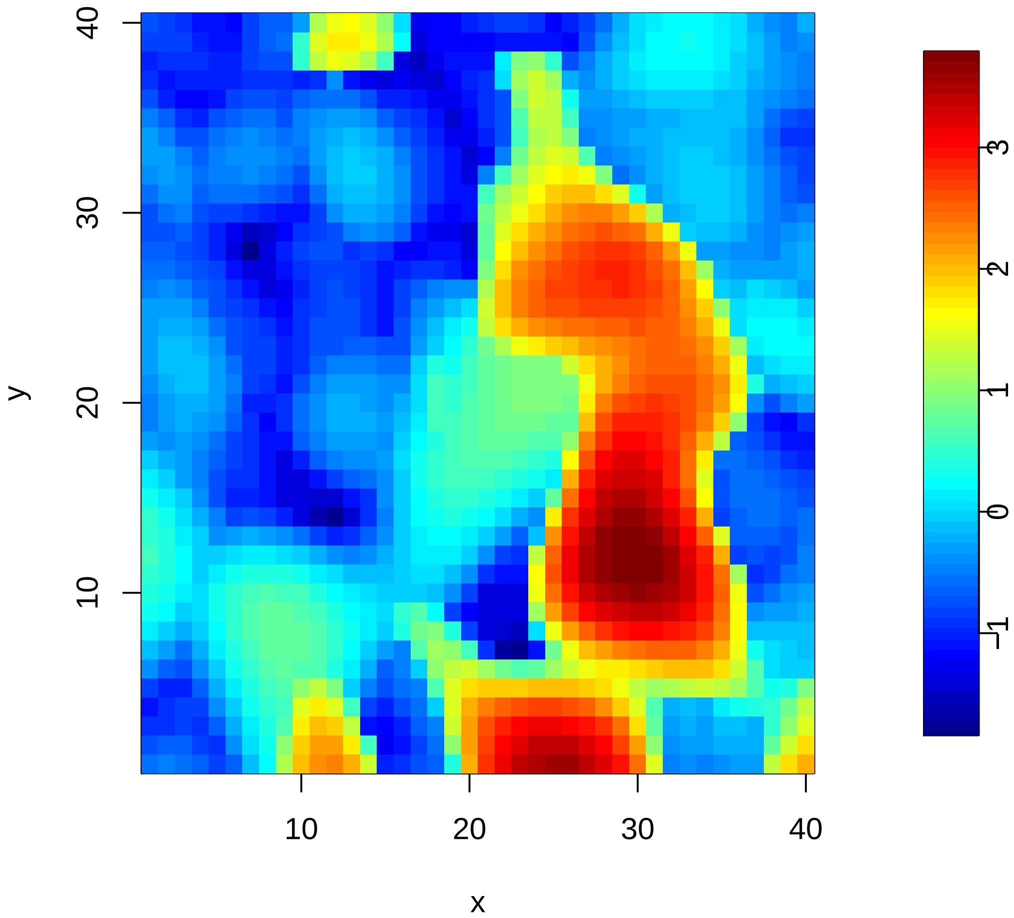
1. Motivations

2. Maxima distribution

3. Geostatistics

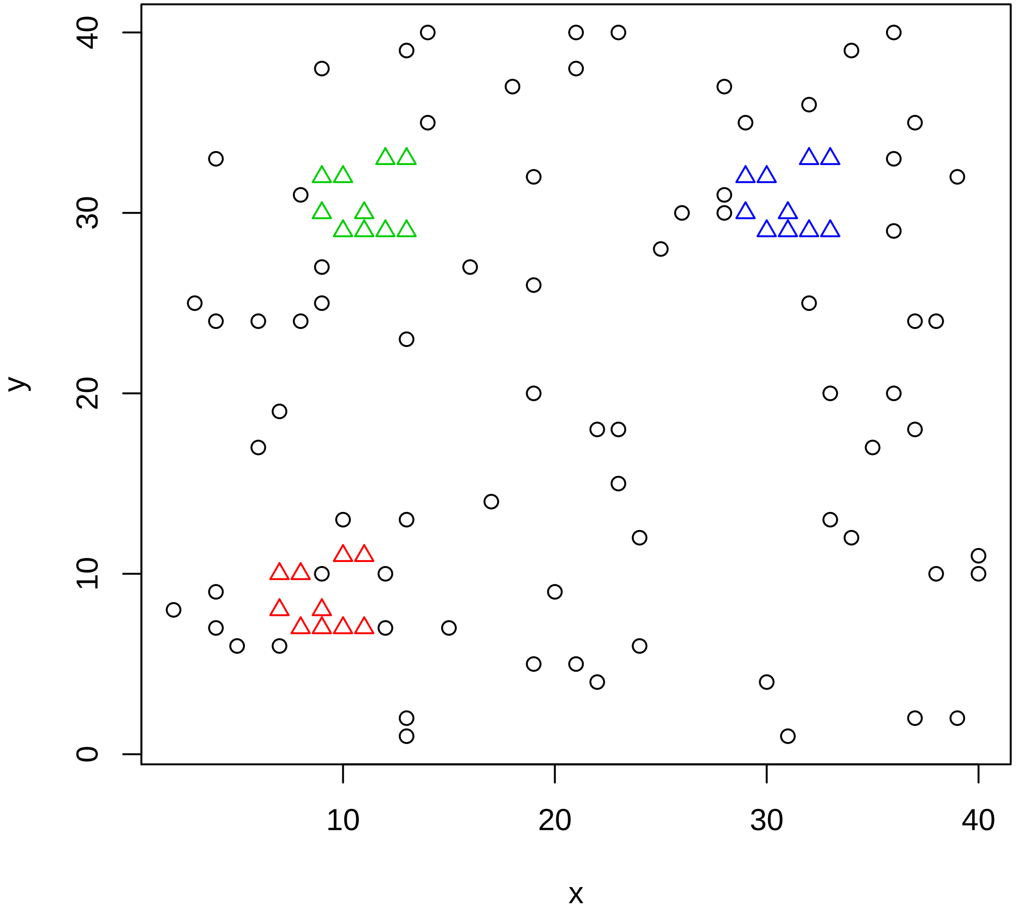
4. Estimation

Spatial Statistics for Extremes



How to describe the **spatial dependence** as a function of the **distance** between two points?

Spatial Statistics for Extremes



How to perform

spatial interpolation

for extreme events?

A few Approaches for modeling spatial extremes

- **Max-stable processes**: Adapting asymptotic results for multivariate extremes
Schlather & Tawn (2003), Naveau et al. (2007), de Haan & Pereira (2005)

- **Bayesian or latent models**: spatial structure **indirectly** modeled via the EVT parameters distribution
Coles & Tawn (1996), Cooley et al. (2005)

- **Linear filtering**: Auto-Regressive spatio-temporal heavy tailed processes,
Davis and Mikosch (2007)

- **Gaussian anamorphosis**: Transforming the field into a Gaussian one
Wackernagel (2003)

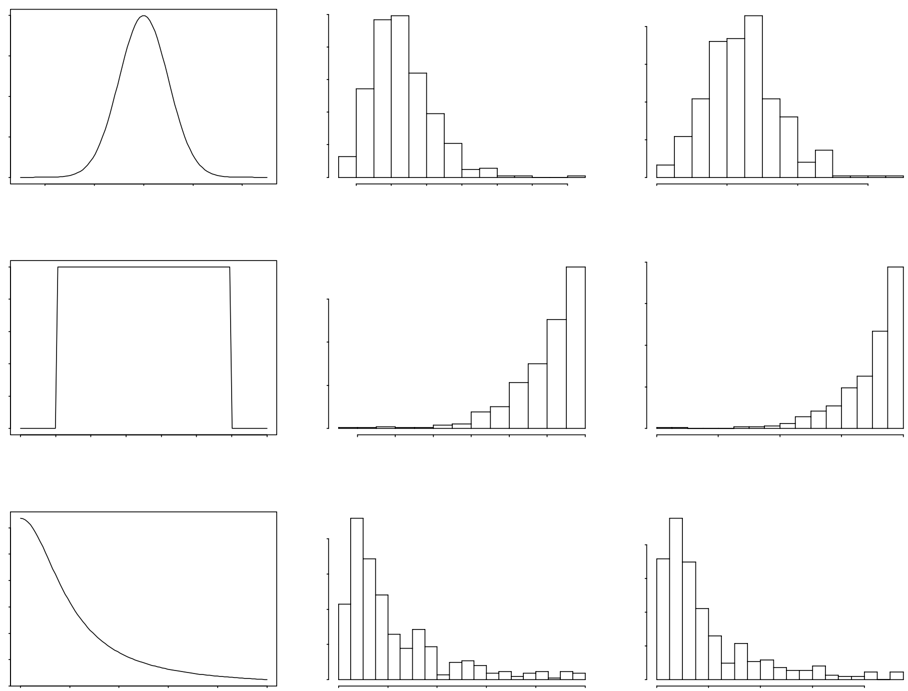
Univariate case for Maxima

Convergence of sample maxima

Normal density ⇒ ⇐ Gumbel density

Uniform density ⇒ ⇐ Weibull density

Cauchy density ⇒ ⇐ Fréchet density



$n = 50$ $n = 100$

Assumptions

- Suppose we know the **marginal** distributions of maxima $M(x)$ with $M(x) =$ the maximum recorded at the location x from a **stationary and isotropic** field.

- Without loss of generality, we first assume that the margins follow an **unit Fréchet**

$$F(u) = \mathbb{P}[M(x) \leq u] = \exp(-1/u)$$

A central question

For large n

$$\mathbb{P}\left[M_n(x) < u, M_n(x+h) < v\right] = ??$$

Bivariate case for Maxima

Asymptotic theory

If one assumes that we have unit Fréchet margins then

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{M_n(x)-a_n}{b_n} \leqslant u,\ \frac{M_n(x+h)-a_n}{b_n} \leqslant v\right] = \exp\left[-V_h(u,v)\right]$$

where

$$V_h(u,v) = 2\int_0^1 \max\left(\frac{w}{u}, \frac{1-w}{v}\right) dL_h(w)$$

with $L_h(.)$ a distribution function on $[0,1]$ such that $\int_0^1 w\, dL_h(w) = 0.5$.

Bivariate case $(M(x), M(x+h))$

Complex non-parametric structure

$$V_h(u,v) = 2\int_0^1 \max\left(\frac{w}{u}, \frac{1-w}{v}\right) dL_h(w)$$

Special case $u = v$

Note $V_h(u,u) = V_h(1,1)/u$

Notations: $\boxed{\theta(h) := V_h(1,1)}$

$$\begin{aligned} \mathbb{P}[M(x) < u, M(x+h) < u] &= \exp(-\theta(h)/u) \\ &= F(u)^{\theta(h)} \end{aligned}$$

because $F(u) = \exp(-1/u)$

$\theta(h)$ = Extremal coefficient

$$\mathbb{P}\left[M(x) < u, M(x+h) < u\right] = F(u)^{\theta(h)}$$

Interpretation

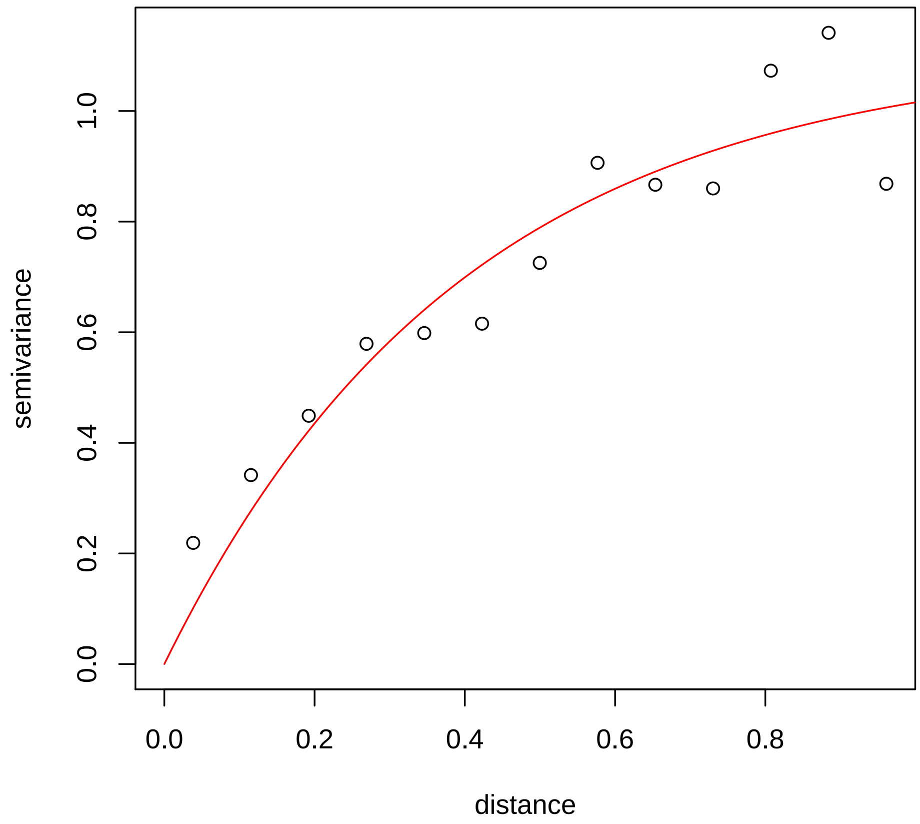
- Independence $\Rightarrow \theta(h) = 2$
- $M(x) = M(x+h) \Rightarrow \theta(h) = 1$
- Similar to correlation coefficients for Gaussian but ...
- No characterization of the **full** bivariate dependence

An important question

(1) How to estimate $\theta(h)$?

Geostatistics: Variograms

$$\gamma(h) = \frac{1}{2}\mathbb{E}|Z(x+h) - Z(x)|^2$$



- Finite if **light** tails
- Capture **all** spatial structure if $\{Z(x)\}$ **Gaussian** fields
- but not well adapted for extremes

A Different Variogram

$$|F(M(x+h)) - F(M(x))|$$

with $F(u) = \exp(-1/u)$

A Different Variogram

$$\nu_h = \frac{1}{2}\mathbb{E}\left|F(M(x+h)) - F(M(x))\right|$$

with $F(u) = \exp(-1/u)$

- Defined for light & heavy tails
- Called a Madogram
- Nice links with extreme value theory

A Different Variogram

$$\nu_h = \frac{1}{2}\mathbb{E}\,|F(M(x+h)) - F(M(x))|$$

Why does it work?

$$\frac{1}{2}|a-b| = \max(a,b) - \frac{1}{2}(a+b)$$

- $a = F(M(x+h))$ and $b = F(M(x))$
- $\mathbb{E}a = \mathbb{E}b = 1/2$

$$\mathbb{E}\max(a,b) = \mathbb{E}F(\underbrace{\max(M(x+h), M(x))}_{\text{max-stable}}) = \frac{\theta(h)}{1+\theta(h)}$$

Madogram $\nu_h \Rightarrow$ Extremal coeff $\theta(h)$

$$\theta(h) = \frac{1 + 2\nu_h}{1 - 2\nu_h}$$

- The madogram ν_h gives the extremal coefficient $\theta(h)$

Comparisons with other estimators

Gumbel (1960)

$$\mathbb{P}(X \leq x, Y \leq y) = \exp\left\{ - \left[\left(\frac{1}{x}\right)^{\frac{1}{\alpha}} + \left(\frac{1}{y}\right)^{\frac{1}{\alpha}} \right]^{\alpha} \right\}$$

Four estimators

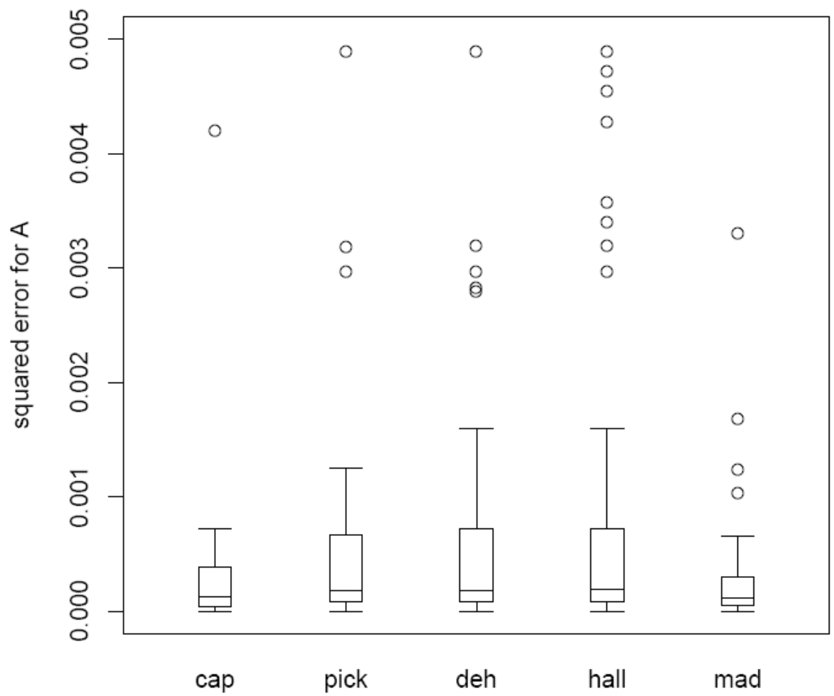
- Pickands' estimator (1975)

- Deheuvels' estimator (1991)

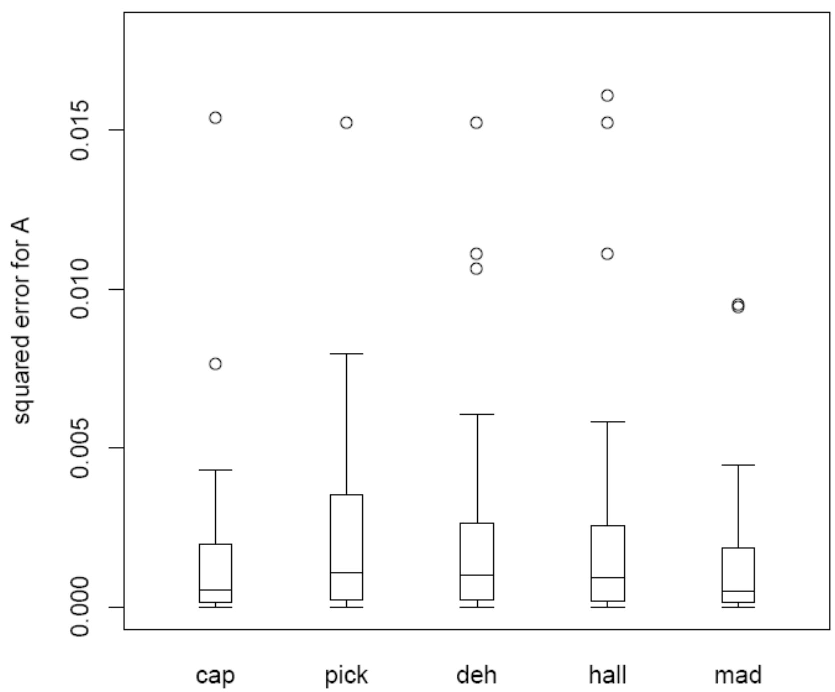
- Hall and Tajvidi's estimator (2000)

- Capéraà *et al.* (1997) estimator

Comparisons with other estimators

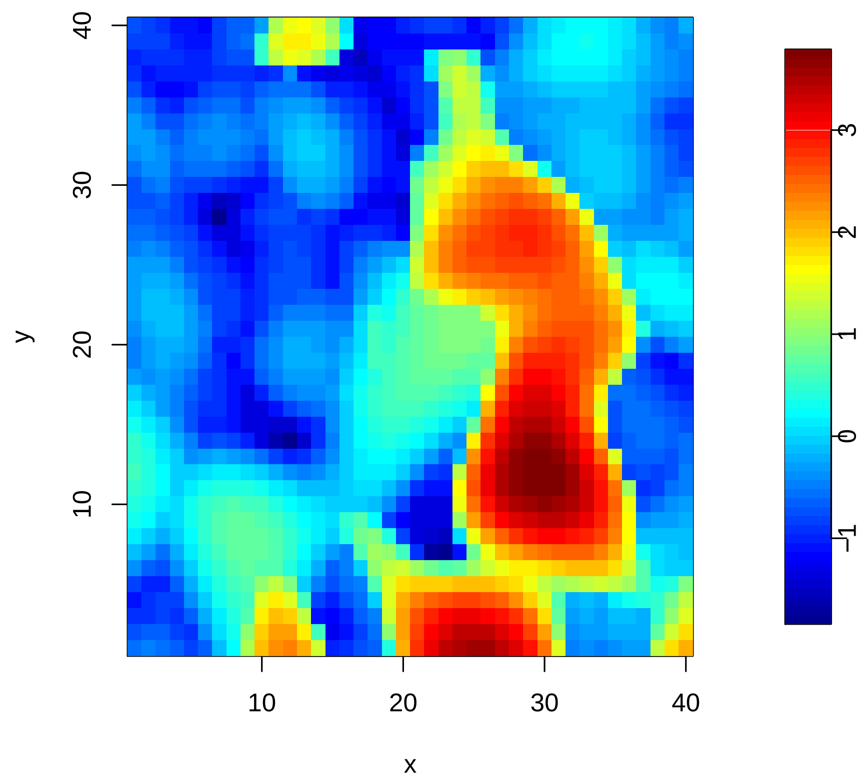


$\alpha = 0.3$



$\alpha = 0.7$

Schlather's models (2003)

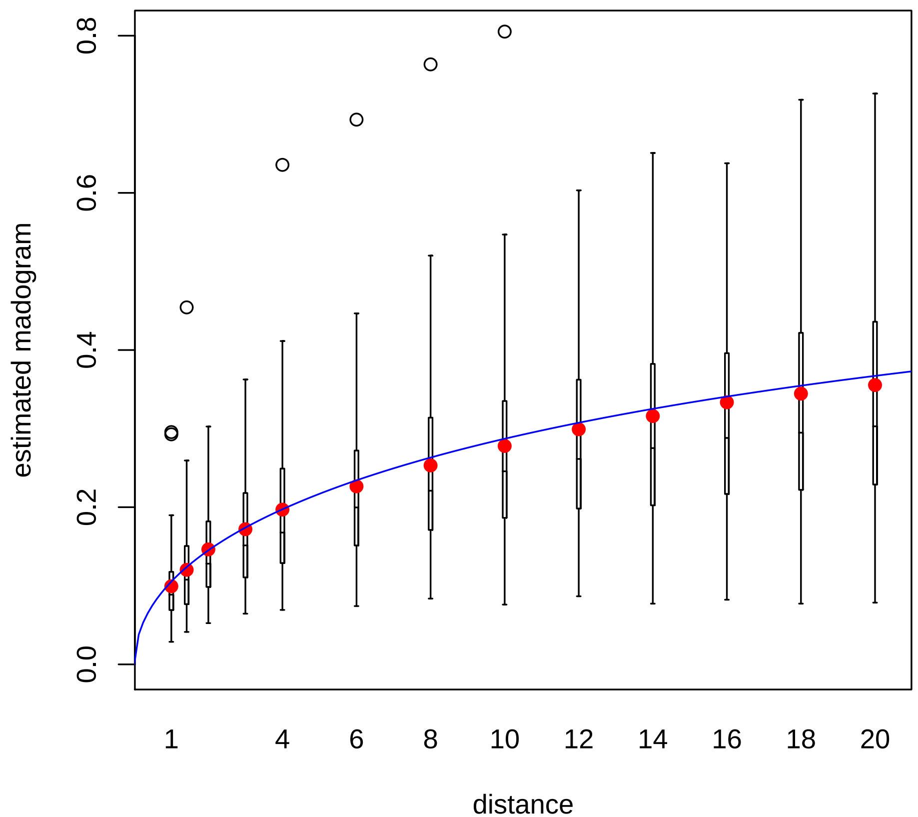


$$\theta(h) = 1 + \sqrt{1 - \frac{1}{2}(\exp(-h/40) + 1)}$$

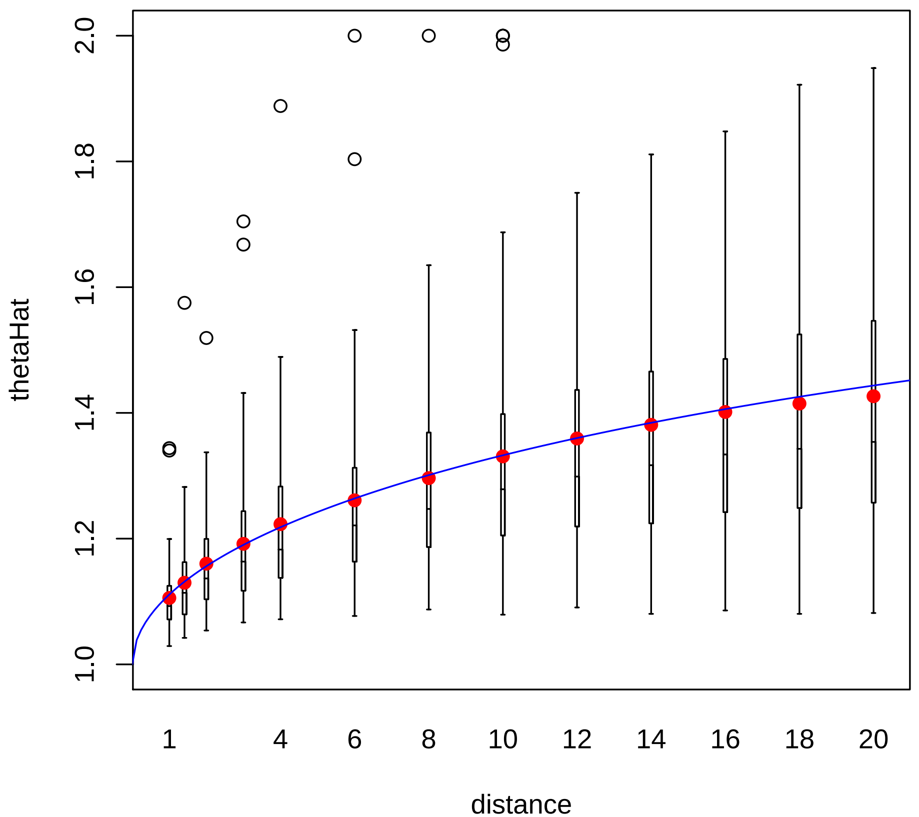
Madogram $\nu_h \Rightarrow$ Extremal coeff $\theta(h)$

Schlather's fields

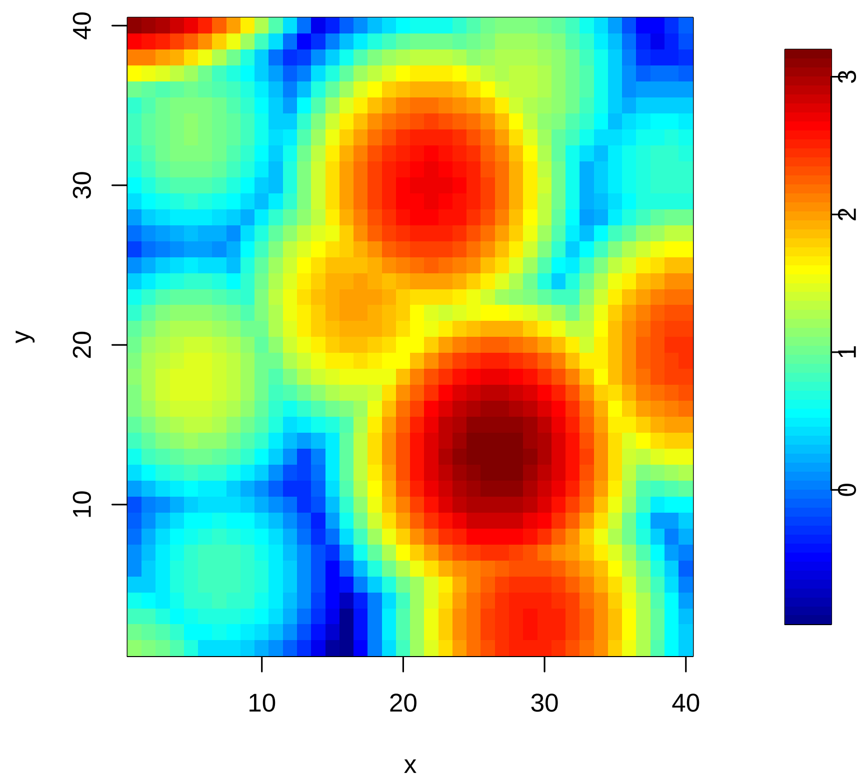
Madogram



Extremal coeff



Smith's models (2003)

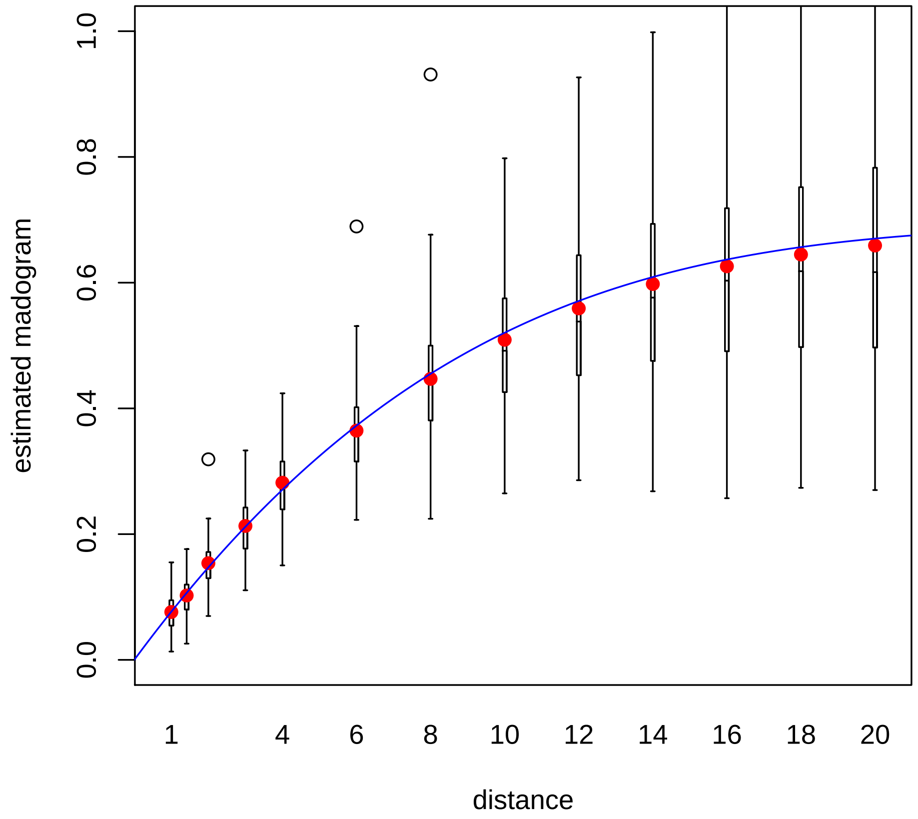


$$\theta(h) = 2\Phi\left(\sqrt{h^T \Sigma^{-1} h}/2\right)$$

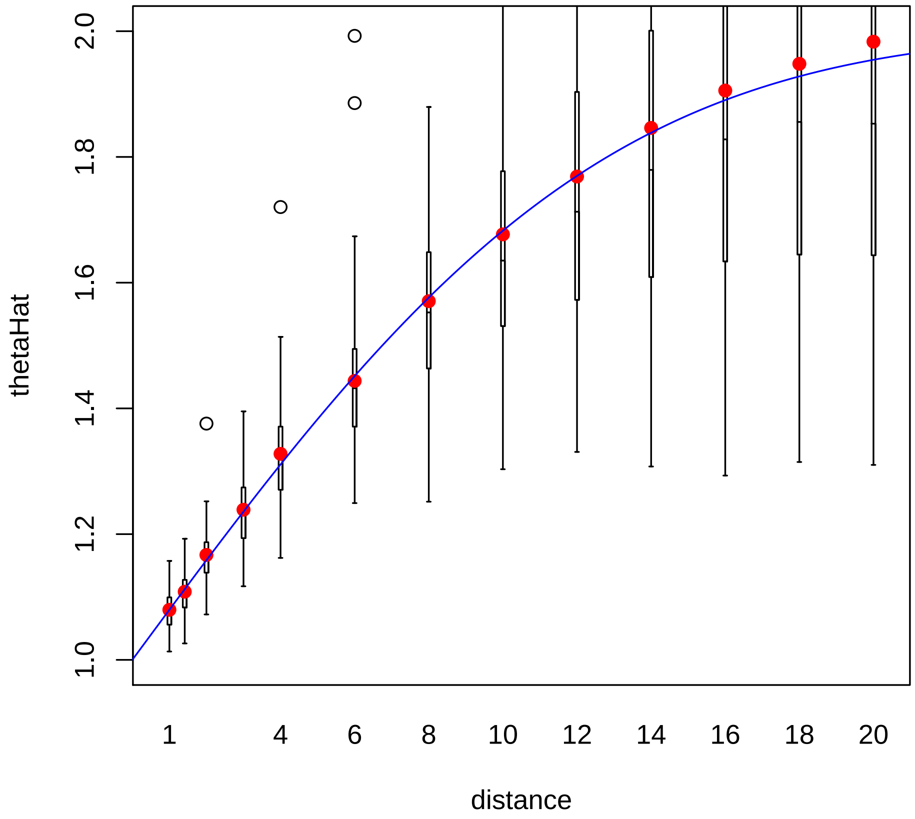
Madogram $\nu_h \Rightarrow$ Extremal coeff $\theta(h)$

Smith's fields

Madogram



Extremal coeff



Building valid Extremal coeff

Proposition A

Any extremal coefficient function $\theta(h)$ is such that $2 - \theta(h)$ is positive semi-definite.

Proposition B

Any extremal coefficient function $\theta(h)$ satisfies the following inequalities

$$\begin{aligned}
\theta(h+k) &\leq \theta(h)\theta(k), \\
\theta(h+k)^{\tau} &\leq \theta(h)^{\tau} + \theta(k)^{\tau} - 1, \text{ for all } 0 \leq \tau \leq 1, \\
\theta(h+k)^{\tau} &\geq \theta(h)^{\tau} + \theta(k)^{\tau} - 1, \text{ for all } \tau \leq 0.
\end{aligned}$$

An important question

(1) How to estimate $\theta(h) = V_h(1,1)$? Done!!

(2) How to estimate $V_h(u,v)$?

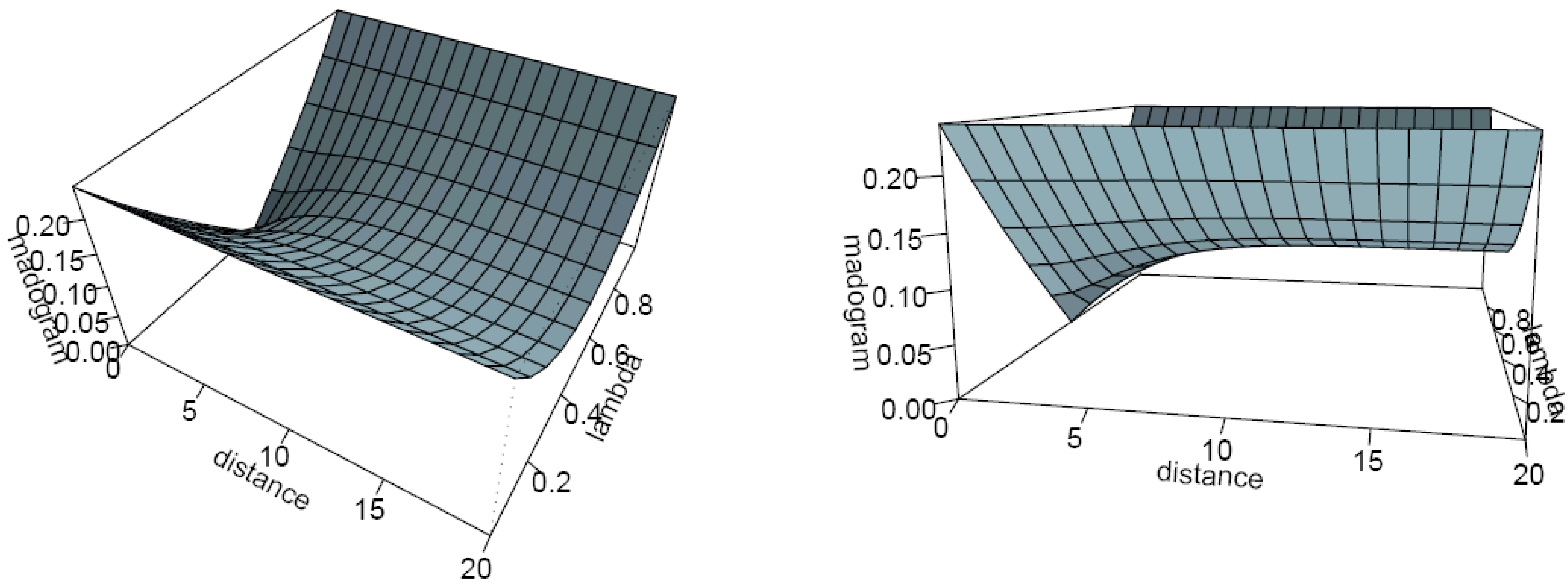
Note:
Because $V_h(u,v) = V_h(u/(u+v), v/(u+v))/(u+v)$ is sufficient to only estimate $V_h(\lambda, 1-\lambda)$ for $\lambda \in [0,1]$.

Extending the madogram

$$\nu_h(\lambda) = \frac{1}{2}\mathbb{E}\left|F^{\lambda}(M(x+h)) - F^{1-\lambda}(M(x))\right|$$

- Defined for light & heavy tails
- Called a λ-Madogram
- Nice links with extreme value theory
- $\nu_h(0) = \nu_h(1) = 0.25$

The λ-madogram



$\nu_h(\lambda) = \frac{1}{2}\mathbb{E}\left|F^{\lambda}(M(x+h)) - F^{1-\lambda}(M(x))\right|$

A fundamental relationship

$$\nu_h(\lambda) = \frac{V_h(\lambda, 1-\lambda)}{1 + V_h(\lambda, 1-\lambda)} - c(\lambda), \text{ with } c(\lambda) = \frac{3}{2(1+\lambda)(2-\lambda)}$$

Conversely,

$$V_h(\lambda, 1-\lambda) = \frac{c(\lambda) + \nu_h(\lambda)}{1 - c(\lambda) - \nu_h(\lambda)}$$

Estimation of $V_h(u, v)$

Suppose that we have T iid years of daily annual maxima fields with unknown margins.

How to estimate $\nu_h(\lambda) = \frac{1}{2}\mathbb{E}\left|F^{\lambda}(M(x+h)) - F^{1-\lambda}(M(x))\right|$?

A naive estimator

$$\widehat{\nu}_h(\lambda) \;=\; \frac{1}{2T}\sum_{t=1}^{T}\left|\mathbb{F}_{n,T}^{\lambda}(M_{n,t}(x+h)) - \mathbb{G}_{n,T}^{1-\lambda}(M_{n,t}(x))\right|$$

with

$$\mathbb{F}_{n,T}(u) = \frac{1}{T}\sum_{t=1}^{T} 1\!\!1_{\{M_{n,t}(x+h)\leq u\}} \text{ and } \mathbb{G}_{n,T}(u) = \frac{1}{T}\sum_{t=1}^{T} 1\!\!1_{\{M_{n,t}(x)\leq u\}}$$

But, the conditions $\mathbb{E}\widehat{\nu}_h(0) = \mathbb{E}\widehat{\nu}_h(1) = 0.25$ are not satisfied

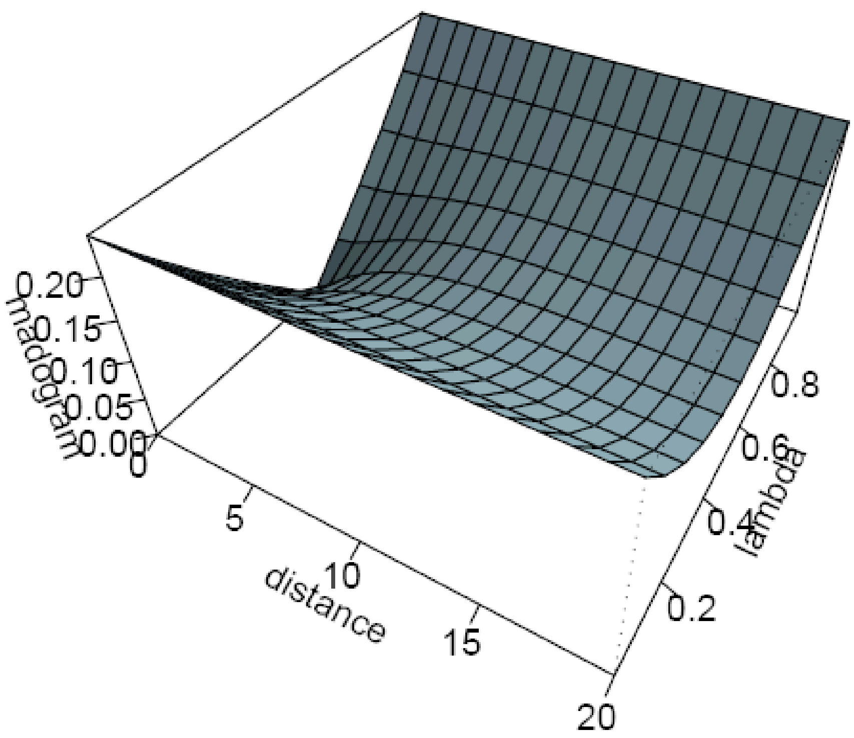
Estimation of $V_h(u, v)$

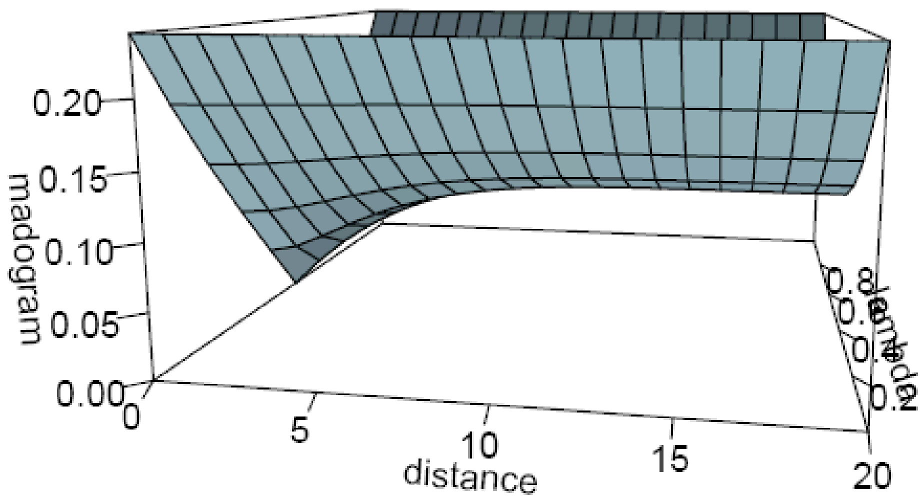
How to estimate $\nu_h(\lambda) = \frac{1}{2}\mathbb{E}\left|F^{\lambda}(M(x+h)) - F^{1-\lambda}(M(x))\right|$?

A modified estimator

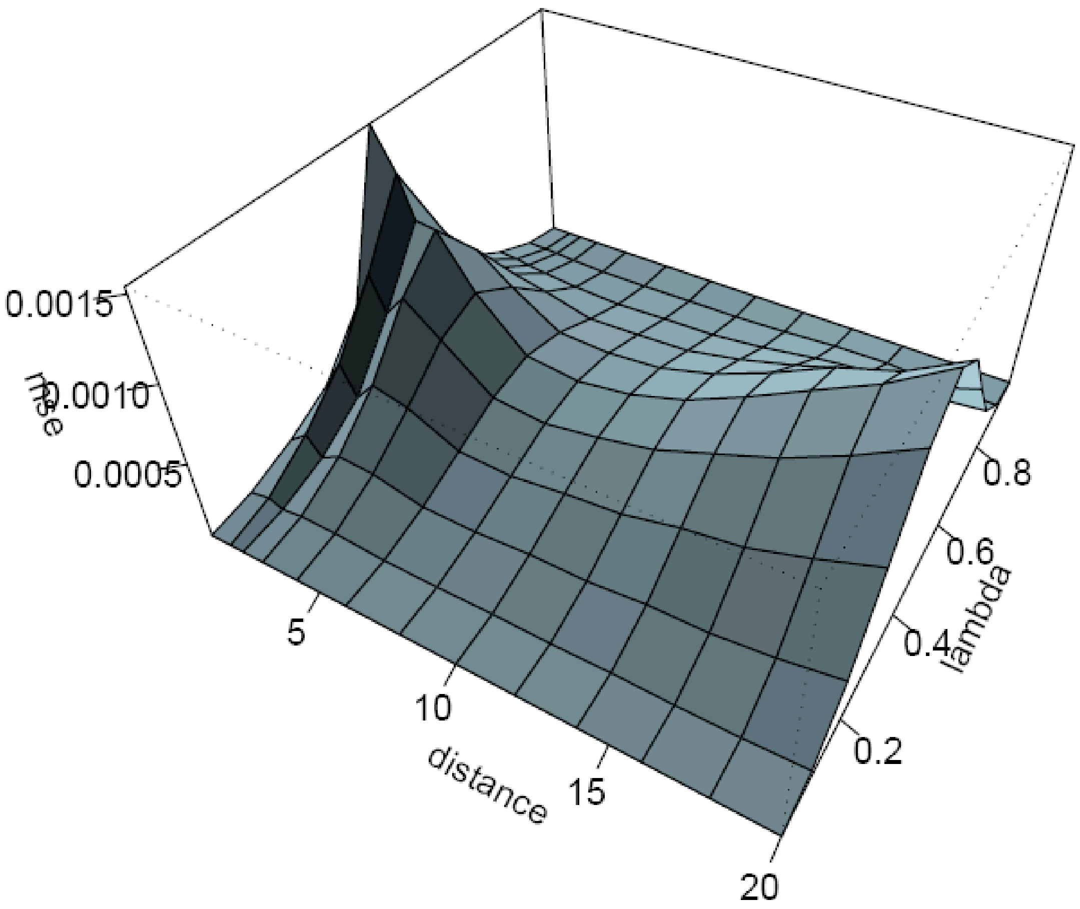
$$
\begin{aligned}
\widehat{\nu}_h(\lambda) \;=\; & \frac{1}{2T}\sum_{t=1}^{T}\left|\mathbb{F}_{n,T}^{\lambda}(M_{n,t}(x+h)) - \mathbb{G}_{n,T}^{1-\lambda}(M_{n,t}(x))\right| \\
& -\frac{\lambda}{2T}\sum_{t=1}^{T}\left(1-\mathbb{F}_{n,T}^{\lambda}(M_{n,t}(x+h))\right) \\
& -\frac{1-\lambda}{2T}\sum_{t=1}^{T}\left(1-\mathbb{G}_{n,T}^{1-\lambda}(M_{n,t}(x))\right) \\
& +\frac{1}{2}\frac{1-\lambda+\lambda^2}{(2-\lambda)(1+\lambda)}
\end{aligned}
$$

Simulations: 300 iid Schalther's fields





Mean Square Error from simulations



300 iid Schalther's fields

Notations for asymptotic results

Margins of (X, Y): unknown continuous margins: F, G

Bivariate distribution H **and copula**:

$$H(x, y) = C(F(x), G(y)) \text{ and } C(u, v) = H\Big(F^{\leftarrow}(u), G^{\leftarrow}(v)\Big)$$

$$\phi(H)(u, v) := H\Big(F^{\leftarrow}(u), G^{\leftarrow}(v)\Big)$$

Bivariate empirical process $\mathbb{Z}_T(u, v)$:

$$\mathbb{Z}_T(u, v) := \sqrt{T}\left(\phi(\mathbb{H}_T)(u, v) - \phi(H)(u, v)\right)$$

with

$$\mathbb{H}_T(u, v) = \frac{1}{T}\sum_{t=1}^{T} 1\!\mathrm{l}_{\{X_t \le u, Y_t \le v\}}$$

Asymptotic properties

$$\mathbb{Z}_T(u,v) = \sqrt{T}\Big(\phi(\mathbb{H}_T)(u,v) - \phi(H)(u,v)\Big)$$

Proposition 1. *Let $(X_t, Y_t)_{t=1,\dots,T}$ be a sample of T bivariate random vectors with df H, continuous margins F and G, and with its associated copula C whose partial derivatives are continuous. Then, the process $\{\mathbb{Z}_T(u,v), 0 \le u,v \le 1\}$ converges weakly to the Gaussian process $\{\mathbb{N}_C(u,v), 0 \le u,v \le 1\}$ in $\ell^\infty([0,1]^2)$ that is defined as*

$$\mathbb{N}_C(u,v) = \mathbb{B}_C(u,v) - \mathbb{B}_C(u,1)\frac{\partial C}{\partial u}(u,v) - \mathbb{B}_C(1,v)\frac{\partial C}{\partial v}(u,v),$$

where $\mathbb{B}_C$ is a Brownian bridge on $[0,1]^2$ with covariance function

$$\mathbb{E}\Big[\mathbb{B}_C(u,v)\cdot\mathbb{B}_C(u',v')\Big] = C(u\wedge u', v\wedge v') - C(u,v)\cdot C(u',v')$$

Convergence of the $\lambda-$madogram

$(X_1,Y_1),...,(X_T,Y_T)$ T bivariate rv with unknown margins F and G

$$\widehat{\nu}_T(\lambda) := \frac{1}{2T}\sum_{t=1}^{T}\left|\Big(\mathbb{F}_T(X_t)\Big)^{\lambda} - \Big(\mathbb{G}_T(Y_t)\Big)^{1-\lambda}\right|$$

Proposition 2. *Under the assumptions of Proposition 1, let J be a function of bounded variation, continuous. Then, we have*

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left\{J\Big(\mathbb{F}_T(X_t),\mathbb{G}_T(Y_t)\Big) - \mathbb{E}J\Big(F(X),G(Y)\Big)\right\}$$

$$\xrightarrow{d} \int_{[0,1]^2}\mathbb{N}_C(u,v)dJ(u,v)$$

The special case, $J(x,y) := \frac{1}{2}|x^{\lambda} - y^{1-\lambda}|$, provides the weak convergence of the $\lambda-$madogram estimator

Madogram & EVT

- $(Z(x), Z(x+h))$= precip. measurements at two nearby locations

$$\Big(M_n(x), M_n(x+h)\Big) = \Big(\max_{i=1,\dots,n} Z_i(x), \max_{i=1,\dots,n} Z_i(x+h)\Big)$$

n = recording unit, either hourly, daily or monthly

- Suppose that such bivariate vectors can be computed for a series of years and that these vectors are assumed to be iid in time

$$\mathbb{F}_{n,T}(u) = \frac{1}{T}\sum_{t=1}^{T} 1\!\!1_{\{M_{n,t}(x+h)\leq u\}} \text{ and } \mathbb{G}_{n,T}(u) = \frac{1}{T}\sum_{t=1}^{T} 1\!\!1_{\{M_{n,t}(x)\leq u\}}$$

$$\widehat{\nu}_{n,T}(h,\lambda) = \frac{1}{2T}\sum_{t=1}^{T}\left|\mathbb{F}^{\lambda}_{n,T}\Big(M_{n,t}(x+h)\Big) - \mathbb{G}^{1-\lambda}_{n,T}(M_{n,t}(x))\right|$$

Madogram & EVT (cont'd)

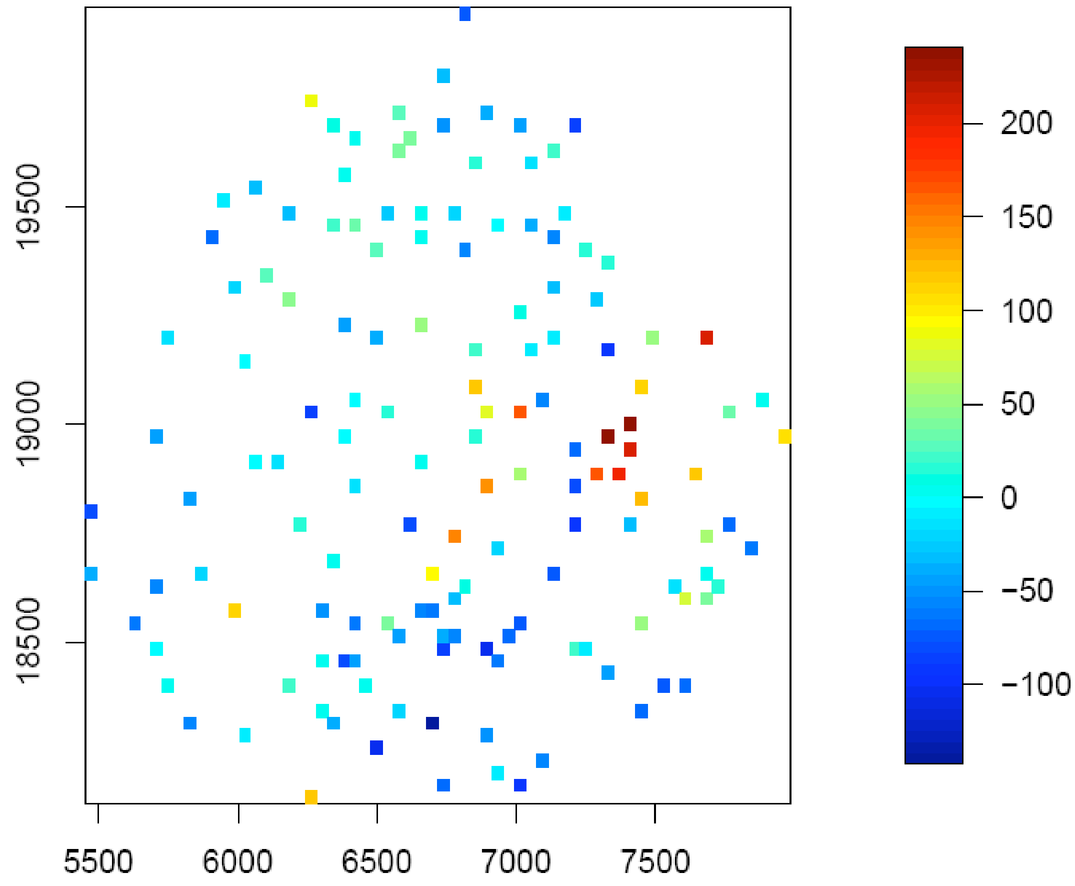
Proposition 3. *Let* $(M_{n,t}(x), M_{n,t}(x+h))$ *be a sample of* T *bivariate vectors with that satisfies the assumptions of Proposition 1 and such that* $\left(\frac{M_{n,t}(x)-a_n}{b_n}, \frac{M_{n,t}(x+h)-a_n}{b_n}\right)$ *converges in distribution to a bivariate EV distribution with an extremal function defined by* $V_h(.,.)$. *Then, we have*

$$\sqrt{T}\left(\widehat{\nu}_{n,T}(h,\lambda) - \frac{1}{2}\mathbb{E}|F^{\lambda}(M(x+h)) - F^{1-\lambda}(M(x))|\right)$$

$$\xrightarrow{d} \int_{[0,1]^2} \mathbb{N}_{C}(u,v)dJ(u,v)$$

where n *tends to* ∞ *as* T *goes to* ∞

An application: in Bourgogne (Dijon)



Locations (in Lambert coordinates). Pre-processed 30-year maxima of daily precipitation

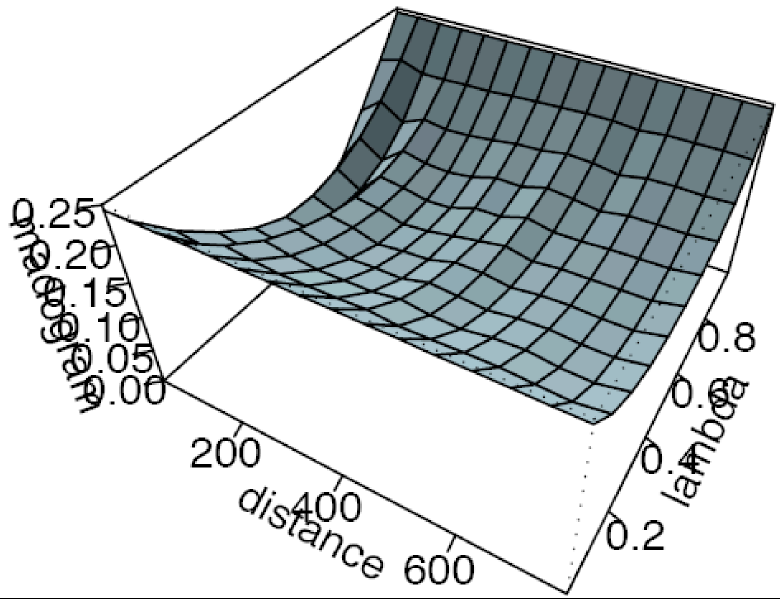
$\lambda-$madogram

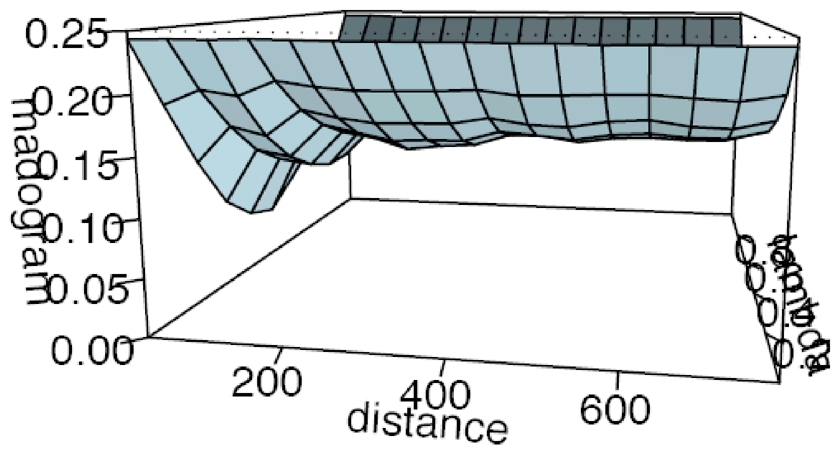
Our estimator of our $\lambda-$madogram $\nu(h,\lambda)$

$$\frac{1}{2|\mathcal{N}_h|}\sum_{(x_i,x_j)\in\mathcal{N}_h}\left|\mathbb{F}^{\lambda}(M(x_j))-\mathbb{F}^{1-\lambda}(M(x_i))\right|+\frac{1}{2}\frac{1-\lambda+\lambda^2}{(2-\lambda)(1+\lambda)}$$
$$-\frac{\lambda}{2|\mathcal{N}_h|}\sum_{(x_i,x_j)\in\mathcal{N}_h}\left(1-\mathbb{F}^{\lambda}(M(x_i))\right)-\frac{1-\lambda}{2|\mathcal{N}_h|}\sum_{(x_i,x_j)\in\mathcal{N}_h}\left(1-\mathbb{F}^{1-\lambda}(M(x_i))\right)$$

where $\mathcal{N}_h$ is the set of sample pairs lagged by the distance h.

λ−madogram





Estimated λ−madogram for the field of maxima of daily precipitation over 1970-1999

Take-home messages

- Fields of maxima $\neq$ Gaussian ones
- Spatial structure defined by the function $V_h(u, v)$
- $\lambda-$**Madogram** $\nu_h \Rightarrow$ **dependence function** $V_h(u, v)$
- We have proposed and study an estimator $\widehat{\nu}_h(\lambda)$

Future research

- Develop spatial interpolation methods for maxima
- Derive statistical schemes for downscaling for maxima