Patchwork copulas

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Outline

1 Introduction

2 Complete dependence and shuffles of Min

3 A measure-theoretic notion of shuffle of Min

4 Multivariate shuffles of copulas

5 The patchwork construction

6 Conclusions
The main goal

Given a copula $C$, a **patchwork copula** derived from $C$ is any copula whose mass probability distribution coincides with the mass distribution of $C$ up to some $d$–dimensional boxes $B_i \subseteq \mathbb{I}^d$ (here $\mathbb{I} := [0, 1]$), in which the probability mass is distributed in a different way.

Applications:
- Modification of tail dependence behaviour
- Approximation of copulas

Patchwork copulas include ordinal sum constructions, orthogonal grid constructions, gluing copulas, upper comonotonicity, piecing–together, etc.

Aim: provide a general framework for patchwork copulas.
The main tool

Given a copula $C$, a probability measure $\mu_C$ is defined on all boxes $B$ of $\mathbb{I}^d$ via

$$\mu_C(B) = V_C(B) = \mathbb{P}(U \in B),$$

where $U \sim C$, and extended by classical arguments to all Borel sets. Moreover, such a measure is $d$–fold stochastic, i.e.

$$\mu_C(p_i^{-1}(A)) = \lambda(A)$$

for any Borel set $A \subseteq \mathbb{I}$ ($\lambda = \text{Lebesgue measure}$) and for any canonical projection $p_i$.

Conversely, given a $d$–fold stochastic measure $\mu$, a copula is defined via

$$C_\mu(u) = \mu([0, u]) \quad \text{for all } u \in \mathbb{I}^d.$$
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Complete dependence

**Definition**

A r.v. $Y$ is defined to be **completely dependent** on a r.v. $X$ if there exists a measurable function $f$ such that

$$
P(Y = f(X)) = 1.
$$

The r.v.’s $X$ and $Y$ are **mutually completely dependent** (in short, MCD) if there exists a bijective measurable function $f$ such that

$$
P(Y = f(X)) = 1.
$$

(Lancaster, 1963)

In other words, two r.v.’s are MCD if one variable is perfectly predictable from the other one, and conversely.
Shuffle of Min

Shuffles of Min are bivariate copulas constructed by means of a rearrangement of the probability mass of the Fréchet upper bound $M_2(u_1, u_2) = \min\{u_1, u_2\}$.

**Proposition**

Let $(X, Y)$ be a random pair distributed according to the copula $C$. Then $C$ is a shuffle of Min iff $\mathbb{P}(Y = f(X)) = 1$ for some bijective piece-wise continuous function $f$.

(Mikusinski, Sherwood and Taylor, 1992)

If two r.v.’s are coupled by means of a shuffle of Min, then they are MCD.
Geometric interpretation of shuffles of Min

Placing the mass of the copula $M_2$ on $\mathbb{I}^2$.
Geometric interpretation of shuffles of Min

Cut $\mathbb{I}^2$ into a finite number of vertical strips.
Eventually, flip some strips around their vertical axis of symmetry.
Geometric interpretation of shuffles of Min

“Shuffle” the strips and reassemble them to reform $\mathbb{I}^2$. The resulting picture represents the probability mass distribution of a “shuffle of Min”.

Kimeldorf & Sampson’s shuffle of Min

Fig. 1. Support of the distribution of $(U_3, V_3)$.
Kimeldorf & Sampson’s shuffle of Min

**Theorem**

There are sequences \((U_n)_n\) and \((V_n)_n\) of r.v.’s all having uniform distribution on \((0, 1)\) such that:

- for each \(n\), \(U_n\) and \(V_n\) are mutually completely dependent,
- the pairs \((U_n, V_n)\) converge in law to a pair \((U, V)\) of independent r.v.’s each having a uniform distribution on \((0, 1)\).

(Kimeldorf and Sampson, 1978; Vitale, 1991)

In other words, independent r.v.’s can be approximated by means of a sequence of MCD r.v.’s.
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The shuffling transformation

Let $T \in \mathcal{T}_p$, the class of measure-preserving bijections of $\mathbb{I}$, i.e. for all Borel sets $A \subseteq \mathbb{I}$,

$$\lambda(T^{-1}(A)) = \lambda(A)$$

We define the shuffling transformation $S_T : \mathbb{I}^2 \rightarrow \mathbb{I}^2$ via

$$S_T(u_1, u_2) = (T(u_1), u_2)$$

for every $(u_1, u_2) \in \mathbb{I}^2$.

Proposition

A copula $C$ is a shuffle of Min iff there exists a piece-wise continuous $T \in \mathcal{T}_p$ such that $\mu_C = S_T \ast \mu_M$, i.e. for all Borel $A \subseteq \mathbb{I}^2$

$$\mu_C(A) = \mu_M(S_T^{-1}(A))$$

(Durante, Sarkoci and Sempi, 2009)
Shuffles of copulas

Definition

Let $C$ be any copula.
A copula $D$ is a shuffle of $C$ if there exists $T \in \mathcal{T}_p$ such that

$$\mu_D = S_T \ast \mu_C.$$  

(Durante, Sarkoci and Sempi, 2009)

In other words, any copula can be modified by cutting in a countable number of stripes its probability mass and by shuffling the resulting stripes.
Representation of shuffles of copulas

We recall that any copula \( C \) can be represented in the form

\[
C(u) = C_{f_1, \ldots, f_d} = \lambda(f_1^{-1}[0, u_1] \cap \cdots \cap f_d^{-1}[0, u_d])
\]

for suitable mpt’s \( f_1, \ldots, f_d \).

(Vitale, 1996; Kolesárová, Mesiar, Sempi, 2008)

**Proposition**

Let \( D = D_{f,g} \) be a copula represented in terms of mpt’s in the following way:

\[
D_{f,g}(u_1, u_2) = \lambda(f^{-1}[0, u_1] \cap g^{-1}[0, u_2])
\]

Then any shuffle of \( D \) via \( T \in \mathcal{T}_p \) (write: \( D_T \)) can be represented in the form

\[
D_T(u_1, u_2) = \lambda((T \circ f)^{-1}[0, u_1] \cap g^{-1}[0, u_2])
\]
Remark

The mapping

\[ \varphi: \mathcal{T}_p \times \mathcal{C}_2 \rightarrow \mathcal{C}_2, \quad \varphi(T, C) = C_T \]

defines an action of the group $\mathcal{T}_p$ on the set of all copulas. The orbit of a copula $C$ with respect to this action is the set

\[ \mathcal{T}_p(C) = \{ C_T \mid T \in \mathcal{T}_p \} \]

formed by all shuffles of $C$.

Properties of shuffles of copulas

- $\mathcal{T}_p(C) = \{ C \}$ iff $C = \Pi_2$.
- If $C$ is abs continuous, then every copula belonging to $\mathcal{T}_p(C)$ is abs continuous.
- If $C \neq \Pi_2$, then $\mathcal{T}_p(C)$ contains non-symmetric copulas.
Proposition

For every copula $C$, the independence copula $\Pi$ can be approximated uniformly by elements of $\mathcal{T}_p(C)$.

(Durante, Sarkoci and Sempi, 2009)

The proof is based on ergodic theory and uses the following facts:

- the existence of suitable weakly mixing transformations in $\mathcal{T}_p$;
- the following Lemma by Walters (1982).

Lemma

Let $(\Omega, \mathcal{F}, \nu)$ be a measure space. Let $T: \Omega \rightarrow \Omega$ be a weakly mixing transformation. Then there exists a subset $D$ of $\mathbb{Z}_+$ of density zero such that

$$\lim_{n \to \infty} \int_{\Omega} (f \circ T^n)(x) g(x) \, d\nu = \int_{\Omega} f(x) \, d\nu \int_{\Omega} g(x) \, d\nu$$

for all real functions $f$ and $g$ in $L^2(\nu)$. 
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Another look at shuffles of Min
Another look at shuffles of Min

Consider two suitable partitions $\mathcal{J}^1 = (J^1_i)$ and $\mathcal{J}^2 = (J^2_i)$ such that $J^1_i \times J^2_i$ is a square.
Another look at shuffles of Min

Plug an affine transformation of the probability mass of $M_2$ or $W_2$ in the square $J_i^1 \times J_i^2$. 
Visual definition of shuffle of copulas

For $i = 1, 2, 3$, take $C_i \in \mathcal{C}_2$ and plug an affine transformation of the measure induced by $C_i$ in the square $J_i^1 \times J_i^2$. The resulting measure is doubly stochastic.
Definition of shuffle of copulas

Let $\mathcal{J}^1, \ldots, \mathcal{J}^d$ be systems of closed and non-empty intervals of $\mathbb{R}$,

$$\mathcal{J}^i = (J^i_n = [a^i_n, b^i_n])_{n \in N}$$

such that:

(S1) $N$ represents a finite or countable index set, i.e. $N = \{0, 1, \ldots, \tilde{n}\}$ or $N = \mathbb{Z}_+$;

(S2) for every $i \in \{1, 2, \ldots, d\}$ and $n, m \in N$, $n \neq m$, $J^i_n$ and $J^i_m$ have at most one endpoint in common;

(S3) for every $i \in \{1, 2, \ldots, d\}$, $\sum_{n \in N} \lambda(J^i_n) = 1$;

(S4) for every $n \in N$, $\lambda(J^1_n) = \lambda(J^2_n) = \cdots = \lambda(J^d_n)$.

Let $(C_n)_{n \in N}$ be a system of $d$–copulas.
Definition of a shuffle of copulas

For all \( u \in \mathbb{I}^d \),

\[
C(u) = \sum_{n \in \mathbb{N}} \lambda(J_n^1) C_n \left( \frac{u_1 - a_1^n}{\lambda(J_n^1)}, \ldots, \frac{u_d - a_d^n}{\lambda(J_n^1)} \right)
\]

is a copula, called **shuffling copula** related to the partitions \((J^i)^d_{i=1}\) and the system \((C_n)_{n \in \mathbb{N}}\).

(Durante, Fernández–Sánchez, 2010)

The multivariate shuffling copula is a patchwork copula since it can be also viewed as a modification of the probability mass of \(M_d\) in some suitable boxes.
Probabilistic interpretation of a shuffle of copulas

Assume that:
- for every \( n \in \mathbb{N} \), \( U^n = (U^n_1, \ldots, U^n_d) \sim C_n \)
- \( Z \) is a discrete random variable assuming values in \( \mathbb{N} \) such that, for every \( n \in \mathbb{N} \), \( P(Z = n) = \lambda(J^1_n) \).

For every \( n \in \mathbb{N} \), consider the random vector

\[
V^n = (V^n_1, \ldots, V^n_d) = (\lambda(J^1_n)U^n_1 + a^n_1, \ldots, \lambda(J^1_n)U^n_d + a^n_d)
\]

Finally, let us consider the random vector \( W \) given by

\[
W = \sum_{n \in \mathbb{N}} \sigma_n(Z)V^n,
\]

where, for every \( n \in \mathbb{N} \), \( \sigma_n(x) = 1 \) if \( x = n \), \( \sigma_n(x) = 0 \) otherwise.
Then \( W \) is distributed according to a shuffle of \((C_n)_n\).
Ordinal sums of bivariate copulas

Let \( (a_i, b_i)_{i \in I} \) be a family of non-empty, pairwise disjoint, open subintervals of \( \mathbb{I} \) and let \( (T_i)_{i \in I} \) be a family of copulas. Then the function

\[
C(u_1, u_2) = \begin{cases} 
  a_i + (b_i - a_i) C_i \left( \frac{u_1 - a_i}{b_i - a_i}, \frac{u_2 - a_i}{b_i - a_i} \right) & \text{if } (u_1, u_2) \in ]a_i, b_i[, \\
  \min(u_1, u_2) & \text{otherwise},
\end{cases}
\]

is a copula, called ordinal sum of the summands \( \langle a_i, b_i, C_i \rangle_{i \in I} \).
Geometric interpretation of ordinal sums

\[ M_2 \]

\[ \langle C_1 \rangle \]

\[ \langle C_2 \rangle \]

\[ \langle C_3 \rangle \]
Multivariate ordinal sum of copulas

An ordinal sum of multivariate copulas can be introduced in the following way:

Let $L$ be a finite or countable set, let $([a_k, b_k])_{k \in L}$ be a system of sub–intervals of $\mathbb{I}$, and let $(C_k)_{k \in L}$ be a system in $\mathcal{C}_d$.

Then the ordinal sum $C$ of $(C_k)_{k \in L}$ wrt the family of intervals $([a_k, b_k])_{k \in L}$ is the $d$–copula defined, for all $u \in \mathbb{I}^d$ by

$$C(u) = \begin{cases} 
    a_k + (b_k - a_k) \ C_k \left( \frac{\min\{u_1, b_k\} - a_k}{b_k - a_k}, \ldots, \frac{\min\{u_d, b_k\} - a_k}{b_k - a_k} \right), \\
    \text{if } \min\{u_1, u_2, \ldots, u_d\} \in ]a_k, b_k[ \text{ for some } k \in L, \\
    \min\{u_1, u_2, \ldots, u_d\}, \quad \text{elsewhere.}
\end{cases}$$

(Mesiar and Sempi, 2010; Jaworski and Rychlik, 2008)
Approximation by means of shuffles

**Proposition**

Fix a $d$–copula $C$. Then any copula can be approximated uniformly by means of shuffles of the system $(C_n)_{n \in \mathbb{N}}$, where $C_n = B$ for every $n$.

(Durante, Fernández–Sánchez, 2010)

**Corollary**

Any copula can be approximated uniformly by means of shuffles of copulas that are absolutely continuous.

(Durante, Fernández–Sánchez, 2010)
Remark

The approximation of copulas by means of shuffles strongly depends on the \textit{topology} over \( C_d \).

Example

Consider the topology induced on \( C_d \) by the Sobolev norm

\[
\|C\| = \left( \int_{\mathbb{R}^d} |\nabla C(u)|^2 du \right)^{1/2}.
\]

If \( C \) is a shuffle of Min, then \( \|C\| = 1 \); but \( \|\Pi_2\| = 2/3 \).

In the Sobolev norm, shuffles of Min do not approximate \( \Pi_2 \).

(Siburg and Stoimenov, 2008)
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Definition

Let $C$ and $C_B$ be $d$–dimensional copulas and let $B = [a, b]$ be a non-empty box contained in $\mathbb{I}^d$ such that $\mu_C(B) = \alpha > 0$. The function $C^* : \mathbb{I}^d \rightarrow \mathbb{I}$ given by

$$C^*(u) = \mu_C \left([0, u] \cap B^c\right) + \alpha C_B \left(\tilde{F}_B^1(u_1), \ldots, \tilde{F}_B^d(u_d)\right)$$

is a copula, where

$$\tilde{F}_B^i(x_i) = \frac{1}{\alpha} \mu_C \left([a_1, b_1] \times \cdots \times [a_{i-1}, b_{i-1}] \times [a_i, x_i] \times [a_{i+1}, b_{i+1}] \times \cdots \times [a_d, b_d]\right),$$

for every $x_i \in [a_i, b_i]$.

The copula $C^*$ is called patchwork of $(B, C_B)$ into $C$ and it is denoted by the symbol $C^* = \langle B, C_B \rangle^C$.

(Durante, Fernández–Sánchez and Sempi, 2013)
Illustration: tail modification

Consider the patchwork \( C^* = \langle B, C_B \rangle^C \), where \( B = [\mathbf{a}, 1] \) given by

\[
C^*(\mathbf{u}) = \mu_C ([0, \mathbf{u}] \setminus [\mathbf{a}, 1]) + \alpha C_B \left( \tilde{F}_B^1(u_1), \ldots, \tilde{F}_B^d(u_d) \right),
\]

where \( \alpha = V_C(B) \) and, for every \( i \in \{1, \ldots, d\} \), one has

\[
\tilde{F}_B^i(x_i) = \frac{1}{\alpha} V_C ([a_1, 1] \times \ldots [a_i, x_i] \times \ldots \times [a_d, 1]).
\]

An algorithm for generating a random sample from \( C^* \) goes as follows.

1. Generate \( \mathbf{u} \) from the copula \( C \).
2. Generate \( \mathbf{v} \) from the copula \( C_B \).
3. For \( i = 1, 2, \ldots, d \) set \( w_i = (\tilde{F}_B^i)^{-1}(v_i) \).
4. If \( \mathbf{u} \in B \), then return \( \mathbf{w} \).
   Otherwise, return \( \mathbf{u} \).
Illustration: tail modification

Random sample of 1000 realizations from the copula \( \langle B, C_B \rangle^C \) where \( B = [0.5, 1]^3 \), \( C \) is the independence copula and \( C_B \) is the comonotone copula.
Illustration: worst-case VaR scenario

Given the vector of losses \((L_1, L_2)\) having fixed marginals, the worst-possible \(VaR\) (at level \(\alpha\)) for the sum \(L^+ = L_1 + L_2\) (write: \(\overline{VaR}_\alpha(L^+)\)) is given when \((L_1, L_2)\) is coupled by \(\langle [\alpha, 1]^2, W_2 \rangle^{M_2}\).

Interestingly, it is well known that

\[
VaR_\alpha(L_1) + VaR_\alpha(L_2) \leq \overline{VaR}_\alpha(L^+),
\]

where the left hand side corresponds to the comonotone case.

Now, for any copula \(C\), the patchwork \(C^* = \langle [\alpha, 1]^2, C \rangle^{M_2}\) can be used in order to interpolate between the comonotonic scenario and the worst-case scenario for \(VaR_\alpha(L^+)\).
Illustration: worst-case VaR scenario
Illustration: worst-case VaR scenario

\[
\begin{array}{ccccc}
\tau & \tau = 1 & \tau = 0.50 & \tau = 0.00 & \tau = -0.50 & \tau = -1 \\
\hline
VaR_\alpha(L^C_1, L^C_2) & 2.5631 & 2.5663 & 2.5749 & 3.0340 & 3.2897 \\
\end{array}
\]

Numerical approximation of \(VaR_{0.90}(L^C_1, L^C_2)\) where \(L_1, L_2, \sim N(0, 1)\), \(C^* = \langle [0.90, 1]^2, C \rangle^{M_2}\) for a Clayton copula \(C\) with Kendall’s \(\tau\) equal to the indicated value. Results based on \(10^6\) simulation from the given copula.
Illustration: upper comonotonicity

Let $C_B$ be an arbitrary $d$–copula and let $M_d$ be the comonotone copula. Consider the patchwork of copulas of type $\langle B, C_B \rangle^{M_d}$, where $B = [0, a]$. Then

$$C^*(u) = \mu_C ([0, u] \cap B^c) + \alpha C_B \left( \frac{\min\{a_1, \ldots, u_1, \ldots, a_d\}}{\alpha}, \ldots, \frac{\min\{a_1, \ldots, u_d, \ldots, a_d\}}{\alpha} \right).$$

Notice that in this case, $\alpha = V_{M_d}(B) = \min\{a_1, \ldots, a_d\}$.

When all the components of $a$ are equal to $a$, constructions of copulas of this type describe upper comonotonic random vectors (Cheung, 2009).
Illustration: upper comonotonicity

Random sample of 1000 realizations from the copula $\langle B, C_B \rangle^{M_2}$ where $B = [0, 0.8]^2$, $C_B$ is a Frank with Kendall’s tau equal to: 0.5 (left) and 0.75 (right).
Definition: the general case

Let $C$ and $C_{B_s}$ ($s \in S$) be $d$–dimensional copulas and let $B_s$ ($s \in S$) be a system (finite or countable) of non–empty boxes contained in $\mathbb{I}^d$ such that $\lambda_d(B_s \cap B_{s'}) = 0$ if $s \neq s'$. Let $B = \bigcup_{s \in S} B_s$. Then the function $C^* : \mathbb{I}^d \to \mathbb{I}$ given by

$$C^*(u) = \mu_C ([0, u] \cap B^c) + \sum_{s \in S} \alpha_s C_s \left( \tilde{F}_{B_s}^1(u_1), \ldots, \tilde{F}_{B_s}^d(u_d) \right),$$

is a copula.

The copula $C^*$ is called patchwork of $(B_s, C_{B_s})_{s \in S}$ into $C$ and it is denoted by the symbol $C^* = \langle B_s, C_{B_s} \rangle_{s \in S}^C$. 
The patchwork construction

Define \( \mathcal{C}_d^S := \{(C_s)_{s \in S}\} \), where, for every \( s \in S \), \( C_s \) is a \( d \)–copula. Let \( C \in \mathcal{C}_d \) and let \( B_s \ (s \in S) \) be a system of \( d \)–boxes.

Formally, the patchwork is defined as the mapping \( T_C : \mathcal{C}_d^S \to \mathcal{C}_d \) given by

\[
T_C ((C_s)_{s \in S}) := \langle B_s, C_s \rangle_{s \in S}^C .
\]

\( T_C \) is uniformly continuous when \( \mathcal{C}_d \) is endowed by the uniform distance \( d_\infty \), and \( \mathcal{C}_d^S \) by the distance

\[
d_S \left( (C_s)_{s \in S}, (\tilde{C}_s)_{s \in S} \right) := \sup_{s \in S} d_\infty (C_s, \tilde{C}_s) .
\]

(Durante, Fernández–Sánchez and Sempi, 2013)
The patchwork construction and approximation

Then patchwork constructions are the general setting when an approximation of a copula $C$ can be considered, by using, for instance, the following scheme:

- Divide the domain $\mathbb{I}^d$ in several boxes $\mathbb{I}^d = \bigcup_i B_i$ such that each $B_i$ is sufficiently small.
- For any $B_i$ approximate $\mu_{C|B_i}$ with another convenient measure $\mu_i$.
- Join all the measure $\mu_i$’s by obtaining a suitable $d$–fold stochastic measure $\mu = \sum_i \mu_i$.

Depending on the expression of $\mu_i$’s several approximations of copulas can be obtained (Bernstein copulas, checkerboard copulas, etc.).
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To summarize

We have revisited the notion of patchwork copulas by using measure-theoretic techniques.

The introduced construction principle:

- works in any dimension
- induces specific tail behaviour in the dependence structure
- can be used in the approximation of copulas


Questions? Comments?

Thanks for your attention!

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