Strong mixing properties of max-stable random fields

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Workshop "Copulas and Extremes"
Grenoble, November, 19-20th, 2013

Structure of the talk

1. Motivations
2. Max-stable processes
3. Ergodic and mixing properties
4. Strong mixing properties

Motivations

- Needs for modeling extremes in environmental sciences :
  - maximal temperatures in a heat wave,
  - intensity of winds during a storm,
  - water heights in a flood ...
- Spatial extreme value theory - geostatistics of extremes :
  - ▷ de Haan & Pereira 2006, Spatial extremes : Models for the stationary case.
  - ▷ Davison, Ribatet & Padoan 2011, Statistical modelling of spatial extremes.
- Max-stable random fields play a crucial role.
- Many developpments in the last decades regarding max-stable processes :
  - theoretical properties,
  - statistics and inference,
  - modelling and applications.

Motivations

- Here we want to consider statistics based not on i.i.d. observations but rather on observations of a stationary weakly dependent max-stable process.
- Recent results for ergodic and mixing properties of stationary max-stable and max-i.d. processes :

  ▷ Stoev ('10) Max-stable processes : representations, ergodic properties and statistical applications.

  ▷ Kabluchko & Schlather ('10) Ergodic properties of max-infinitely divisible processes.
- Ergodicity and mixing are important to derive strong law of large numbers and hence consistency of estimators.

Motivations

- Ergodicity and mixing are not enough to get central limit theorems or asymptotic normality of estimators.
- Central limit theorems for stationary processes are available under various weak dependence assumptions. We consider here strong mixing assumptions (e.g. α-mixing or β-mixing).
- Can we derive some estimates for the mixing coefficients of a max-stable process ?

Structure of the talk

Definition and first properties

Let T be a locally compact parameter space.
Let $\eta = (\eta(t))_{t\in T}$ be a sample continuous process.

Definition

- η is max-stable if for any $n \geq 1$, there are continuous functions $a_n(\cdot) > 0$ and $b_n(\cdot)$ such that

$$\left(\frac{\bigvee_{i=1}^{n} \eta_i(t) - b_n(t)}{a_n(t)}\right)_{t\in T} \stackrel{\mathcal{L}}{=} (\eta(t))_{t\in T}$$

  with $\eta_1, \ldots, \eta_n$ i.i.d. copies of η.

Definition and first properties

- Max-stable processes arrise as limit of maxima of i.i.d. processes : if $(X_i)_{i\geq 1}$ are i.i.d. random processes on T such that there exists normalization functions $a_n(\cdot) > 0$ and $b_n(\cdot)$

$$\left(\frac{\bigvee_{i=1}^n X_i(t) - b_n(t)}{a_n(t)}\right)_{t\in T} \Longrightarrow \big(\eta(t)\big)_{t\in T},$$

  then η must is max-stable.
- For each $t \in T$, $\eta(t)$ has a GEV distribution.
- We say that η is simple max-stable if the margins are 1-Fréchet :

$$\mathbb{P}[\eta(t) \leq y] = \exp(-1/y).$$

Structure of max-stable processes

Theorem (de Haan '84, Penrose '92)

Let η be a simple max-stable process with continuous sample path. Then η can be represented as

$$\Big(\eta(t)\Big)_{t\in T} \stackrel{\mathcal{L}}{=} \Big(\bigvee_{i\geq 1} U_i Y_i(t)\Big)_{t\in T}$$

where

- $(Y_i)_{i\geq 1}$ are i.i.d. copies of a random process Y with path in $\mathcal{C}_0 = \mathcal{C}(T,[0,+\infty)) \setminus \{0\}$ and such that
$$\mathbb{E}[Y(t)] = 1, \quad t \in T$$
$$\mathbb{E}[\sup_{t\in K} Y(t)] < \infty, \quad K \text{ compact};$$
- $\{U_i, i \geq 1\}$ is a Poisson Point Process on $(0,+\infty)$ with intensity $u^{-2}du$;
- $(Y_i)_{i\geq 1}$ and $\{U_i, i \geq 1\}$ are independent.

Finite dimensional distributions

- Finite dimensional distribution of η : for all $t_1, \dots, t_k \in T$ and $y_1, \dots, y_k > 0$

$$\mathbb{P}[\eta(t_i) \le y_i, 1 \le i \le k] = \exp\Big(- \mathbb{E}\big[\bigvee_{i=1}^{k} \frac{Y(t_i)}{y_i} \big] \Big).$$

- In particular for $k = 2$ and $y_1 = y_2 = y$

$$\mathbb{P}[\eta(t_1) \le y, \eta(t_2) \le y] = \exp\Big(- \mathbb{E}[Y(t_1) \vee Y(t_2)]/y \Big)$$

Pair extremal coefficient

- The quantity

$$\theta(t_1, t_2) = \mathbb{E}[Y(t_1) \vee Y(t_2)] \in [1, 2]$$

  is called the pair extremal coefficient.
- It gives some insight into the bivariate dependence structure :
  - $\theta(t_1, t_2) = 2$ iff $\eta(t_1)$ and $\eta(t_2)$ are independent.
  - $\theta(t_1, t_2) = 1$ iff $\eta(t_1)$ and $\eta(t_2)$ are equal a.s.

Functional Poisson Point Process

- From $\eta = \bigvee_{i \geq 1} U_i Y_i$ we construct the $\mathcal{C}_0$-valued point process

$$\Phi = \{\phi_i, i \geq 1\} \quad \text{where } \phi_i = U_i Y_i \in \mathcal{C}_0.$$

- Then Φ is a Poisson Point Process with intensity

$$\mu(A) = \int_0^\infty \mathbb{P}[uf \in A] u^{-2} du \quad A \subset \mathcal{C}_0 \text{ Borel set.}$$

- The measure μ is called the exponent measure and is homogeneous of prder -1 :

$$\mu(uA) = u^{-1} \mu(A), \quad u > 0.$$

Example : Brown-Resnick processes

Theorem (de Haan, Kabluchko & Schlather '10)

if $(W_i)_{i\geq 1}$ are i.i.d. copies of continuous stationary increments centered Gaussian processes on $\mathbb{R}^d$ with variance $\sigma^2(t)$, then

$$\eta(t) \;=\; \bigvee_{i=1}^{\infty} \Gamma_i^{-1} e^{W_i(t)-\sigma^2(t)/2}, \quad t \in \mathbb{R}^d,$$

is a stationary max-stable process. Its law depends only on the variogram

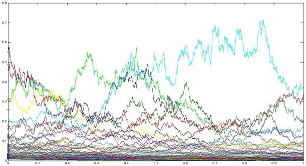
$$\gamma(h) = \mathbb{E}[(W(t+h) - W(t))^2], \quad h \in \mathbb{R}^d.$$

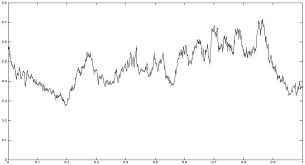
The pair extremal coefficient is easily computed :

$$\theta(t_1, t_2) = 2 - \bar{\Phi}(\sqrt{\gamma(t_2 - t_1)/2}).$$

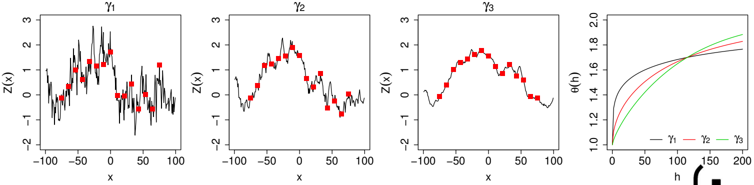
Example : Brown-Resnick processes

- "Historical" Brown-Resnick process : $\gamma(h) = |h|$.





- Brown-Resnick processes directed by fractional Brownian motions, $\gamma(h) = |h|^{2H}$, $0 < H \leq 1$.



Structure of the talk

Ergodicity and mixing

Let $X = (X(t))_{t \in \mathbb{Z}}$ be a stationary sequence and denote by τ the shift operator :

$$\tau^h X(\cdot) = X(\cdot + h), \quad h \in \mathbb{Z}.$$

Definition

We say that X is mixing if for all $A, B \subset \mathbb{R}^{\mathbb{Z}}$ Borel sets

$$\mathbb{P}[X \in A,\ \tau^n X \in B] \to \mathbb{P}[X \in A]\mathbb{P}[X \in B] \quad as n \to +\infty.$$

We say that X is ergodic if for all $A, B \subset \mathbb{R}^{\mathbb{Z}}$ Borel

$$\frac{1}{n}\sum_{h=1}^{n} \mathbb{P}[X \in A,\ \tau^h X \in B] \to \mathbb{P}[X \in A]\mathbb{P}[X \in B] \quad as n \to +\infty.$$

Clearly, mixing implies ergodicity.

The ergodic theorem

The ergodic theorem

Let $X = (X(t))_{t\in\mathbb{Z}}$ be an ergodic stationary sequence.
Consider $F : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}$ such that $F(X)$ is integrable.
Then, as $n \to +\infty$,

$$\frac{1}{n}\sum_{h=1}^{n} F(\tau^h X) \to \mathbb{E}[F(X)] \quad \text{a.s. and in } L^1.$$

Useful to prove consistency of estimators based on ergodic means.

Mixing and ergodicity of max-stable random fields

Theorem (Stoev '10, Kabluchko & Schlather '10)

Let $\eta = (\eta(t))_{t\in\mathbb{Z}}$ be a stationary simple max-stable random sequence. We note $\theta(t_1, t_2) = \theta(t_2 - t_1)$.

- η is mixing if and only if

$$\theta(n) \to 2 \quad \textit{as } n \to +\infty.$$

- η is ergodic if and only if

$$\frac{1}{n}\sum_{h=1}^{n}\theta(h) \to 2 \quad \textit{as } n \to +\infty.$$

Structure of the talk

α- and β-mixing

- Various ways to measure dependence (or rather distance to independance) : we consider here α-mixing (Rosenblatt '56) and β-mixing (Volkonskii et Rozanov '59) coefficients associated to a random process $X = (X(t))_{t\in T}$.
- For $S \subset T$, we denote by X_S the restriction on S of X and $\mathcal{P}_{X_S}$ the law of X_S. For disjoint subsets $S_1, S_2 \subset T$:

$$\alpha_X(S_1,S_2) = \sup\left\{\left|\mathbb{P}(X_{S_1}\in A_1, X_{S_2}\in A_2) - \mathbb{P}(X_{S_1}\in A_1)\mathbb{P}(X_{S_2}\in A_2)\right|;\ A_i \subset \mathbb{R}^{S_i} \text{ Borel}\right\},$$

$$\begin{aligned}\beta_X(S_1,S_2) &= \sup\left\{\left|\mathcal{P}_{(X_{S_1},X_{S_2})}(C) - \mathcal{P}_{X_{S_1}}\otimes\mathcal{P}_{X_{S_2}}(C)\right|;\ C\subset \mathbb{R}^{S_1}\times\mathbb{R}^{S_1} \text{ Borel}\right\}\\ &= \|\mathcal{P}_{(X_{S_1},X_{S_2})} - \mathcal{P}_{X_{S_1}}\otimes\mathcal{P}_{X_{S_2}}\|_{var}.\end{aligned}$$

It holds $\alpha_X(S_1,S_2) \le \frac{1}{2}\beta_X(S_1,S_2)$.

A simple central limit theorem under α-mixing

Theorem (Ibragimov '62)

Let $X = (X(t))_{t\in\mathbb{Z}}$ be a stationary sequence and denote by

$$\alpha_n = \alpha_X((-\infty, 0]\!], [\![n, +\infty)), \quad n \geq 1.$$

Assume that there is $\delta > 0$ such that

$$\mathbb{E}[X(0)^{2+\delta}] < \infty \quad \text{and} \quad \sum_{n\geq 1} \alpha_n^{2/(2+\delta)} < +\infty.$$

Then $S_n = \sum_{t=1}^{n} X(t)$ satisfies

$$n^{-1/2}\Big(S_n - \mathbb{E}[S_n]\Big) \Longrightarrow \mathcal{N}(0, \sigma^2)$$

with $\sigma^2 = \sum_{t\in\mathbb{Z}} \mathrm{cov}(X(0), X(t))$.

β-mixing for max-stable processes

- Consider a simple max-stable process $\eta = (\eta(t))_{t\in T}$ given by the representation
$$\eta(t) = \bigvee_{i\geq 1} U_i Y_i(t), \quad t \in T.$$
- For disjoint subsets $S_1, S_2 \subset T$, can we get an estimate for $\beta_\eta(S_1, S_2)$?
- We work on the level of the $\mathcal{C}_0$-valued point process
$$\Phi = \{\phi_i,\ i \geq 1\}, \quad \phi_i = U_i Y_i \in \mathcal{C}_0.$$

A decomposition of the point process

- For any $S \subset T$, $\Phi = \Phi_S^+ \cup \Phi_S^-$ with

$$\begin{aligned} \Phi_S^+ &= \{\phi \in \Phi;\ \exists s \in S,\ \phi(s) = \eta(s)\} \\ \Phi_S^- &= \{\phi \in \Phi;\ \forall s \in S,\ \phi(s) < \eta(s)\}. \end{aligned}$$

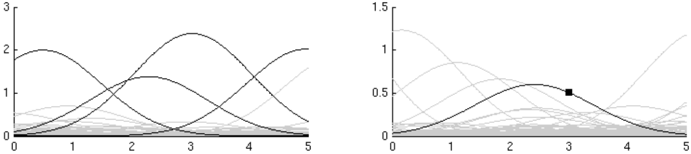


FIGURE: Realizations of the decomposition $\Phi = \Phi_S^+ \cup \Phi_S^-$, with $S = [0, 5]$ (left) or $S = \{3\}$ (right).

- Clearly for $s \in S$, $\eta(s) = \bigvee_{\phi \in \Phi_S^+} \phi(s)$ whence

$$\beta_\eta(S_1, S_2) \leq \beta(\Phi_{S_1}^+, \Phi_{S_2}^+).$$

Recall that Φ is a Poisson Point Process with intensity μ.

A simple upper bound for $\beta(S_1, S_2)$

Theorem

The following upper bound holds true

$$\beta(\Phi^+_{S_1}, \Phi^+_{S_2}) \leq 2\,\mathbb{P}[\Phi^+_{S_1} \cap \Phi^+_{S_2} \neq \emptyset].$$

- The result holds also for max-infinitely divisible processes.
- Elements for the proof :
  - a coupling lemma for the estimate of β-coefficient ;
  - a result on conditional distribution for the law of Φ^-_S given Φ^+_S ;
  - Palm calculus for Poisson point processes (e.g. Slyvniak formula).

A simple upper bound for $\beta(S_1, S_2)$

Theorem

$$\beta_\eta(S_1, S_2) \leq 2\,[C(S_1) + C(S_2)]\,[\theta(S_1) + \theta(S_2) - \theta(S_1 \cup S_2)]$$

with $C(S) = \mathbb{E}\big[\sup_S \eta^{-1}\big]$ and $\theta(S) = -\log \mathbb{P}\big[\sup_S \eta \leq 1\big]$ the areal coefficient (Coles & Tawn '96).

Corollary

If S_1 and S_2 are finite or countable (e.g. in the discrete case $T = \mathbb{Z}^d$),

$$\beta_\eta(S_1, S_2) \leq 4 \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} [2 - \theta(s_1, s_2)].$$

A CLT for max-i.d. random fields

Theorem

Let η be stationary simple max-stable on $T = \mathbb{Z}^d$ and define

$$X(h) = g(\eta(t_1 + h), \dots, \eta(t_p + h)), \quad h \in \mathbb{Z}^d.$$

Assume that there is $\delta > 0$ such that $\mathbb{E}[X(0)^{2+\delta}] < \infty$ and

$$\theta(h) = 2 - o(\|h\|^{-b}) \quad \text{for some } b > d \max\left(2, \frac{2+\delta}{\delta}\right).$$

Then $S_n = \sum_{\|h\| \leq n} X(h)$ satisfies the central limit theorem :

$$c_n^{-1/2}\left(S_n - \mathbb{E}[S_n]\right) \Longrightarrow \mathcal{N}(0, \sigma^2)$$

with $c_n = \mathrm{card}\{\|h\| \leq n\}$ and $\sigma^2 = \sum_{h \in \mathbb{Z}^d} \mathrm{cov}(X(0), X(h))$.

An application

- Estimation of the pair extremal coefficient $\theta(h_0)$ of a stationary simple max-stable process η on $\mathbb{Z}^d$
- Recall that

$$\mathbb{P}[\eta(t) \le y, \eta(t+h_0) \le y] = \exp(-\theta(h_0)/y), \quad y > 0,$$

whence we deduce the naive estimator

$$\hat{\theta}_n^{(1)}(h_0) = -y \log \left(c_n^{-1} \sum_{\|h\| \le n} 1_{\{\eta(h) \le y,\ \eta(h+h_0) \le y\}} \right).$$

where $c_n = \mathrm{card}\{\|h\| \le n\}$.

An application

- To avoid the arbitrary choice of the arbitrary truncation level $y > 0$, Smith ('90) uses

$$\mathbb{E}\Big[\min\Big(\frac{1}{\eta(0)}, \frac{1}{\eta(h_0)}\Big)\Big] = \frac{1}{\theta(h_0)}$$

  and suggests the estimator

$$\hat{\theta}_n^{(2)}(h_0) = \Big(c_n^{-1}\sum_{\|h\|\le n} \min\Big(\tfrac{1}{\eta(h)}, \tfrac{1}{\eta(h+h_0)}\Big)\Big)^{-1}.$$

- Cooley, Naveau & Poncet ('06) suggest the use of the F-madogram

$$\mathbb{E}[|F(\eta(0)) - F(\eta(h_0))|] = \frac{1}{2}\frac{\theta(h_0)-1}{\theta(h_0)+1}$$

  with $F(y) = exp(-1/y)$ and the alternative estimator

$$\hat{\theta}_n^{(3)}(h_0) = \frac{1 + 2c_n^{-1}\sum_{\|h\|\le n}|F(\eta(h)) - F(\eta(h+h_0))|}{1 - 2c_n^{-1}\sum_{\|h\|\le n}|F(\eta(h)) - F(\eta(h+h_0))|}$$

An application

Proposition

Assume that $\theta(h) = 2 - o(\|h\|^{-b})$ for some $b > 2d$.
Then, the estimators $\hat{\theta}_n^{(i)}(h_0)$ $(i = 1, 2, 3)$ are asymptotically normal :

$$c_n^{-1/2}\left(\hat{\theta}_n^{(i)}(h_0) - \theta(h_0)\right) \Longrightarrow \mathcal{N}(0, \sigma_i^2).$$

with variance

$$\begin{aligned}
\sigma_1^2 &= y^2 \sum_{t\in\mathbb{Z}^d} \left(\exp[(2\theta(h_0) - \theta(\{0, h_0, t, t+h_0\}))y^{-1}] - 1\right), \\
\sigma_2^2 &= \theta(h_0)^4 \sum_{t\in\mathbb{Z}^d} \mathrm{Cov}\left[\min(\eta(0)^{-1}, \eta(h_0)^{-1}), \min(\eta(t)^{-1}, \eta(t+h_0)^{-1})\right], \\
\sigma_3^2 &= (\theta(h_0)+1)^4 \sum_{t\in\mathbb{Z}^d} \mathrm{Cov}\left[|F(\eta(0)) - F(\eta(h_0))|, |F(\eta(t)) - F(\eta(t+h_0))|\right].
\end{aligned}$$

Proof : apply the above CLT and Cramer's delta-method.

Thank you for your attention !