Mapping extreme snowfall in the French Alps using Max-Stable Processes

J.Gaume[1], **N.Eckert**[1], L. Bel[2], G.Chambon[1], M.Naaim[1]

[1] IRSTEA - Unité ETNA - Grenoble
[2] AgroParisTech, MMIP Department

Extremes – 26/09/2012

Context

Avalanche release depth

- Input parameter of avalanche propagation models (Naaim & al. 2003; Lachamp & al., 2002)
- Traditional engennering assumption: correspond to fresh snow depth (1 or 3 days).

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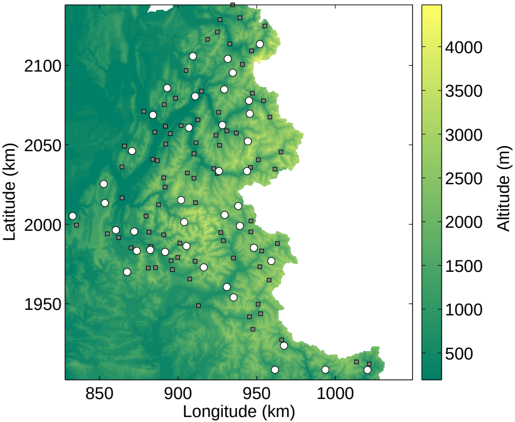
Objective: **Long term** management

- Obtain fresh snow depths maps for different return periods and accumulation periods.
- Improve the setting up of territorial planification tools such as hazard maps or risk prevention plans (PPR)
- Improve the conception of protection tools in terms of functional characteristics (height, orientation...)

Data

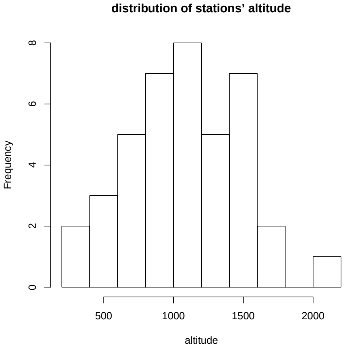


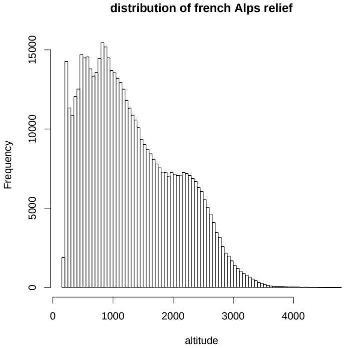
- $K = 40$ sites among 110 sites
- $N = 44$ years of mesurement (from 1966 to 2009)
- Daily measurements of snowfall (in mm w.eq)
- Data source: **MétéoFrance** Clim network.

Problematic

Weather station elevation

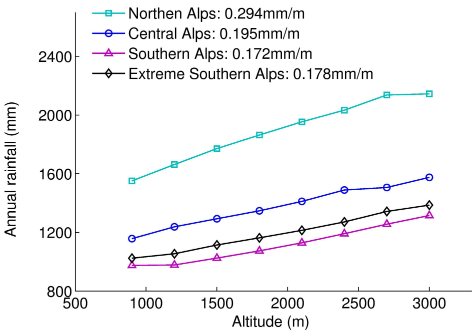
Weather stations are usually located at low altitude.

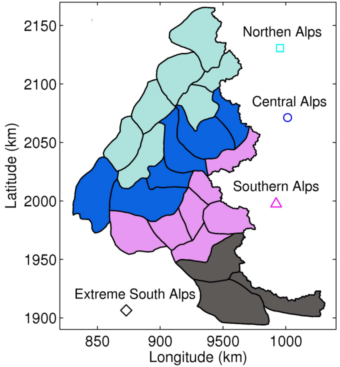




→ Need to take into account orographic gradient of altitude.

Orographic gradient of altitude: Data transformation $\rightarrow 2000m$ (Durand et al 2009)





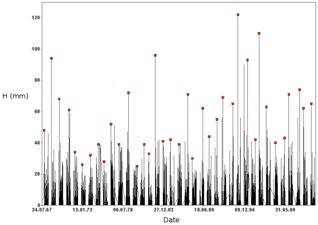
$$H_{2000}(x) = H_e(x) + \gamma(x)\frac{H_e(x)}{\mathrm{WMS}(x)}(2000 - e(x))$$

WMS: annual accumulation

Problematic

Spatial interpolation of extreme values

Avalanches are rare events
$\rightarrow$ Analysis of extreme snowfalls



$\Rightarrow$ Need for spatial interpolation techniques specific to extreme values $\neq$ means
$\neq$ regional homogeneity (Salm et al 1990; Bocchiola et al 2006)
$\neq$ quantile smoothing (Weisse et Bois 2002)

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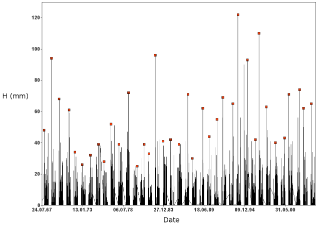

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Max-Stable Processes

New well established theoretical framework adapted for the spatial interpolation of extreme value.

Definition of a Max-Stable Process (de Haan, 1984)

$$\{H(x)\}_{x\in\mathcal{X}} \text{ has the same distribution as } \left\{\frac{max_{i=1}^{n} Y_i(x) - b_n(x)}{a_n(x)}\right\}_{x\in\mathcal{X}},$$

$Y_i(x)_{x\in\mathbb{R}^d}$, n independent realisations of a continuous stochastic process, $a_n(x) > 0$ et $b_n(x) > 0$ two sequences of continuous functions.

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Consequences in the monovariate case

$H \equiv GEV(\mu, \sigma, \xi)$ with the following distribution function:

$$F(h; \mu, \sigma, \xi) = exp\left(-\left(1 + \frac{\xi(h-\mu)}{\sigma}\right)_{+}^{-1/\xi}\right)$$

$a_+ = max(0; a)$
μ: location parameter
σ: scale parameter
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Consequences in the spatial case (multivariate)

- Infinity of Max-Stable processes difficult to obtain and to use.
- The most known and used MS processes are Smith, Schlather and Brown-Resnick.

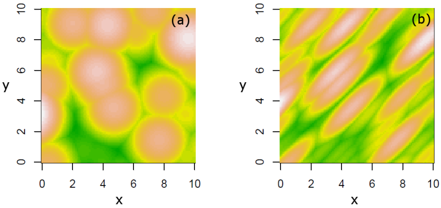
Smith's MSP (rainfall-storm process)

The Smith Max-Stable Process (Smith 1991) with unit Frechet margins is defined as:

$$H(x) = \max_i(\xi_i f(y_i, x)) \qquad x \in \mathcal{X},$$

where $(\xi_i, y_i), i \geq 1$ are the points of a Poisson process on $(0, +\infty) \times \mathcal{X}$ with intensity measure $\xi^{-2} d\xi \nu(dy)$, where $\nu(dy)$ is a positive measure on $\mathcal{X}$. The function f is non-negative such that

$$\int_{\mathcal{X}} f(x,y)\nu(dy) = 1, \qquad x \in \mathcal{X},$$



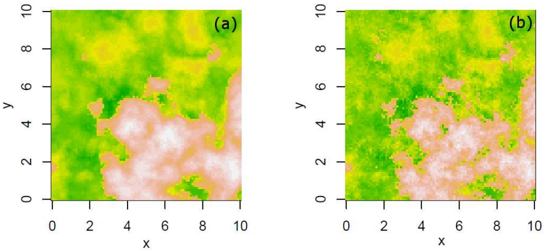
Isotropic Anisotropic

Schlather's MSP

The Schlather Max-Stable Process (Schlather 2002) with unit Frechet margins is defined as:

$$Z(x) = \max_i \xi_i \max(0, Y_i(x)) \qquad x \in \mathcal{X},$$

where Y_i are i.i.d copies of a stationnary standard Gaussian process with correlation function $\rho(||h||)$ on $\mathcal{X}$ such that $\mathbb{E}[\max(0, Y(x))] = 1$ and $\xi_i, i \geq 1$ the points of a Poisson process on $\mathbb{R}_*^+$ with intensity $\xi^{-2} d\xi$.



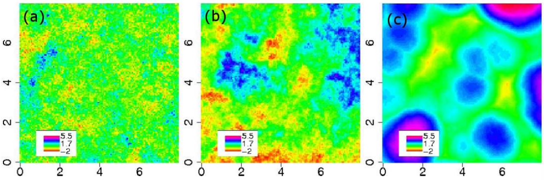
ρ Wittle-Mattern $\qquad$ ρ Power-exponential

Brown-Resnick's MSP

The Brown-Resnick model (Brown 1977), generalized by (Kabluchko2009) is defined as:

$$Z(x) = \max_i (\xi_i \exp(W_i(x) - \sigma^2(x)/2) \qquad x \in \mathcal{X},$$

where ξ_i is a Poisson process on $\mathbb{R}^{*+}$ of intensity $\frac{1}{\xi^2} d\xi$ and W_i are independent Gaussian fields, with stationary increments, variance $\sigma^2(x)$ and variogram $\gamma(h) = ||h||^\alpha$.



$\alpha = 0.4$ $\qquad$ $\alpha = 1$ $\qquad$ $\alpha = 1.95$

Spatial dependance: extremal coefficient

We know how to build these 3 processes, but we don't know analytically the joint density. However, the bivariate density is known.

Definition

Notion of variogram or spatial correlation applied to extremes.

If $H(x)$ is a **Max-Stable process** with unit Frechet margins

$$\Downarrow$$

$$\boxed{P(H(x) \leq u \text{ and } H(x+h) \leq u) = P(H(x) \leq u)^{\theta(h)}}$$

$\theta(h)$: extremal coefficient $\quad \theta = 1 \Rightarrow$ Perfect dependence; $\theta = 2 \Rightarrow$ Total independence

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Smith's MSP

$\Rightarrow \boxed{\theta(h) = 2\Phi(a(h)/2)}$ avec $a(h) = (h^t \Sigma^{-1} h)^{1/2}$, Σ: covariance matrix and Φ: cumulative normal distribution.

Spatial dependance: extremal coefficient

Schlather's MSP

$\Rightarrow$ $\boxed{\theta(h) = 1 + \sqrt{1 - \frac{1}{2}(\rho(\|h\|) + 1)}}$ with $-1 \leq \rho(\|h\|) \leq 1$, a valid correlation function (Wittle-Matern, Cauchy, Exponentiel, Bessel...)

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Brown-Resnick's MSP

$\Rightarrow \boxed{\theta(h) = 2\Phi\left(\sqrt{\frac{\gamma(\|h\|)}{2}}\right)}$

with Φ the cumulative normal distribution
and γ the variogram.

Solution to the problem of isotropy: Transformed space (Blanchet and Davison 2010)

Isotropy/Anisotropy

- Smith: Anisotropic (Mahalanobis distance: $d = a(h) = \sqrt{h^t \Sigma^{-1} h}$)
- Brown-Resnick & Schlather: Isotropic ($d = \|h\|$)

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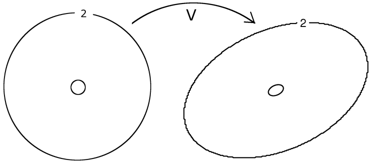
Space transformation

$$\left\{\text{Anisotropic in } \tilde{X}\right\} \Rightarrow \{\text{Isotropic in } X\}$$

$$\tilde{X} = VX \text{ avec } X = (x, y)$$

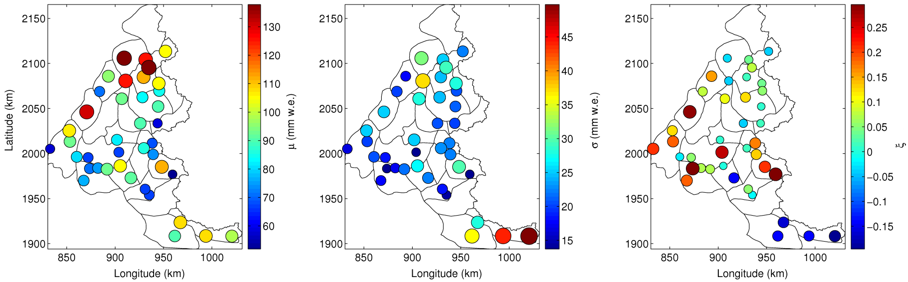
$$V = \begin{pmatrix} \cos\psi & \sin\psi \\ -\rho\sin\psi & \rho\cos\psi \end{pmatrix}$$



$\Rightarrow$ The directional effect is taken into account

Empirical analysis

GEV parameters (constant altitude of 2000 m)



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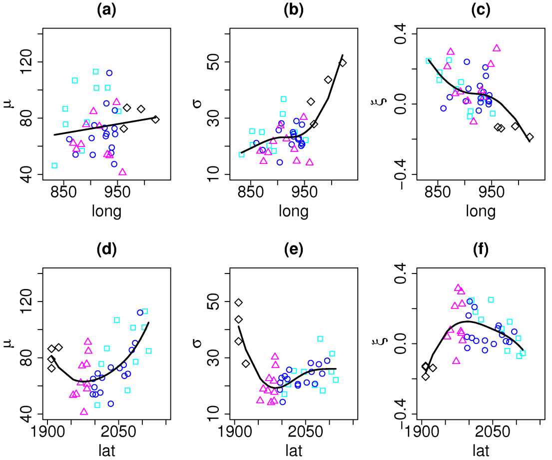
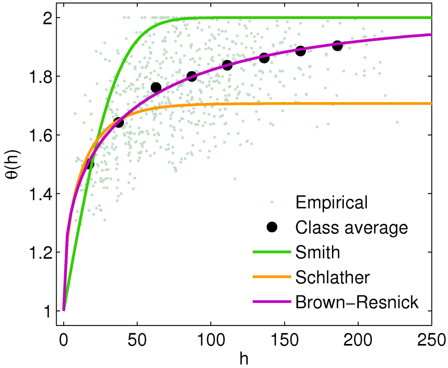
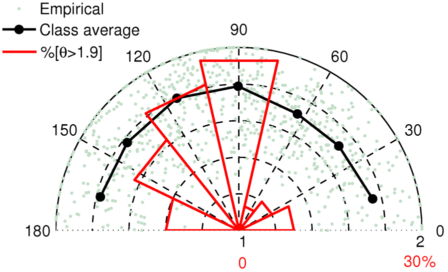


Empirical analysis

Extremal coefficient (constant altitude of 2000 m)





Spatial evolution models for the GEV parameters

Description of GEV parameters (μ, σ, ξ) through linear models or cubic splines which can be function of space (altitude, longitude, latitude), environment (orientation of the path, slope angle...) or random effects.
$\eta(x) = BX(x)$

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Interest

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Model selection

Need a criterion for the selection of the model

Example of spatial evolution: Linear

$$\begin{pmatrix} \mu \\ \sigma \\ \xi \end{pmatrix} = \begin{pmatrix} \beta_{\mu 0} & \beta_{\mu 1} & \beta_{\mu 2} \\ \beta_{\sigma 0} & \beta_{\sigma 1} & \beta_{\sigma 2} \\ \beta_{\xi 0} & \beta_{\xi 1} & \beta_{\xi 2} \end{pmatrix} \begin{pmatrix} 1 \\ \text{Lat} \\ \text{Long} \end{pmatrix}$$

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Example of spatial evolution: Spline

Penalized splines with radial basis functions (Ruppert 2003) of order 3:

$$\eta(x) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \sum_{r=0}^{R-1} \beta_{3+r} ||x - \kappa^r||^3,$$

where κ^r are the coordinates of the r^{th} knot of the spline, R is the number of knots and η is one of the 3 GEV parameters.

The complete joint density is not known but the bivariate density is known analytically. Need a criterion for model selection adapted to this case.

Likelihood: Definition in the context of the pairwise analysis of spatial extremes (Padoan et al 2009)

$$l_c(\beta, H) = \sum_{n=1}^{N} \sum_{i=1}^{K} \sum_{j=i+1}^{K-1} \log f(H_{n,i}, H_{n,j}; \beta)$$

- $\cdot K$: number of sites
- $\cdot N$: number of years
- $\cdot i \in [1, ..., K]$
- $\cdot j \in [i+1, ..., K-1]$
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Estimation: Likelihood maximisation (MLE)

$$D_\beta l_c(\hat{\beta}_{MLE}, H) = 0 \rightarrow \hat{\beta}_{MLE}$$

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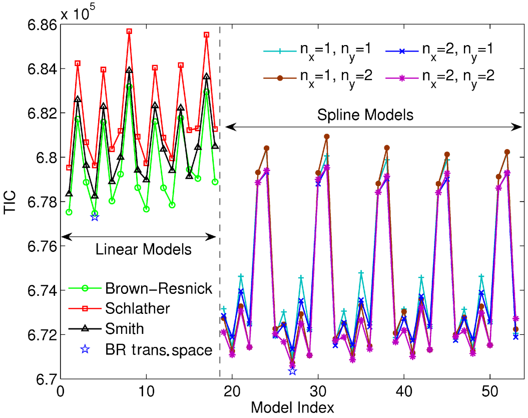
Criterion: TIC

$$\text{TIC} \approx \text{AIC} = -2 l_c(\hat{\beta}_{MLE}, y) + 2p$$

TIC Criterion (Takeuchi information criterion) for model selection

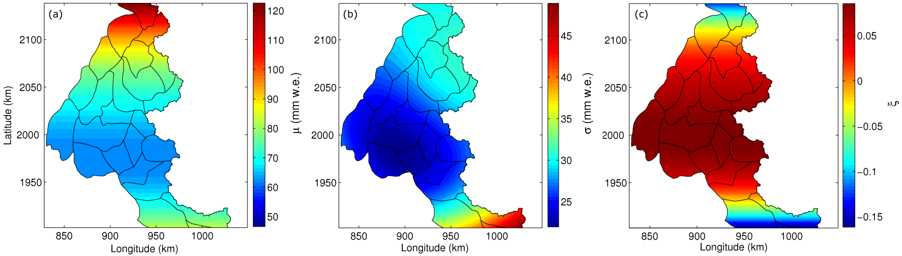
Best model: **Lower TIC value** (Takeuchi 1976)

$$\mathrm{TIC} \approx \mathrm{AIC} = -2 l_c(\hat{\beta}_{MLE}, y) + 2p$$



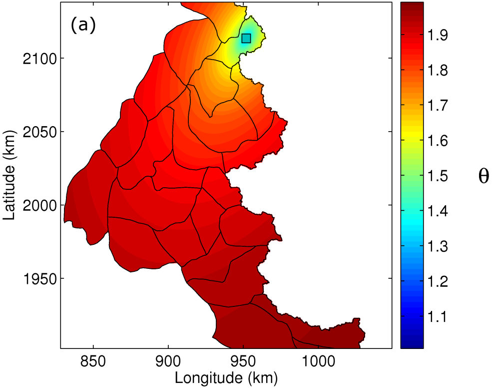
- Linear models: BR > Smith > Schlather
- Anisotropy → reduction of the TIC
- Spline > Linear → Important reduction of the TIC

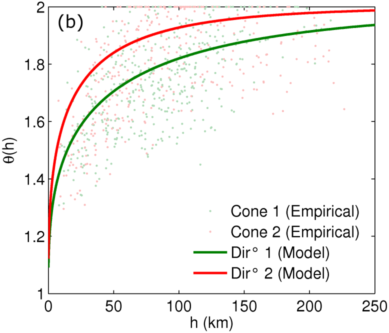
Spatial evolution of μ and σ and ξ



- Highest "means": North
- Highest "variances": South-East (Mediterranean effect)
- $\xi > 0$ except in the extreme North and South.

Spatial dependence: Extremal coefficient $\theta(h)$





$$l_1 \approx 85 \text{ km},\ l_2 \approx 185 \text{ km } (\theta = 1.9)$$
$$\psi = 62.5^\circ,\ \rho = 2.05$$

Anisotropy angle = Local alpin axis angle (Rhone, Isère and Durance valleys)

Quantile

Quantile calculation y_T

$$P(H \leq h_T) = \exp\left[-\left(1+\frac{\xi(h_T-\mu)}{\sigma}\right)_+^{-1/\xi}\right] = 1-\frac{1}{T}$$

$$\Rightarrow \boxed{h_T = \mu + \frac{\sigma}{\xi}\left[\left(-\ln\left(1-\frac{1}{T}\right)\right)^{-\xi} - 1\right]}$$

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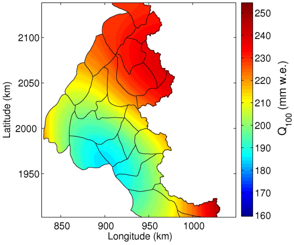
Calculation of the error on the quantile at 2000m

$$\Delta h_T = \Delta\mu + \left|\frac{1}{\xi}\left(\eta^{-\xi} - 1\right)\right|\Delta\sigma + \left|\frac{\sigma}{\xi}\left(\frac{1}{\xi}\left(\eta^{-\xi} - 1\right) + \eta^{-\xi}\ln\eta\right)\right|\Delta\xi$$

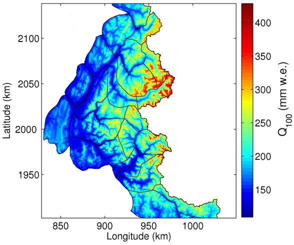
Avec $\eta = -\ln\left(1 - \frac{1}{T}\right)$

Quantile

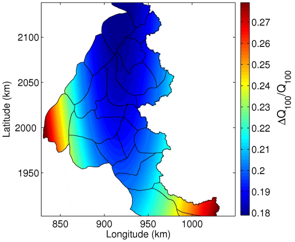
T =100ans



100-year quantile at 2000 m

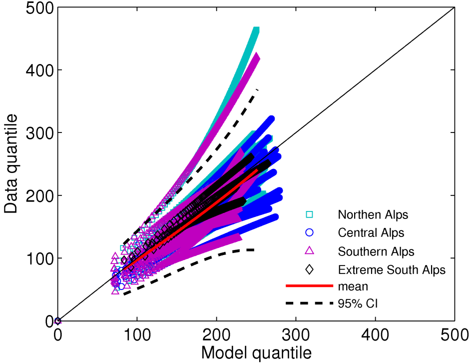


100-year quantile at real altitude

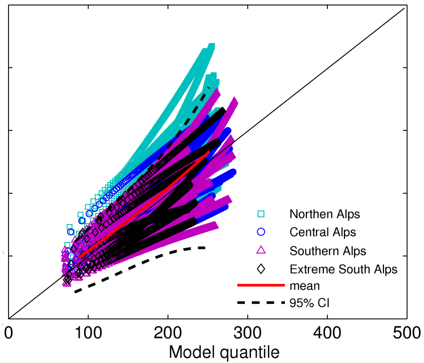


100-year quantile standard error

Validation



Local-spatial comparison at 2000 m



Cross validation for non-used stations at real altitude

Influence of the accumulation period on the quantile: IDF curves

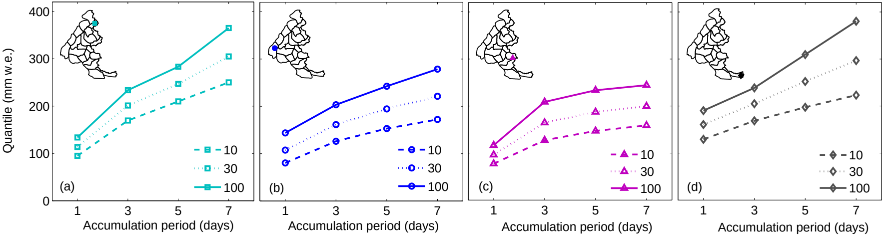
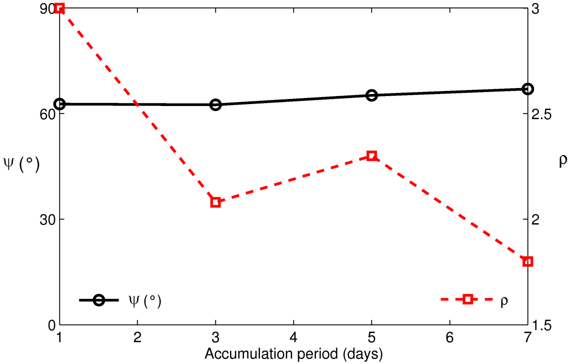


Figure: IDF curves for Chamonix-Mont-Blanc (Mont-Blanc massif) (a), Villard-de-Lans (Vercors massif) (b), Saint Veran (Queyras massif) (c) and Tende (Alpes-Azureennes massif) (d).

Influence of the accumulation period on the directional effect



- Anisotropy angle $\psi \approx$ cste
- Elongation $\rho \searrow$ with accumulation period

Application: Joint analysis

$$P\left(Z(x) > z_T(x)\,;\; Z(x') > z_T(x')\right) = P\left(Z^*(x) > z_T^*\,;\; Z^*(x') > z_T^*\right)$$

$$= 1 - 2\left(1 - \frac{1}{T}\right) + \left(1 - \frac{1}{T}\right)^{\theta(x,x')}$$

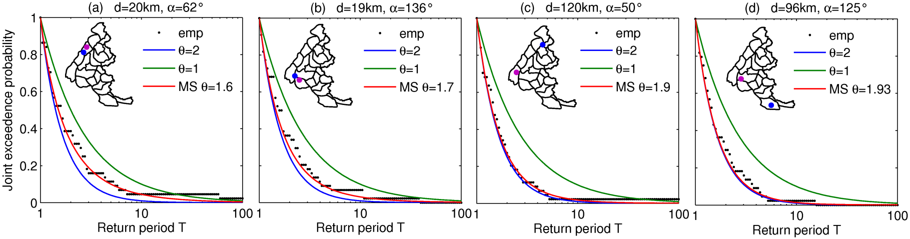


Figure: Joint exceedence probability $P(Z^*(x) > z_T; Z^*(x') > z'_T)$ for 4 couples of stations. (a) Challes-les-Eaux (291 m, Bauges) – Lescheraines (590 m, Bauges), (b) Monestier de Clermont (800 m, Oisan)– Pellafol (930 m, Devoluy), (c) Megeve (1104 m, Mont-Blanc) – Villard-de-Lans (1050 m, Vercors), (d) La Mure (856 m, Oisan) – Guillaumes (620 m, Alpes-Azureennes)

Application: Conditional quantile estimation

Conditional probability of exceeding the T-year return level in location x knowing that the T'-year return level was exceeded in location x':

$$P\left(Z(x) > z_T(x) \mid Z(x') > z_{T'}(x')\right)$$

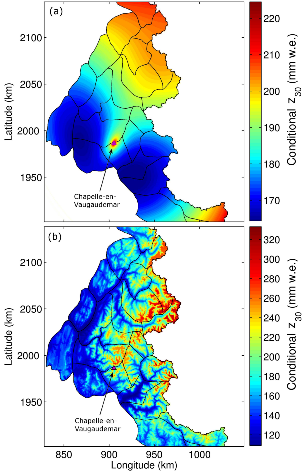
$$= P\left(Z^*(x) > z_T^* \mid Z^*(x') > z_{T'}^*\right) = \frac{1}{T}$$

with $z_{T'}^* = -1/\ln(1 - 1/T')$. Using the general expression of the bivariate probability (Brown 1977, Kabluchko 2009, Davison 2012):

$$-\ln P\left(Z^*(x) \le z_T^* \; ; \; Z^*(x') \le z_{T'}^*\right) = \frac{1}{z_T^*}\Phi(c) + \frac{1}{z_{T'}^*}\Phi\left(c'\right)$$

with $c = d/2 + 1/d \ln(z_{T'}^*/z_T^*)$, $d = \sqrt{2\gamma(h)}$ and $c' = d - c$, we can determine the conditional quantile z_T^* by numerically solving the following equation:

$$-\frac{1}{T} + T'\left(\frac{1}{T'} - e^{-\frac{1}{z_T^*}} + e^{-\frac{\Phi(c)}{z_T^*} - \frac{\Phi(c')}{z_{T'}^*}}\right).$$

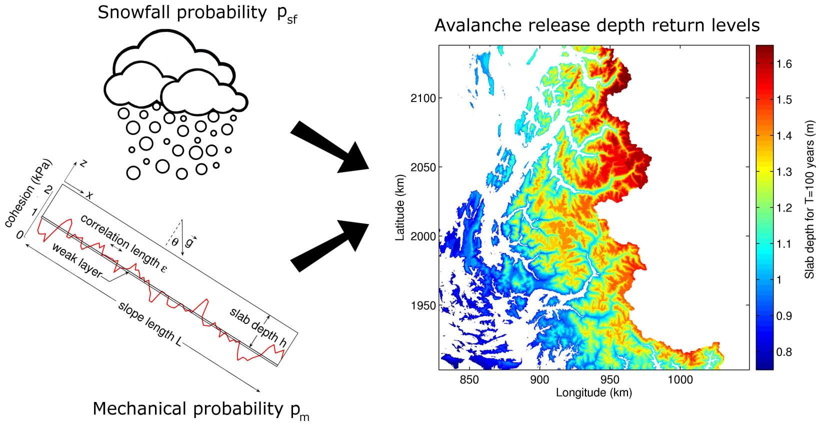


$T = T' = 30$ ans

Conclusions

- Practical implementation of the **Max-Stable formalism** with a **Brown-Resnick Process** with **spline** evolution models for the GEV parameters and taking into account **anisotropy** for the **mapping of extreme snowfalls in water equivalent**.
- Snowfall quantile maps for different return periods.
- Highest μ: North / Highest σ: South-East
- The spatial dependence depends strongly on the orientation of the Alps in accordance with other results (Swiss Alps: Blanchet and Davison 2010; Apalaches: Padoan et al 2009)
- Brown-Resnick MSP: more flexible at short distances + asymptotic independence ($\theta \to 2$ for large distances) $\Rightarrow$ well adapted for snowfalls and presumably for other hydrological variables
- Risk management: Snowfall quantile maps – Joint analysis – Conditional snowfall quantile maps

Coupling with a mechanical model



Invitation

- PhD thesis of Johan Gaume in IRSTEA (salle écrin) – 30 October 2012

Thanks for your attention!

Reference: J.Gaume, N.Eckert, G.Chambon, M.Naaim, L.Bel (2012), Mapping extreme snowfalls in the French Alps using Max-Stable processes, submitted to *Water Resources Research*