Strong mixing properties of max-stable random fields

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Structure of the talk

1. Motivations
2. Max-stable processes
3. Ergodic and mixing properties
4. Strong mixing properties
Motivations

- Needs for modeling extremes in environmental sciences:
  - maximal temperatures in a heat wave,
  - intensity of winds during a storm,
  - water heights in a flood ...

- Spatial extreme value theory - geostatistics of extremes:
  - de Haan & Pereira 2006, Spatial extremes: Models for the stationary case.

- Max-stable random fields play a crucial role.

- Many developments in the last decades regarding max-stable processes:
  - theoretical properties,
  - statistics and inference,
  - modelling and applications.
Motivations

- Here we want to consider statistics based not on i.i.d. observations but rather on observations of a stationary weakly dependent max-stable process.

- Recent results for ergodic and mixing properties of stationary max-stable and max-i.d. processes:
  - Stoev (’10) Max-stable processes: representations, ergodic properties and statistical applications.
  - Kabluchko & Schlather (’10) Ergodic properties of max-infinitely divisible processes.

- Ergodicity and mixing are important to derive strong law of large numbers and hence consistency of estimators.
Motivations

- Ergodicity and mixing are not enough to get central limit theorems or asymptotic normality of estimators.
- Central limit theorems for stationary processes are available under various weak dependence assumptions. We consider here strong mixing assumptions (e.g. $\alpha$-mixing or $\beta$-mixing).
- Can we derive some estimates for the mixing coefficients of a max-stable process?
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Definition and first properties

Let $\mathcal{T}$ be a locally compact parameter space.
Let $\eta = (\eta(t))_{t \in \mathcal{T}}$ be a sample continuous process.

**Definition**

- $\eta$ is **max-stable** if for any $n \geq 1$, there are continuous functions $a_n(\cdot) > 0$ and $b_n(\cdot)$ such that

$$
\left( \frac{\bigvee_{i=1}^n \eta_i(t) - b_n(t)}{a_n(t)} \right)_{t \in \mathcal{T}} \overset{\mathcal{L}}{=} (\eta(t))_{t \in \mathcal{T}}
$$

with $\eta_1, \ldots, \eta_n$ i.i.d. copies of $\eta$. 

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Max-stable processes arise as limit of maxima of i.i.d. processes: if \((X_i)_{i \geq 1}\) are i.i.d. random processes on \(T\) such that there exists normalization functions \(a_n(\cdot) > 0\) and \(b_n(\cdot)\)

\[
\left( \frac{\vee_{i=1}^n X_i(t) - b_n(t)}{a_n(t)} \right)_{t \in T} \Longrightarrow (\eta(t))_{t \in T},
\]

then \(\eta\) must is max-stable.

For each \(t \in T\), \(\eta(t)\) has a GEV distribution.

We say that \(\eta\) is simple max-stable if the margins are 1-Fréchet:

\[
\mathbb{P}[\eta(t) \leq y] = \exp(-1/y).
\]
Theorem (de Haan ’84, Penrose ’92)

Let \( \eta \) be a simple max-stable process with continuous sample path. Then \( \eta \) can be represented as

\[
\left( \eta(t) \right)_{t \in \mathcal{T}} \overset{\mathcal{L}}{=} \left( \bigvee_{i \geq 1} U_i Y_i(t) \right)_{t \in \mathcal{T}}
\]

where

- \((Y_i)_{i \geq 1}\) are i.i.d. copies of a random process \(Y\) with path in \(C_0 = C(T, [0, +\infty)) \setminus \{0\}\) and such that
  \[
  \mathbb{E}[Y(t)] = 1, \quad t \in \mathcal{T}
  \]
  \[
  \mathbb{E}\left[\sup_{t \in K} Y(t)\right] < \infty, \quad K \text{ compact};
  \]
- \(\{U_i, i \geq 1\}\) is a Poisson Point Process on \((0, +\infty)\) with intensity \(u^{-2} du\);
- \((Y_i)_{i \geq 1}\) and \(\{U_i, i \geq 1\}\) are independent.
Finite dimensional distributions

- Finite dimensional distribution of $\eta$: for all $t_1, \ldots, t_k \in T$ and $y_1, \ldots, y_k > 0$

  \[ \mathbb{P}[\eta(t_i) \leq y_i, 1 \leq i \leq k] = \exp \left( - \mathbb{E} \left[ \bigvee_{i=1}^{k} \frac{Y(t_i)}{y_i} \right] \right). \]

- In particular for $k = 2$ and $y_1 = y_2 = y$

  \[ \mathbb{P}[\eta(t_1) \leq y, \eta(t_2) \leq y] = \exp \left( - \mathbb{E}[Y(t_1) \vee Y(t_2)]/y \right). \]
Pair extremal coefficient

- The quantity

\[ \theta(t_1, t_2) = \mathbb{E}[Y(t_1) \vee Y(t_2)] \in [1, 2] \]

is called the pair extremal coefficient.

- It gives some insight into the bivariate dependence structure:
  - \( \theta(t_1, t_2) = 2 \) iff \( \eta(t_1) \) and \( \eta(t_2) \) are independent.
  - \( \theta(t_1, t_2) = 1 \) iff \( \eta(t_1) \) and \( \eta(t_2) \) are equal a.s.
Functional Poisson Point Process

From $\eta = \bigvee_{i \geq 1} U_i Y_i$ we construct the $C_0$-valued point process

$$\Phi = \{ \phi_i, i \geq 1 \} \quad \text{where} \quad \phi_i = U_i Y_i \in C_0.$$ 

Then $\Phi$ is a Poisson Point Process with intensity

$$\mu(A) = \int_0^{\infty} P[uf \in A] u^{-2} du \quad A \subset C_0 \text{ Borel set}.$$ 

The measure $\mu$ is called the exponent measure and is homogeneous of prder $-1$:

$$\mu(uA) = u^{-1} \mu(A), \quad u > 0.$$
Example: Brown-Resnick processes

Theorem (de Haan, Kabluchko & Schlather ’10)

if \((W_i)_{i \geq 1}\) are i.i.d. copies of continuous stationary increments centered Gaussian processes on \(\mathbb{R}^d\) with variance \(\sigma^2(t)\), then

\[
\eta(t) = \bigvee_{i=1}^{\infty} \Gamma_i^{-1} e^{W_i(t) - \sigma^2(t)/2}, \quad t \in \mathbb{R}^d,
\]

is a stationary max-stable process. Its law depends only on the variogram

\[
\gamma(h) = \mathbb{E}[(W(t + h) - W(t))^2], \quad h \in \mathbb{R}^d.
\]

The pair extremal coefficient is easily computed:

\[
\theta(t_1, t_2) = 2 - \Phi(\sqrt{\gamma(t_2 - t_1)/2}).
\]
Example: Brown-Resnick processes

- "Historical" Brown-Resnick process: $\gamma(h) = |h|$. 

- Brown-Resnick processes directed by fractional Brownian motions, $\gamma(h) = |h|^{2H}$, $0 < H \leq 1$. 

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Ergodicity and mixing

Let \( X = (X(t))_{t \in \mathbb{Z}} \) be a stationary sequence and denote by \( \tau \) the shift operator:

\[
\tau^h X(\cdot) = X(\cdot + h), \quad h \in \mathbb{Z}.
\]

**Definition**

We say that \( X \) is mixing if for all \( A, B \subset \mathbb{R}^\mathbb{Z} \) Borel sets

\[
\mathbb{P}[X \in A, \tau^n X \in B] \to \mathbb{P}[X \in A]\mathbb{P}[X \in B] \quad \text{asn} \to +\infty.
\]

We say that \( X \) is ergodic if for all \( A, B \subset \mathbb{R}^\mathbb{Z} \) Borel

\[
\frac{1}{n} \sum_{h=1}^{n} \mathbb{P}[X \in A, \tau^h X \in B] \to \mathbb{P}[X \in A]\mathbb{P}[X \in B] \quad \text{asn} \to +\infty.
\]

Clearly, mixing implies ergodicity.
The ergodic theorem

Let \( X = (X(t))_{t \in \mathbb{Z}} \) be an ergodic stationary sequence. Consider \( F : \mathbb{R}^\mathbb{Z} \to \mathbb{R} \) such that \( F(X) \) is integrable. Then, as \( n \to +\infty \),

\[
\frac{1}{n} \sum_{h=1}^{n} F(\tau^h X) \to \mathbb{E}[F(X)] \quad \text{a.s. and in } L^1.
\]

Useful to prove consistency of estimators based on ergodic means.
Theorem (Stoev ’10, Kabluchko & Schlather ’10)

Let \( \eta = (\eta(t))_{t \in \mathbb{Z}} \) be a stationary simple max-stable random sequence. We note \( \theta(t_1, t_2) = \theta(t_2 - t_1) \).

- \( \eta \) is mixing if and only if
  \[
  \theta(n) \to 2 \quad \text{as } n \to +\infty.
  \]

- \( \eta \) is ergodic if and only if
  \[
  \frac{1}{n} \sum_{h=1}^{n} \theta(h) \to 2 \quad \text{as } n \to +\infty.
  \]
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α- and β-mixing

- Various ways to measure dependence (or rather distance to independance) : we consider here α-mixing (Rosenblatt '56) and β-mixing (Volkonskii et Rozanov '59) coefficients associated to a random process $X = (X(t))_{t \in T}$.

- For $S \subset T$, we denote by $X_S$ the restriction on $S$ of $X$ and $\mathcal{P}_{X_S}$ the law of $X_S$. For disjoint subsets $S_1, S_2 \subset T$ :

$$
\alpha_X(S_1, S_2) = \sup \left\{ \left| \mathbb{P}(X_{S_1} \in A_1, X_{S_2} \in A_2) - \mathbb{P}(X_{S_1} \in A_1) \mathbb{P}(X_{S_2} \in A_2) \right| ; A_i \subset \mathbb{R}^{S_i} \text{ Borel} \right\},
$$

$$
\beta_X(S_1, S_2) = \sup \left\{ \left| \mathbb{P}(x_{S_1}, x_{S_2})(C) - \mathbb{P}_{X_{S_1}} \otimes \mathbb{P}_{X_{S_2}}(C) \right| ; C \subset \mathbb{R}^{S_1} \times \mathbb{R}^{S_1} \text{ Borel} \right\}
\leq \| \mathbb{P}(x_{S_1}, x_{S_2}) - \mathbb{P}_{X_{S_1}} \otimes \mathbb{P}_{X_{S_2}} \|_{\text{var}}.
$$

It holds $\alpha_X(S_1, S_2) \leq \frac{1}{2} \beta_X(S_1, S_2)$. 

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A simple central limit theorem under $\alpha$-mixing

**Theorem (Ibragimov ’62)**

Let $X = (X(t))_{t \in \mathbb{Z}}$ be a stationary sequence and denote by

$$\alpha_n = \alpha_X((-\infty, 0], [n, +\infty)), \quad n \geq 1.$$

Assume that there is $\delta > 0$ such that

$$\mathbb{E}[X(0)^{2+\delta}] < \infty \quad \text{and} \quad \sum_{n \geq 1} \alpha_n^{2/(2+\delta)} < +\infty.$$

Then $S_n = \sum_{t=1}^{n} X(t)$ satisfies

$$n^{-1/2} \left( S_n - \mathbb{E}[S_n] \right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2)$$

with $\sigma^2 = \sum_{t \in \mathbb{Z}} \text{cov}(X(0), X(t))$. 
Consider a simple max-stable process $\eta = (\eta(t))_{t \in T}$ given by the representation

$$\eta(t) = \bigvee_{i \geq 1} U_i Y_i(t), \quad t \in T.$$ 

For disjoint subsets $S_1, S_2 \subset T$, can we get an estimate for $\beta_{\eta}(S_1, S_2)$?

We work on the level of the $C_0$-valued point process

$$\Phi = \{\phi_i, \ i \geq 1\}, \quad \phi_i = U_i Y_i \in C_0.$$
A decomposition of the point process

For any $S \subset T$, $\Phi = \Phi^+_S \cup \Phi^-_S$ with

$$\Phi^+_S = \{ \phi \in \Phi; \exists s \in S, \phi(s) = \eta(s) \}$$

$$\Phi^-_S = \{ \phi \in \Phi; \forall s \in S, \phi(s) < \eta(s) \}.$$

**Figure**: Realizations of the decomposition $\Phi = \Phi^+_S \cup \Phi^-_S$, with $S = [0, 5]$ (left) or $S = \{3\}$ (right).

Clearly for $s \in S$, $\eta(s) = \bigvee_{\phi \in \Phi^+_S} \phi(s)$ whence

$$\beta_{\eta}(S_1, S_2) \leq \beta(\Phi^+_S, \Phi^+_S).$$

Recall that $\Phi$ is a Poisson Point Process with intensity $\mu$. 

\[
\text{Figure}: \quad \text{Realizations of the decomposition } \Phi = \Phi^+_S \cup \Phi^-_S, \text{ with } S = [0, 5] \text{ (left) or } S = \{3\} \text{ (right).}
\]

Recall that $\Phi$ is a Poisson Point Process with intensity $\mu$. 

\[
\text{Figure}: \quad \text{Realizations of the decomposition } \Phi = \Phi^+_S \cup \Phi^-_S, \text{ with } S = [0, 5] \text{ (left) or } S = \{3\} \text{ (right).}
\]
A simple upper bound for $\beta(S_1, S_2)$

Theorem

The following upper bound holds true

$$\beta(\Phi_{S_1}^+, \Phi_{S_2}^+) \leq 2 \mathbb{P}[\Phi_{S_1}^+ \cap \Phi_{S_2}^+ \neq \emptyset].$$

- The result holds also for max-infinitely divisible processes.
- Elements for the proof:
  - a coupling lemma for the estimate of $\beta$-coefficient;
  - a result on conditional distribution for the law of $\Phi_S^-$ given $\Phi_S^+$;
  - Palm calculus for Poisson point processes (e.g. Slyvniak formula).
A simple upper bound for $\beta(S_1, S_2)$

**Theorem**

$$\beta_\eta(S_1, S_2) \leq 2 \left[ C(S_1) + C(S_2) \right] \left[ \theta(S_1) + \theta(S_2) - \theta(S_1 \cup S_2) \right]$$

with $C(S) = \mathbb{E}\left[ \sup_S \eta^{-1} \right]$ and $\theta(S) = -\log \mathbb{P}\left[ \sup_S \eta \leq 1 \right]$ the areal coefficient (Coles & Tawn ’96).

**Corollary**

If $S_1$ and $S_2$ are finite or countable (e.g. in the discrete case $T = \mathbb{Z}^d$),

$$\beta_\eta(S_1, S_2) \leq 4 \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} \left[ 2 - \theta(s_1, s_2) \right].$$
A CLT for max-i.d. random fields

Theorem

Let $\eta$ be stationary simple max-stable on $T = \mathbb{Z}^d$ and define

$$X(h) = g(\eta(t_1 + h), \ldots, \eta(t_p + h)), \quad h \in \mathbb{Z}^d.$$ 

Assume that there is $\delta > 0$ such that $\mathbb{E}[X(0)^{2+\delta}] < \infty$ and

$$\theta(h) = 2 - o(\|h\|^{-b}) \quad \text{for some } b > d \max\left(2, \frac{2+\delta}{\delta}\right).$$

Then $S_n = \sum_{\|h\| \leq n} X(h)$ satisfies the central limit theorem:

$$c_n^{-1/2} \left( S_n - \mathbb{E}[S_n] \right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$$

with $c_n = \text{card}\{\|h\| \leq n\}$ and $\sigma^2 = \sum_{h \in \mathbb{Z}^d} \text{cov}(X(0), X(h))$. 
An application

- Estimation of the pair extremal coefficient $\theta(h_0)$ of a stationary simple max-stable process $\eta$ on $\mathbb{Z}^d$

Recall that

$$\mathbb{P}[\eta(t) \leq y, \eta(t + h_0) \leq y] = \exp(-\theta(h_0)/y), \quad y > 0,$$

whence we deduce the naive estimator

$$\hat{\theta}_n^{(1)}(h_0) = -y \log \left( c_n^{-1} \sum_{\|h\| \leq n} 1\{\eta(h) \leq y, \eta(h+h_0) \leq y\} \right).$$

where $c_n = \text{card}\{\|h\| \leq n\}$. 

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An application

- To avoid the arbitrary choice of the arbitrary truncation level $y > 0$, Smith ('90) uses

$$\mathbb{E} \left[ \min \left( \frac{1}{\eta(0)}, \frac{1}{\eta(h_0)} \right) \right] = \frac{1}{\theta(h_0)}$$

and suggests the estimator

$$\hat{\theta}_n^{(2)}(h_0) = \left( c_n^{-1} \sum_{\|h\| \leq n} \min \left( \frac{1}{\eta(h)}, \frac{1}{\eta(h+h_0)} \right) \right)^{-1}.$$ 

- Cooley, Naveau & Poncet ('06) suggest the use of the $F$-madogram

$$\mathbb{E}[|F(\eta(0)) - F(\eta(h_0))|] = \frac{1}{2} \frac{\theta(h_0) - 1}{\theta(h_0) + 1}$$

with $F(y) = \exp(-1/y)$ and the alternative estimator

$$\hat{\theta}_n^{(3)}(h_0) = \frac{1 + 2c_n^{-1} \sum_{\|h\| \leq n} |F(\eta(h)) - F(\eta(h + h_0))|}{1 - 2c_n^{-1} \sum_{\|h\| \leq n} |F(\eta(h)) - F(\eta(h + h_0))|}.$$
An application

**Proposition**

Assume that $\theta(h) = 2 - o(\|h\|^{-b})$ for some $b > 2d$.

Then, the estimators $\hat{\theta}_n^{(i)}(h_0)$ ($i = 1, 2, 3$) are asymptotically normal:

$$c_n^{-1/2} \left( \hat{\theta}_n^{(i)}(h_0) - \theta(h_0) \right) \implies \mathcal{N}(0, \sigma_i^2).$$

with variance

$$\sigma_1^2 = \left( \exp[(2\theta(h_0) - \theta(\{0, h_0, t, t + h_0\}))y^{-1} - 1),
\sigma_2^2 = \theta(h_0)^4 \sum_{t \in \mathbb{Z}^d} \text{Cov} \left[ \min(\eta(0)^{-1}, \eta(h_0)^{-1}), \min(\eta(t)^{-1}, \eta(t + h_0)^{-1}) \right],
\sigma_3^2 = (\theta(h_0) + 1)^4 \sum_{t \in \mathbb{Z}^d} \text{Cov} \left[ |F(\eta(0)) - F(\eta(h_0))|, |F(\eta(t)) - F(\eta(t + h_0))| \right].$$

Proof: apply the above CLT and Cramer's delta-method.
Thank you for your attention!