Estimation of the multivariate Conditional Tail Expectation, an approach based on the Kendall's process

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Outline

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1. Multivariate Conditional Tail Expectation
2. The Kendall's process
3. A new non parametric estimator of the multivariate CTE_α
4. Simulations and study on real data
5. Conclusion, perspectives

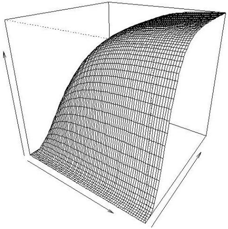
Quantile curve and risk measures: possible extensions for dim ≥ 2

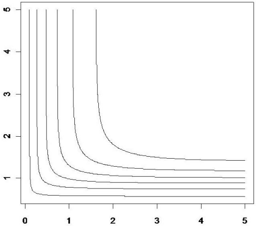
Notation, definitions :

Let $X = (X_1, \ldots, X_d)$ be a random vector with continuous distribution function $F : \mathbb{R}_+^d \to [0, 1]$.

Let $\alpha \in (0, 1)$, $d \geq 2$. Define the upper α-level set (*resp.* the α-quantile curve) of F by

$$L(\alpha) = \{x \in \mathbb{R}_+^d : F(x) \geq \alpha\}, \quad \partial L(\alpha) = \{x \in \mathbb{R}_+^d : F(x) = \alpha\}.$$





Quantile curve and risk measures: possible extensions for dim ≥ 2

The α-quantile curve has been proposed to generalize the Value-at-Risk (VaR) in dimension $d \geq 2$ (see e.g., Embrechts & Puccetti, 2006; Nappo & Spizzichino, 2009).

Advantages :

• "metric-free",

• provides a data segmentation of predefined size,

• valid for symmetric as far as non-symmetric distribution functions,

• De Haan & Huang (1995), Chebana & Ouarda (2011) used quantile curves to model hydrological events.

Quantile curve and risk measures: possible extensions for dim ≥ 2

Definition (Di Bernardino *et al.*, 2012)

Consider a random vector $\mathbf{X}$ with continuous distribution function $F : \mathbb{R}_+^d \to [0,1]$. For $\alpha \in (0,1)$, we define the Multivariate Conditional Tail Expectation by

$$\mathrm{CTE}_\alpha(\mathbf{X}) = \begin{pmatrix} \mathbb{E}[\,X_1 \,|\, \mathbf{X} \in L(\alpha)\,] \\ \vdots \\ \mathbb{E}[\,X_d \,|\, \mathbf{X} \in L(\alpha)\,] \end{pmatrix}$$

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Remark

CTE_α does not use an aggregate variable (sum, min, max, *. . .) to analyse the multivariate risk's issue. This measure is of particular interest when factors of risk are heterogeneous and can therefore not be aggregated.*

Quantile curve and risk measures: possible extensions for dim ≥ 2

Definition (Di Bernardino *et al.*, 2011; Cousin *et al*, 2012)

Consider a random vector $\mathbf{X}$ with continuous distribution function $F : \mathbb{R}_+^d \to [0,1]$. Define $U = (F_1(X_1), \ldots, F_d(X_d))$. For $\alpha \in (0,1)$, we define the Multivariate Conditional Tail Expectation by

$$\mathrm{CTE}_\alpha(\mathbf{X}) = \begin{pmatrix} \mathbb{E}[\,X_1 \,|\, C(U) \geq \alpha\,] \\ \vdots \\ \mathbb{E}[\,X_d \,|\, C(U) \geq \alpha\,] \end{pmatrix}$$

Quantile curve and risk measures: possible extensions for dim ≥ 2

Main objective of our work : estimating the Multivariate Conditional Tail Expectation and derive the properties of our estimate.

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Level sets-based plug-in, Di Bernardino *et al.* (2011) :

For $\alpha \in (0, 1)$ and $T > 0$, define

$$L_n(\alpha)^T = \{x \in [0, T]^2,\ F_n(x) \geq \alpha\}.$$

Let (T_n) be an increasing positive sequence. Let $X^1, \ldots X^n$ be a sample of the d-variate distribution F.

Di Bernardino *et al.* (2011) define and study properties of

$$\widehat{\mathrm{CTE}}_\alpha^{T_n}(X) = \left(\frac{\sum_{j=1}^n X_1^j 1_{\{X^j \in L_n(\alpha)^{T_n}\}}}{\sum_{j=1}^n 1_{\{X^j \in L_n(\alpha)^{T_n}\}}}, \ldots, \frac{\sum_{j=1}^n X_d^j 1_{\{X^j \in L_n(\alpha)^{T_n}\}}}{\sum_{j=1}^n 1_{\{X^j \in L_n(\alpha)^{T_n}\}}} \right)'.$$

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A difficulty is the way to adjust the sequence (T_n). Therefore we propose here a new estimator of CTE_α.

Outline

A major tool, the Kendall's process

Let $X^1, \dots, X^n$ a random sample of size $n \geq 2$ from the d-variate continuous distribution function F. Let $V_{i,n}$ denote the proportion of observations X^j, $j \neq i$, such that $X^j \leq X^i$ componentwise. We then define K_n the empirical distribution function derived from the (dependent) pseudo-observations $V_{i,n}$.

Remark

Let $F_n(x) = \frac{1}{n}\sum_{j=1}^n \mathbf{1}_{X^j \leq x}$, we can write

$$K_n(t) = \frac{1}{n}\sum_{j=1}^n \mathbf{1}_{F_n(X^j) \leq t + \frac{1-t}{n}} .$$

We also define the Kendall's distribution by $K(t) = \mathbb{P}[F(X) \leq t]$, and C will denote the copula associated to F.

A major tool, the Kendall's process

I : the distribution function $K(t)$ of $F(X)$ admits a continuous density $k(t)$ on $(0,1]$ that verifies $k(t) = o\left(t^{-1/2}\, log^{-1/2-\epsilon}\left(\frac{1}{t}\right)\right)$, for some $\epsilon > 0$ as $t \to 0$,

II : there exists a version of the conditional distribution of the vector $U := (F_1(X_1), ..., F_d(X_d))$ given $C(U) = t$ and a countable family $\mathcal{P}$ of partitions $\mathcal{C}$ of $[0,1]^d$ into a finite number of Borel sets satisfying:

$$\inf_{\mathcal{C}\in\mathcal{P}} \max_{A\in\mathcal{C}} \mathrm{diam}(A) = 0,$$

such that for all $A \in \mathcal{C}$ the mapping

$$t \mapsto \eta_t(A) = k(t)\, \mathbb{P}[U \in A \,|\, C(U) = t]$$

is continuous on $(0,1]$ with $\eta_1(A) = k(1)\mathbf{1}_{\{(1,\ldots,1)\in A\}}$.

A major tool, the Kendall's process

Remark

- *A necessary condition for Assumption* **II** *is that F is partially strictly increasing. In particular, all copulas whose density function is continuous and positive on $(0,1)^d$ sastisfy* **II**.
- *Several examples for assumptions* **I** *and* **II** *are derived in Barbe & Genest (1996).*

Theorem (Barbe & Genest, 1996)

Define the centered Kendall's process

$$\alpha_n(t) = \sqrt{n}\left(K_n(t) - K(t)\right).$$

Under assumptions **I** *and* **II**, $\alpha_n \xrightarrow[n\to+\infty]{\mathcal{D}} \alpha$ *where α is a continuous Gaussian process with zero mean and covariance function Γ.*

Outline

A new estimator

Let $X = (X_1, \ldots, X_d)$. X is said to satisfy the "regularity conditions" if

- $F : \mathbb{R}_+^d \to [0,1]$ is partially strictly increasing,
- there exists $r > 2$ such that $\mathbb{E}(|X_i|^r) < \infty$, for $i = 1, \ldots, d$,
- assumption **I** is satisfied.

Let $X^1, \dots, X^n$ be a sample of the d-variate distribution F. We define $U^i = (F_1(X_1^i), \dots, F_d(X_d^i))$. Let C_n the empirical distribution function associated to C.

Definition (Di Bernardino & Prieur (2012))

The Kendall-based estimator for the Multivariate α-Conditional Tail Expectation is defined by

$$\widehat{\mathrm{CTE}}_\alpha(X) = \frac{1}{1-K_n(\alpha)} \begin{pmatrix} \frac{1}{n}\sum_{i=1}^n X_1^i \mathbb{1}_{\{C_n(U^i)\geq\alpha\}} \\ \vdots \\ \frac{1}{n}\sum_{i=1}^n X_d^i \mathbb{1}_{\{C_n(U^i)\geq\alpha\}} \end{pmatrix}.$$

Properties of our estimator

Let $\widehat{\mathrm{CTE}}_\alpha = \left(\widehat{\mathrm{CTE}}_{\alpha,1}, \ldots, \widehat{\mathrm{CTE}}_{\alpha,d}\right)'$.

Define $\alpha_n^{CTE}(\alpha) = \left(\alpha_{n,1}^{CTE}(\alpha), \ldots, \alpha_{n,d}^{CTE}(\alpha)\right)'$ by

$$\alpha_{n,j}^{\mathrm{CTE}}(\alpha) = \sqrt{n}\,(\widehat{\mathrm{CTE}}_{\alpha,j} - \mathrm{CTE}_{\alpha,j}), \qquad j = 1, \ldots, d.$$

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Theorem (Di Bernardino & Prieur, 2012)

Under the "regularity conditions", $\alpha_n^{CTE} \xrightarrow[n\to+\infty]{\mathcal{D}} \alpha^{CTE}$ where α^{CTE} is a continuous Gaussian process with zero mean and (cross-)covariance function Γ_{CTE}.

The expression for $\Gamma_{CTE}^{i,j}(s,t)$, $i, j = 1, \ldots, d$, $s, t \in [0,1]$ is complex and depends on the limit covariance function Γ (see Theorem on the Kendall's process).

Sketch of the proof

The proof is strongly based on the one of the convergence of the Kendall's process (see Barbe & Genest, 1996). We first write

$$\alpha_{n,j}^{\mathrm{CTE}}(\alpha) = (1 - K_n(\alpha))^{-1}(1 - K(\alpha))^{-1}\left(\zeta_{n,j}(\alpha) + \phi_{n,j}(\alpha) + \psi_{n,j}(\alpha)\right)$$

with

$$\zeta_{n,j}(\alpha) = \sqrt{n}\,\overline{K}(\alpha)\left(n^{-1}\sum_{i=1}^{n} X_j^i\left(\mathbf{1}_{\{C_n(U^i)\geq\alpha\}} - \mathbf{1}_{\{C(U^i)\geq\alpha\}}\right)\right),$$

$$\phi_{n,j}(\alpha) = \sqrt{n}\,\overline{K}(\alpha)\left(n^{-1}\sum_{i=1}^{n} X_j^i\,\mathbf{1}_{\{C(U^i)\geq\alpha\}} - \mathbb{E}[X_j\,\mathbf{1}_{\{C(U)\geq\alpha\}}]\right),$$

$$\psi_{n,j}(\alpha) = \sqrt{n}\;\mathbb{E}[X_j\,\mathbf{1}_{\{C(U)\geq\alpha\}}]\,(K_n(\alpha) - K(\alpha)).$$

Sketch of the proof (2)

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We then make use of the continuous mapping theorem to prove the convergence of the finite dimensional distributions of the sum.

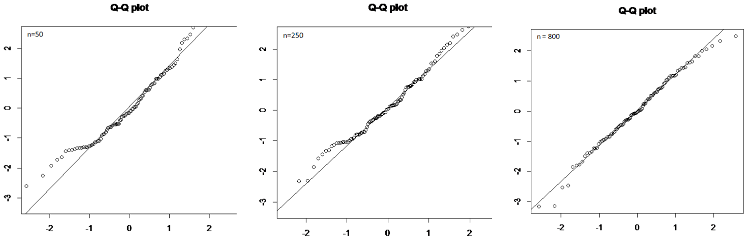
The tightness is deduced as the limit of each process in the sum is continuous.

Outline

Asymptotic normality

$\mathbf{X} = (X_1, X_2)$, independent and exponentially distributed marginals with parameter 2.

Q-Q plot for $\sqrt{n}\,(\widehat{\mathrm{CTE}}^{K}_{\alpha,1} - \mathrm{CTE}_{\alpha,1})$ on 100 simulations, with $\alpha = 0.38$, $n = 50,\ 250,\ 800$.



Comparaison of the two estimators

For the level sets-based estimator, we do the "best choice" for T_n. This is a compromise between

• the rate of convergence of $L_n^{T_n}(\alpha)$ to $L(\alpha)$ (which decreases with T_n),

• the tail behavior of $\mathbf{X}$, i.e. $(\mathbb{P}(X_1 \geq T_n \text{ or } X_2 \geq T_n))^{-1}$, which increases with T_n.

We consider

1) Independent copula with exponentially distributed marginals

2) Clayton copula with parameter 1, with exponential and Burr$(4, 1)$ univariate marginals.

sample size $n = 1000$, number of replications $r = 100$,
$\alpha = 0.10,\ 0.24,\ 0.38\ 0.52,\ 0.66, 0.80$

Comparaison of the two estimators

α	CTE_α	$\widehat{CTE}_{\alpha L_\alpha}$	$\widehat{CTE}_{\alpha K}$	$\hat{\sigma}_{L_\alpha}$	$\hat{\sigma}_K$	$RMSE_{L_\alpha}$	$RMSE_K$
0.10	(1.255, 0.627)	(1.222, 0.638)	(1.259, 0.628)	(0.044, 0.022)	(0.039, 0.021)	(0.043, 0.039)	(0.032, 0.036)
0.24	(1.521, 0.761)	(1.488, 0.811)	(1.524, 0.761)	(0.069, 0.023)	(0.053, 0.023)	(0.051, 0.042)	(0.035, 0.037)
0.38	(1.792, 0.896)	(1.797, 0.911)	(1.791, 0.895)	(0.075, 0.038)	(0.068, 0.037)	(0.044, 0.046)	(0.037, 0.043)
0.52	(2.102, 1.051)	(2.082, 1.047)	(2.113, 1.056)	(0.104, 0.052)	(0.094, 0.045)	(0.052, 0.052)	(0.045, 0.044)
0.66	(2.492, 1.246)	(2.461, 1.255)	(2.507, 1.259)	(0.139, 0.071)	(0.137, 0.071)	(0.057, 0.056)	(0.056, 0.052)
0.80	(3.061, 1.531)	(3.011, 1.544)	(3.105, 1.535)	(0.251, 0.125)	(0.248, 0.122)	(0.084, 0.082)	(0.083, 0.081)

Table: X with independent and exponentially distributed components with parameter 1 and 2 respectively.

Comparaison of the two estimators

α	CTE_α	$\widehat{CTE}_{\alpha L_\alpha}$	$\widehat{CTE}_{\alpha K}$	$\hat{\sigma}_{L_\alpha}$	$\hat{\sigma}_K$	$RMSE_{L_\alpha}$	$RMSE_K$
0.10	(1.188, 1.229)	(1.049, 1.192)	(1.179, 1.231)	(0.032, 0.021)	(0.031, 0.021)	(0.019, 0.033)	(0.013, 0.018)
0.24	(1.448, 1.366)	(1.283, 1.379)	(1.442, 1.372)	(0.053, 0.224)	(0.039, 0.023)	(0.019, 0.063)	(0.014, 0.017)
0.38	(1.727, 1.505)	(1.525, 1.471)	(1.724, 1.506)	(0.046, 0.031)	(0.041, 0.029)	(0.019, 0.031)	(0.017, 0.022)
0.52	(2.049, 1.666)	(1.803, 1.625)	(2.065, 1.667)	(0.058, 0.041)	(0.048, 0.039)	(0.023, 0.034)	(0.021, 0.031)
0.66	(2.454, 1.875)	(2.129, 1.823)	(2.479, 1.873)	(0.071, 0.054)	(0.069, 0.046)	(0.035, 0.039)	(0.029, 0.033)
0.80	(3.039, 2.202)	(2.591, 2.144)	(3.029, 2.252)	(0.111, 0.105)	(0.103, 0.103)	(0.055, 0.054)	(0.041, 0.049)

Table: X with Clayton copula with parameter 1, F_1 exponential distribution with parameter 1, F_2 Burr$(4,1)$ distribution.

What happens for high levels

Independent and exponentially distributed marginals with parameters 1 (*resp. 2)*, $\alpha = 0.9$. *Then* $CTE_{0.9} = (3.78, 1.89)$.

n	1000	1500	2000	2500
$\widehat{\sigma}_K$	*(0.416, 0.299)*	*(0.411, 0.256)*	*(0.368, 0.155)*	*(0.221, 0.113)*
$\widehat{\sigma}_{L_\alpha}$	*(0.444, 0.308)*	*(0.431, 0.295)*	*(0.377, 0.168)*	*(0.241, 0.123)*
$RMSE_K$	*(0.113, 0.158)*	*(0.111, 0.135)*	*(0.095, 0.087)*	*(0.072, 0.063)*
$RMSE_{L_\alpha}$	*(0.123, 0.163)*	*(0.115, 0.161)*	*(0.099, 0.089)*	*(0.077, 0.079)*

We need 2500 *data to obtain performances similar to the ones when* $\alpha = 0.8$ *and* $n = 1000$.

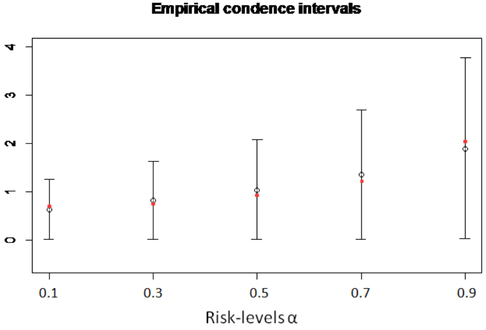
A challenge is the estimation of extreme risks $\alpha \geq 0.95$.

Deterioration when α increases

With $r = 100$ replications and sample size equal to $n = 1000$, we derive empirical confidence intervals for $\widehat{\mathrm{CTE}_{\alpha,2}}_K$:

$$\left[\overline{\widehat{\mathrm{CTE}_{\alpha,2}}_K} - u_{0.95}\,\frac{\widehat{\sigma}_K}{\sqrt{n}}\,,\ \overline{\widehat{\mathrm{CTE}_{\alpha,2}}_K} + u_{0.95}\,\frac{\widehat{\sigma}_K}{\sqrt{n}}\right]$$

with $u_{0.95}$ the quantile of order 0.95 of the standard gaussian distribution.



Influence of the dependence

Clayton family $C(u,v) = \left(\max(u^{-\theta} + v^{-\theta} - 1\,,\,0)\right)^{-1/\theta}$ with $-1 \leq \theta \leq +\infty$. Uniform marginals, $r = 100$ and $n = 1000$.

α \ θ	-0.95	0	1	10^4
0.10	CTE_α 0.6419 $\widehat{CTE}_{\alpha K}$ 0.6397 $\hat\sigma_K$ 0.0337 $RMSE_K$ 0.0538	CTE_α 0.6047 $\widehat{CTE}_{\alpha K}$ 0.6062 $\hat\sigma_K$ 0.0106 $RMSE_K$ 0.0177	CTE_α 0.5827 $\widehat{CTE}_{\alpha K}$ 0.5831 $\hat\sigma_K$ 0.0102 $RMSE_K$ 0.0176	CTE_α 0.5500 $\widehat{CTE}_{\alpha K}$ 0.5495 $\hat\sigma_K$ 0.0091 $RMSE_K$ 0.0165
0.38	CTE_α 0.7757 $\widehat{CTE}_{\alpha K}$ 0.7723 $\hat\sigma_K$ 0.0475 $RMSE_K$ 0.0611	CTE_α 0.7617 $\widehat{CTE}_{\alpha K}$ 0.7644 $\hat\sigma_K$ 0.0127 $RMSE_K$ 0.0179	CTE_α 0.7494 $\widehat{CTE}_{\alpha K}$ 0.7521 $\hat\sigma_K$ 0.0108 $RMSE_K$ 0.0178	CTE_α 0.6900 $\widehat{CTE}_{\alpha K}$ 0.6903 $\hat\sigma_K$ 0.0105 $RMSE_K$ 0.0171
0.66	CTE_α 0.8825 $\widehat{CTE}_{\alpha K}$ 0.8936 $\hat\sigma_K$ 0.1261 $RMSE_K$ 0.1442	CTE_α 0.8789 $\widehat{CTE}_{\alpha K}$ 0.8848 $\hat\sigma_K$ 0.0181 $RMSE_K$ 0.0184	CTE_α 0.8754 $\widehat{CTE}_{\alpha K}$ 0.8799 $\hat\sigma_K$ 0.0119 $RMSE_K$ 0.0182	CTE_α 0.8300 $\widehat{CTE}_{\alpha K}$ 0.8305 $\hat\sigma_K$ 0.0117 $RMSE_K$ 0.0176

River flow data-set

data-set from the National River Flow Archive of the Center for Ecology & Hydrology in UK
http://www.ceh.ac.uk/index.html

hydrological data-set recorded in the uplands of mid-Wales : river flow data measured at the Hore site and at the Tanllwyth site from '85 to '03 ($m^3 s^{-1}$), $n = 2134$.

α	0.45	0.625	0.8
$\widehat{\mathrm{CTE}}_{\alpha,K}$	(0.2099, 0.1339)	(0.2795, 0.1831)	(0.4775, 0.2652)
$\widehat{\mathrm{CTE}}_{L_\alpha}$	(0.1388, 0.1683)	(0.1662, 0.1941)	(0.3621, 0.2863)

River flow data-set

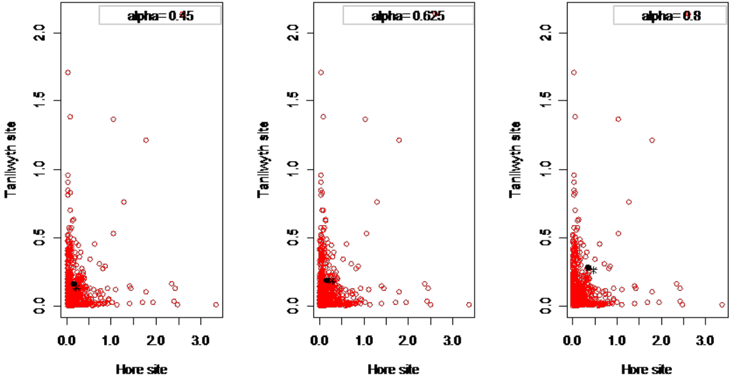


Figure: River flow data; $\widehat{\mathrm{CTE}}_{\alpha,K}$ (black star), $\widehat{\mathrm{CTE}}_{L_\alpha}$ (black dot) for different values of α.

Outline

Conclusion, perspectives

Conclusion :

- a new non parametric estimator,
- no extra parameter to fix,
- a functional central limit theorem.

Perspective :

A main issue is to derive estimators for the Multivariate CTE_α whose properties are good even for high levels α.

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