# Estimating metrics suitable to an empirical manifold of shapes, using transport against the curse of dimensionality

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Pulsar Project

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IHP

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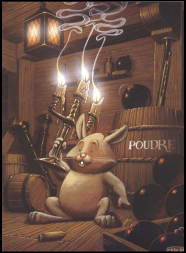
# Map

- **Introduction**
  - Motivation
  - Issues
- **Searching for solutions**
  - Main existing approaches and their limitations
  - Main idea
- **The approach**
  - Shape matching
  - Transport
  - Metric estimation (statistics on deformations)
  - Theory
- **Future work**

# Introduction

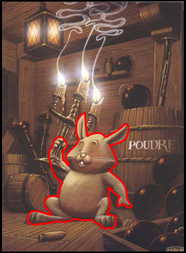
## Image Segmentation

- Find a contour in a given image
- The best curve for a given segmentation criterion
- Criterion based on color homogeneity, texture, edge detectors, etc.




Image

$\longrightarrow$




Segmentation

# Introduction

## Image Segmentation

- Find the best contour for a given criterion

## Variational Method

- Energy $E$ to minimize with respect to a curve $C$
- Compute the derivative of the energy
- Gradient descent: $\partial_t C = -\nabla E(C)$
- Initialization $\longrightarrow$ local minimum
- Other methods: graph cuts (suitable for few energies)

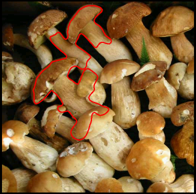
# Introduction

## Image Segmentation

- Find the best contour for a given criterion

## Variational Method

- Minimize criterion by gradient descent with respect to the contour
- Most criteria: no shape information



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- Find the best contour for a given criterion

## Variational Method

- Minimize criterion by gradient descent with respect to the contour

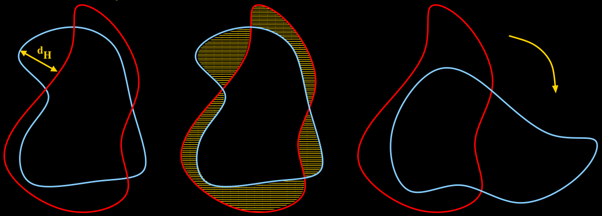
## Shape Statistics

- Sample set of contours from already segmented images
- Shape variability ?
- Shape prior ?

# Introduction

## Motivation

- Shape spaces : which metric ?
  (to define similarity/distance between shapes)
  - Hausdorff distance
  - Symmetric difference area
  - Quotients by transformation groups (rotation, translation, scaling, affine...)



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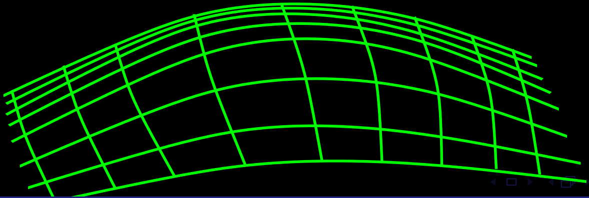

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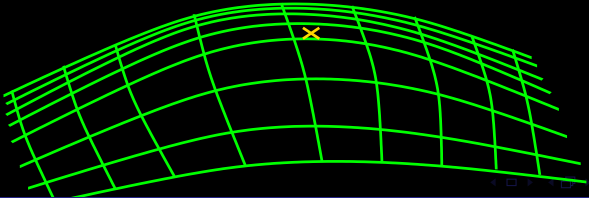

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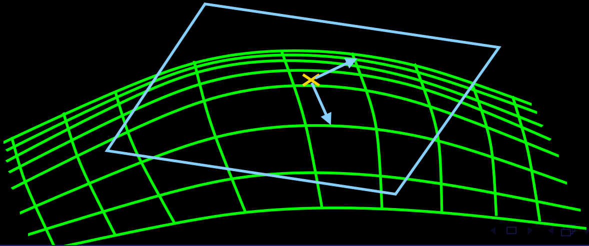

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  - rigid motion more probable $\Longrightarrow$ associated metric

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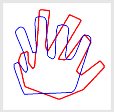
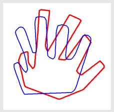
$L^2$ inner product  vs.  rigidifying inner product

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  - rigid motion more probable $\Longrightarrow$ associated metric
- $\Longrightarrow$ learn the suitable metric from examples (datasets of shapes)

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  - e.g. human silhouettes
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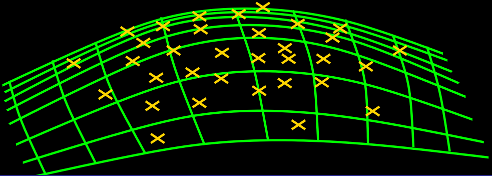

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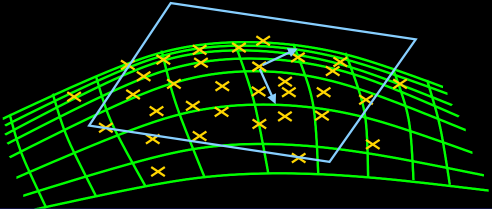

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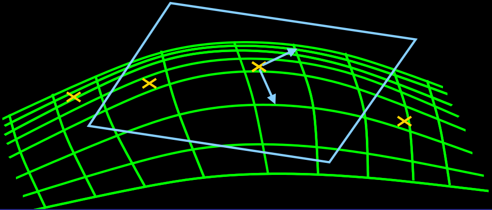

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- to compare quantities defined on different shapes :
  need for correspondences
  - match shape with different topologies ?
  - very frequent topological changes

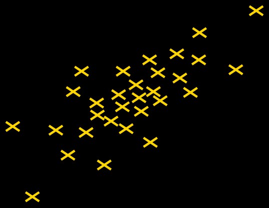
# Searching for solutions

## Main existing approaches and their limitations

Approach 1 : *mean* + *modes* model
Approach 2 : distance-based approaches, such as kernel methods

## Mean + modes model : PCA in tangent space (Gaussian distribution)

- Mean $M$, shapes $S_i$, warpings $W_{M \to S_i}$



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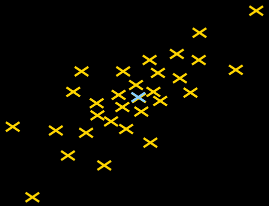

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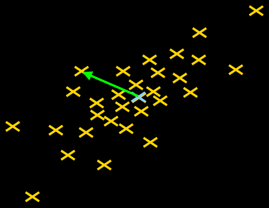

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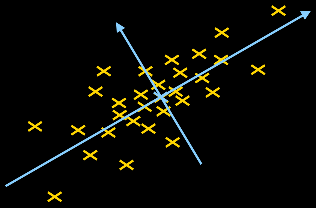
  $\implies$ eigenmodes $e_k$ with eigenvalues $\lambda_k$ : best coordinate system



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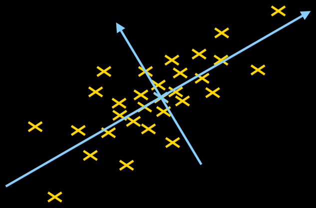
- any new deformation $W$ of $M$ :
  $W = \sum_k \alpha_k e_k +$ noise



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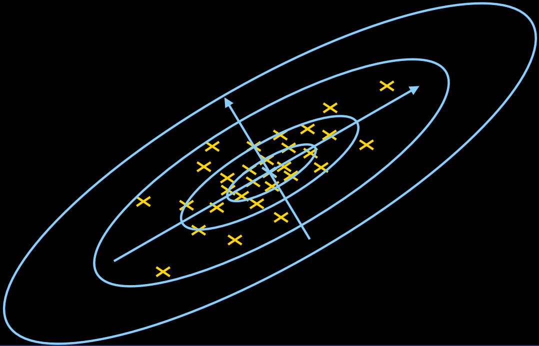

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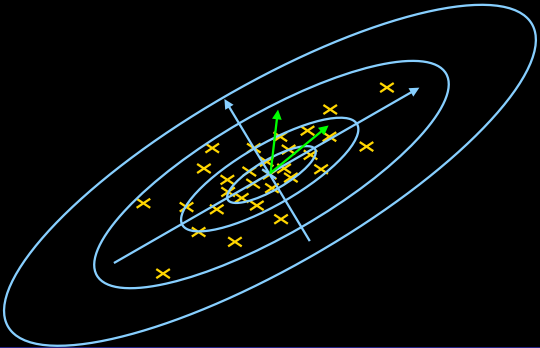
- associated inner product on deformations, in the tangent space of $M$:
  $\langle W_1 | W_2 \rangle = \sum_k \frac{1}{\lambda_k^2} \alpha_{1,k} \alpha_{2,k}$



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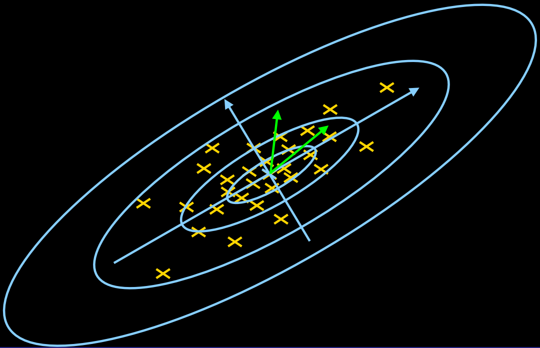
- defines a deformation cost $\|W\|^2 = \langle W | W \rangle$



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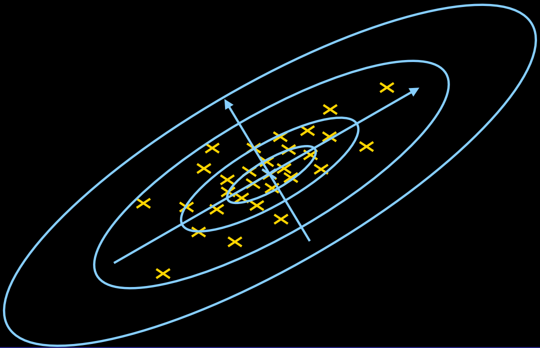

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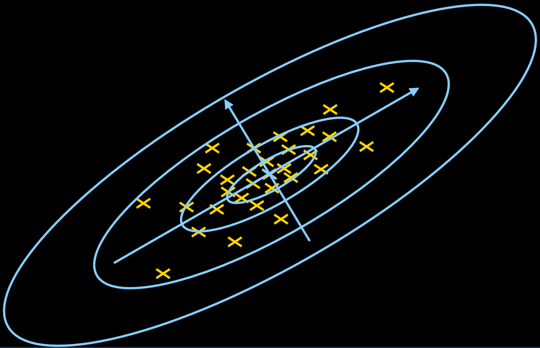
- defines a Gaussian shape prior



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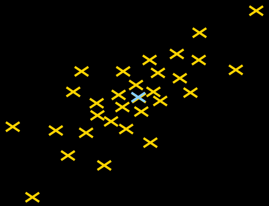
- Empirical distribution : $\mathcal{D}_{emp} = \sum_i \delta_{W_{M \to S_i}}$
  (possibly smoothed by a kernel)



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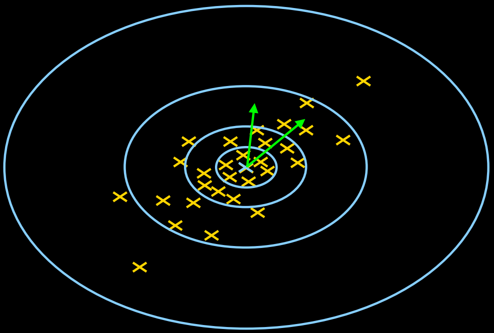
  $\Longrightarrow$ Gaussian distribution $\mathcal{D}_P(W) \propto \exp(-\|W\|_P^2)$



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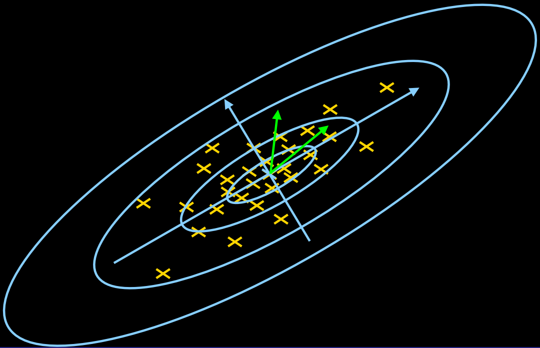
- Best $P$ for Kullback-Leibler$(\mathcal{D}_P|\mathcal{D}_{emp})$ : PCA!



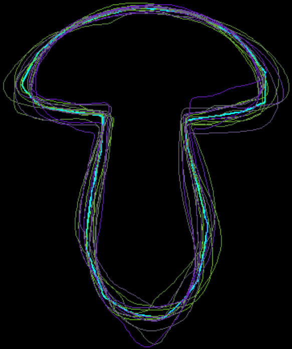
# Approach 1 : mean + modes model

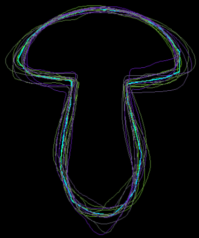
$\hookrightarrow$ example from my PhD thesis


Automatic alignment

$\longrightarrow$

and average shape computation

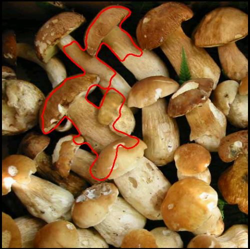




Statistics (PCA) on deformation fields
$\longrightarrow$
between the mean shape and each sample

modes of deformation
= deformation prior
= Gaussian probabilistic model

Example of application : image segmentation with shape prior



without shape prior

with shape prior

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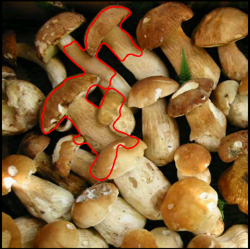

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- requires a mean shape (does not always make sense, e.g. person walking)

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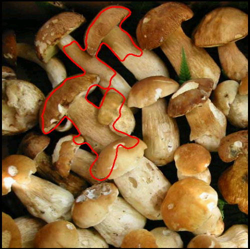
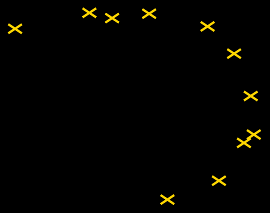

without shape prior


with shape prior

- requires a mean shape (does not always make sense, e.g. person walking)
- requires all deformations between the mean and samples :
  $\Longrightarrow$ relatively similar sample shapes (otherwise, not reliable)

# Approach 2 : distance-based methods (e.g. kernel methods)



- Kernel : symmetric definite positive function $k(x, y)$
- Expresses the similarity between $x$ and $y$
- Typically, the Gaussian kernel : $k(x, y) = exp(-d(x, y)^2)$
- For each point $x_i$ : $k_i(y) := k(x_i, y)$

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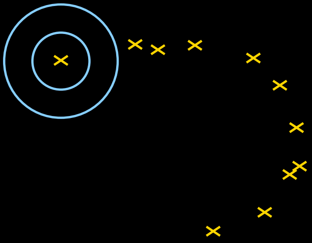

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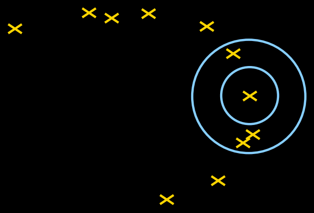

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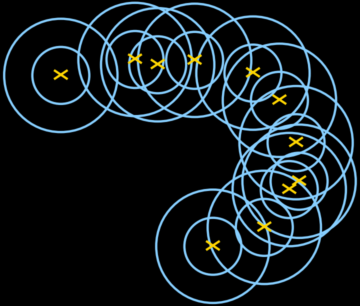

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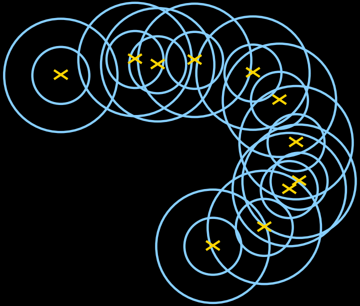

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- distance between 2 shapes : not much informative (wrt deformations)
- rebuild geometry of space of shapes from distances ?
- distances are not reliable/meaningful for far shapes
- $\Longrightarrow$ needs for a representative neighborhood, i.e. a high dataset density (not affordable)

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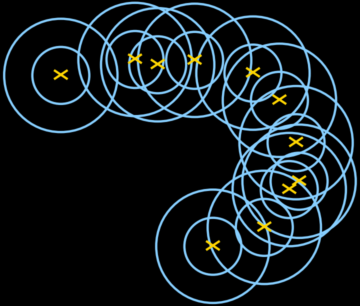

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- distances are not reliable/meaningful for far shapes
- $\Longrightarrow$ needs for a representative neighborhood, i.e. a high dataset density
- in a high-dimensional manifold, all distances are similar, and all points are on the boundary of the manifold
- $\Longrightarrow$ cannot work, need for more information than distances

# Main idea

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- should not require high density of training set
- no magic (to handle/interpolate sparse sets) : add a prior

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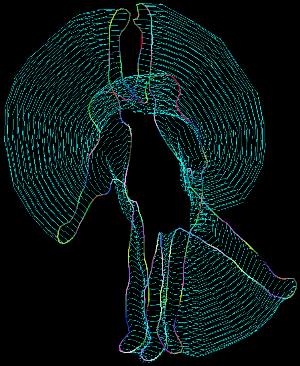

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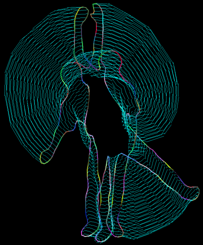

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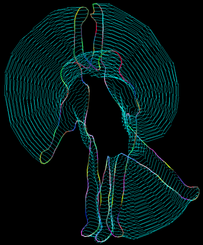

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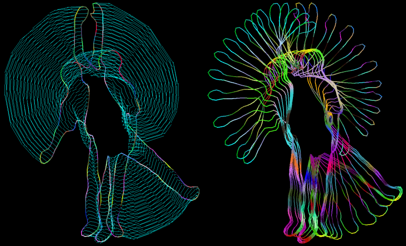


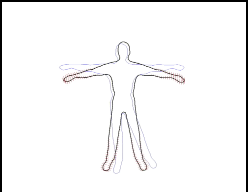
- transport requires correspondences
- but shape matching reliable only for close shapes
- $\Longrightarrow$ compute correspondences between close shapes only, and combine small steps of reliable correspondences to build longer-distance correspondences

# Map

- Close shape matching
- Transport
- Metric estimation (statistics on transported deformations)
- Theoretical justifications

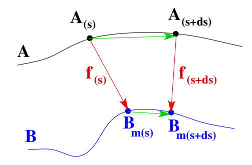
# Shape matching

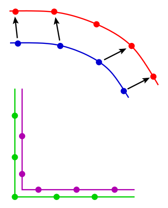
## Simple case : two shapes, A and B, with one connected component



$$\inf_{f:A\to B} \int_A \|f\|^2 + \alpha \|\nabla f\|^2 dA$$

- shape sampling
- dynamic time warping
- theory & experiments :
  higher sampling rate on target



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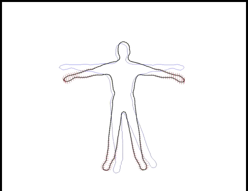

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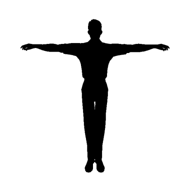
## Usual case : random topologies



Usual cases = more complex
(more than 10 connected components in this silhouette)
but
one connected component $\to \bigcup_i$ connected components
= the same

## Further possible improvements

- as such, allows appearing points (mismatches)
- allows disappearing points : matching to $\emptyset$ with a fixed high cost
- pb : better matchings, but energy value loses meaning

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### Drawbacks

- specific to planar curves
- not symmetric : $m_{A \to B} \neq m^{-1}_{B \to A}$

# Transport

## Local transport

- Set of shapes $(S_i)_{i \in I}$ (e.g. from a video segmentation)

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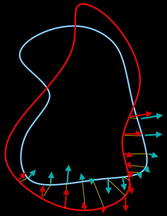
- Transport (translation, naive) :

$$\begin{aligned} \forall\, h : S_j \to \mathcal{X}, \quad T^L_{j\to i}(h) : \quad S_i \quad &\to \quad \mathcal{X} \\ \left(T^L_{j\to i}(h)\right)(s) \quad &= \quad h\left(m_{i\to j}(s)\right) \end{aligned}$$



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## Global transport

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- close shapes : reliable; distant shapes : not reliable
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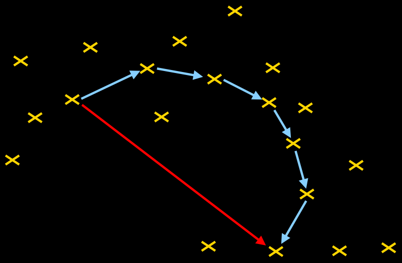
- graph : nodes = shapes, edges = transport, weights = transport cost
- shortest path between pairs of shapes : global transport



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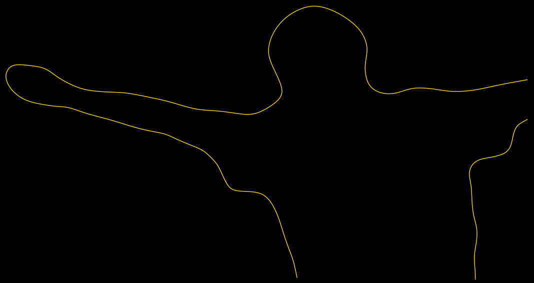
- compose : $T^G_{i\to j} = T^L_{i_n\to j}\ o\ T^L_{i_{n-1}\to i_n}\ o\ \dots\ o\ T^L_{i_1\to i_2}\ o\ T^L_{i\to i_1}$
- reliability : $w^G_{i\to j} = \prod_k w^L_{i_k\to i_{k+1}}$
- use transport to propagate information

Example : colored walker

## Global transport



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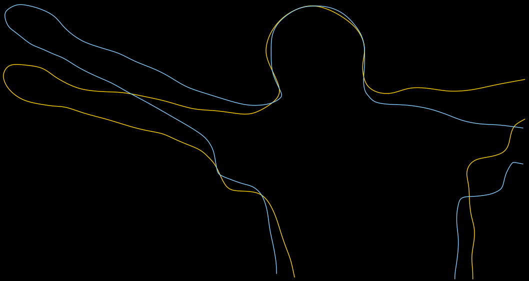

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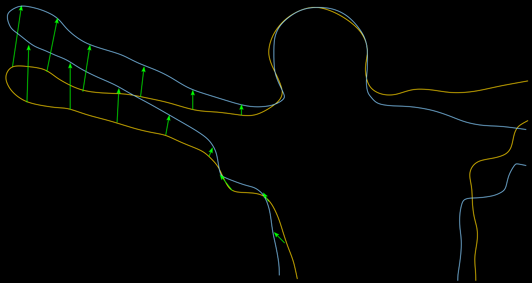
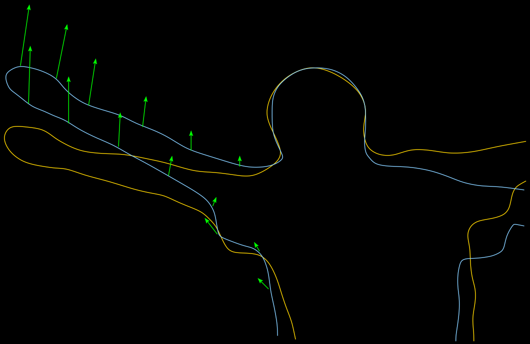


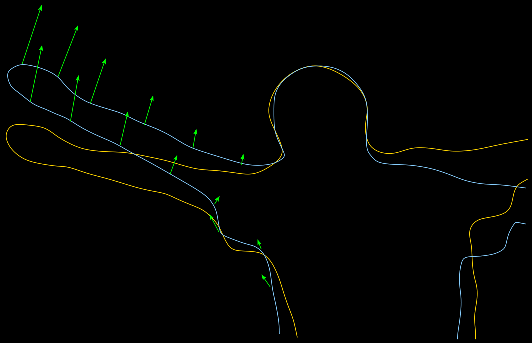
Correspondence field

## Global transport



Transported arm rotation (translation)

## Global transport

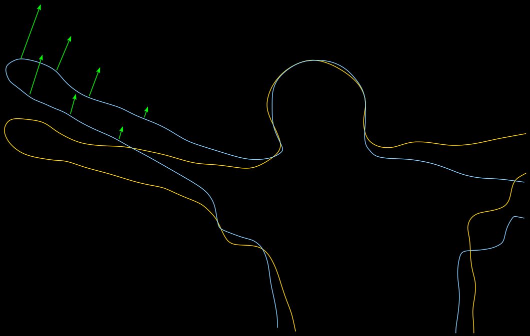


Transported
arm rotation
(better)

## Global transport

Forearm rotation

## Global transport



Transported
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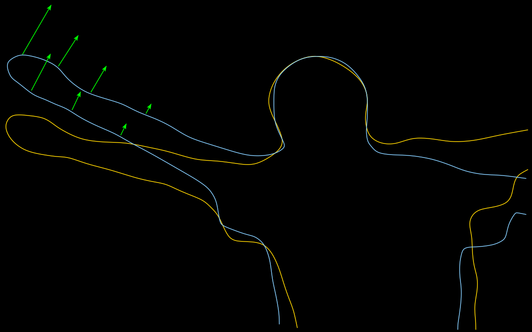
## Global transport

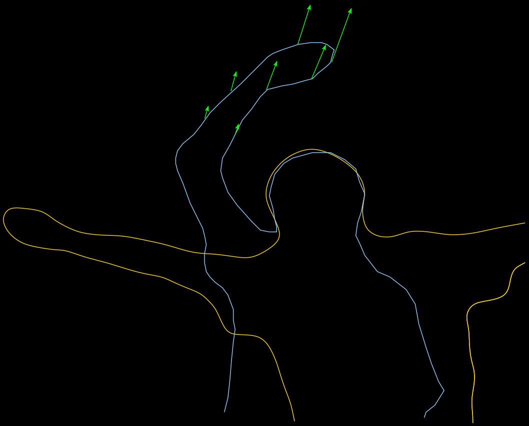


Transported forearm rotation (better)

## Global transport

Transport to another shape

## Global transport



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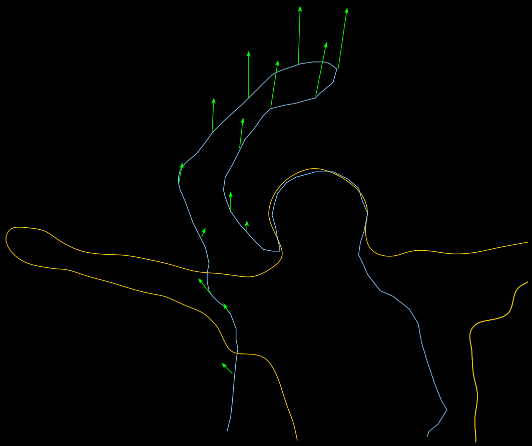

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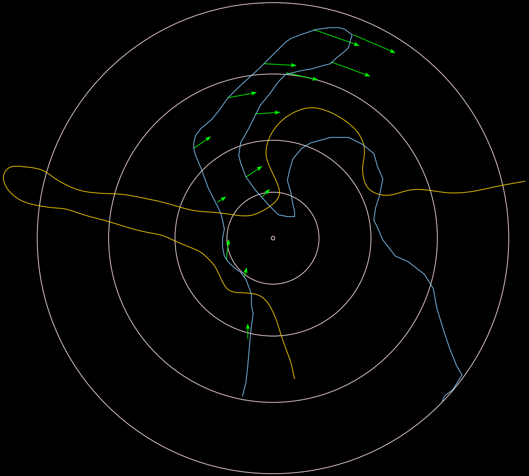

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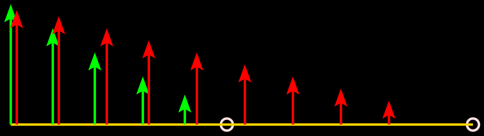
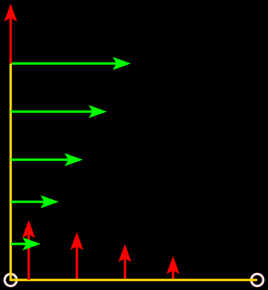


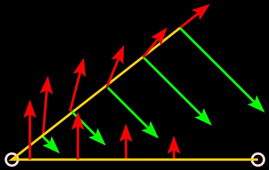
$L^2$
inner-product
$> 0$

## Global transport



$L^2$
inner-product
$= 0$

## Global transport



$L^2$
inner-product
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## Remarks about transport

Why ?

- transport : of deformations : needed to increase training set density

Which ?

- "translation" : ok for short pathes
- transport : not obvious (muscles + T-shirt artifacts)
- criterion to assess transport quality / suitability ?
- transport : should be learned (from video sequences ?)
- path could depend on deformation transported

What properties ?

- probability of a deformation transported : can differ
- inner product : no reason to be transport-invariant

## Transport in differential geometry

A connection $\nabla$ is **Riemannian** if the parallel transport it defines preserves the metric $g$. Metric connection :

$$\nabla_X \, g(\cdot,\cdot) = 0 \quad \text{for all vector fields } X \text{ on } \mathcal{M}$$

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Given transport, under few hypotheses (e.g. smoothness), it is possible to recover the associated infinitesimal connection :

$$\nabla_X V = \lim_{h \to 0} \frac{T_\gamma^{h\to 0} V_{\gamma(h)} - V_{\gamma(0)}}{h} = \frac{d}{dt} T_\gamma^{t \to 0} V_{\gamma(t)} \Big|_{t=0} .$$

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## Connection $\Longrightarrow$ transport :

Given a covariant derivative $\nabla$, the transport along a curve $\gamma$ is obtained by integrating the condition $\nabla_{\dot\gamma} = 0$.

# Metric estimation (statistics on deformations)

- set of shapes $(S_i)$, local deformations $m_{i \to j}$, transport $T^G_{i \to k}$
- $\implies$ transport deformations to a particular shape $S_k$ :
  $f^{i \to k}_{i \to j} = T^G_{i \to k}(m_{i \to j})$ are, $\forall i, j$, deformations defined on the same shape $S_k$
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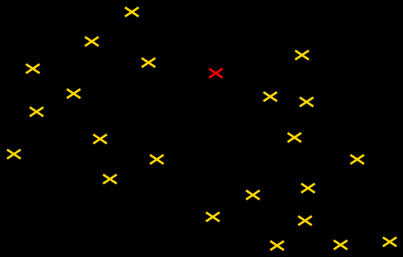

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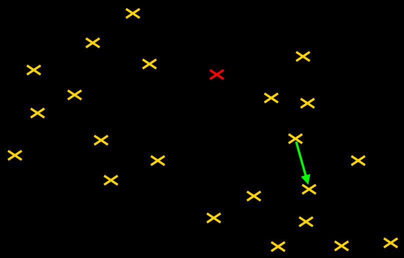

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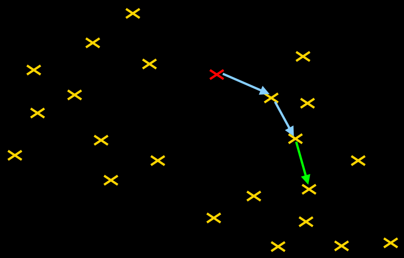

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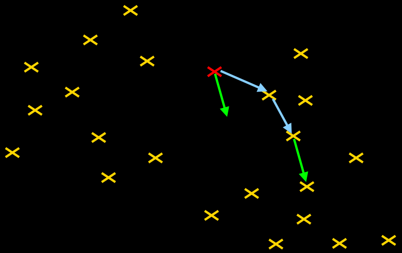

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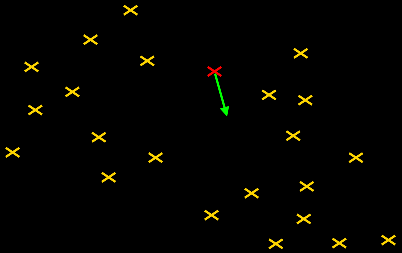

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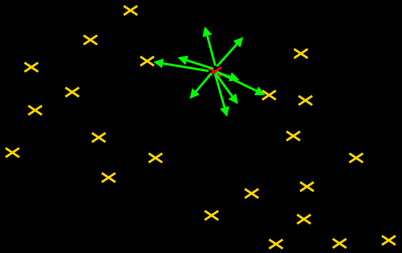

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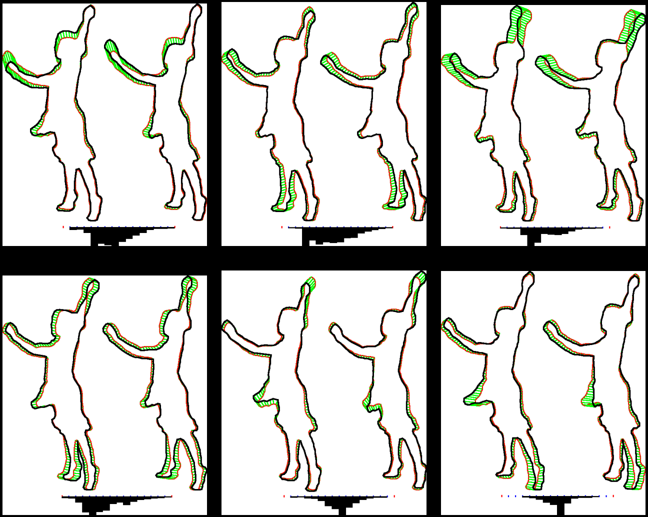
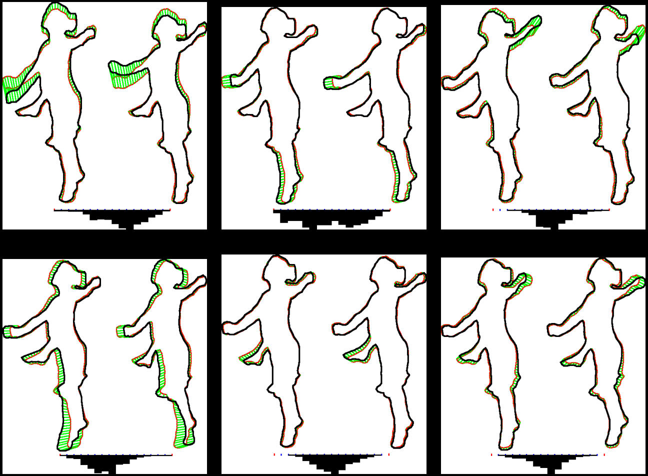
- statistics, for $k$ fixed : PCA
- PCA with weights, and with $H^1$-norm
- $\Longrightarrow$ eigenmodes $e_n$ (= principal deformations) with eigenvalues $\lambda_n$
- $\Longrightarrow$ defines an inner product $P_k$ = metric in the tangent space of the shape $S_k$
- $P_k$ varies smoothly as a function of $k$

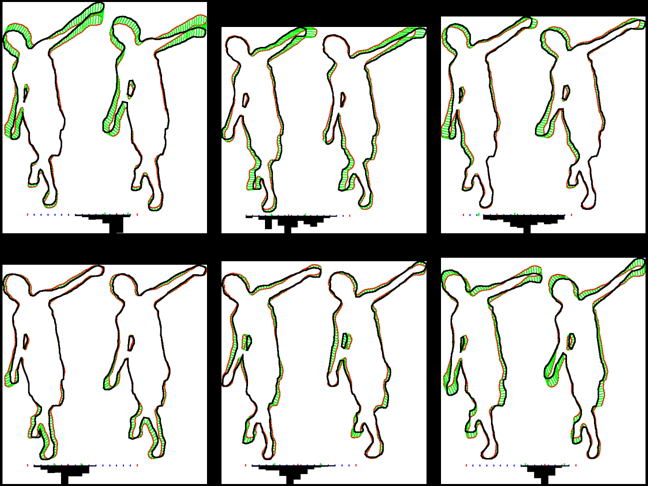
## Example of results : dancing sequence (9s, 24Hz), shape 1



## Example of results : shape 2



## Example of results : shape 3



# Theoretical justifications

## The best metric ?

Searching for principal modes of deformations which vary smoothly (as a function of the shape $S_k$) ?

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Best metric for a given empirical manifold (all shapes together) ?

- needs a smoothness criterion ( $\Longrightarrow$ transport)
- $\Longrightarrow$ best metric for a criterion involving transport & K-L divergence.
- $\Longrightarrow$ best metric for another criterion involving transport & $L^2$-norm of distributions.

# Best metric for a given empirical manifold

- set of shapes $(S_i)$, local deformations $\mathbf{f}_{i \to j}$, transport $T^G_{i \to k}$
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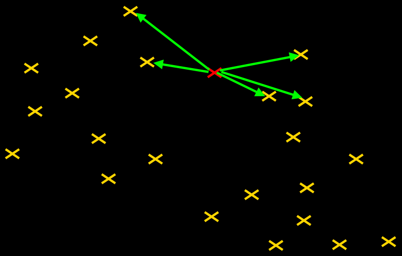

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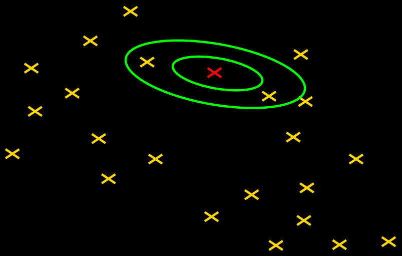

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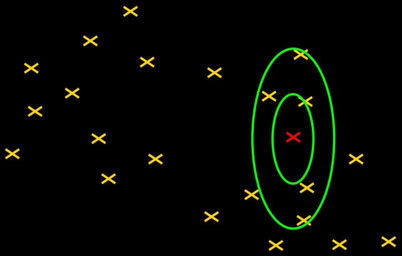

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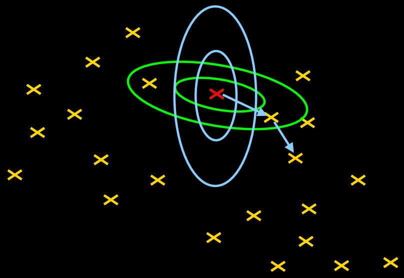

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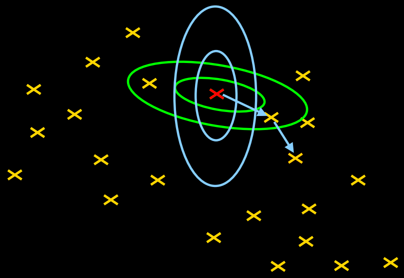
- Criterion : best $(P_k)$ for $\displaystyle\sum_{i,k} \; w^G_{ik} \;\; KL\left(\mathcal{D}_{P_k} \mid T_{i\to k}(\mathcal{D}_{emp_i})\right)$



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- $=$ best $(P_k)$ for $\displaystyle\sum_k KL(\mathcal{D}_{P_k}|\mathcal{D}^T_{emp_k})$

  where $\mathcal{D}^T_{emp_k} = \displaystyle\sum_{i,j} w^k_{i\to j}\;\delta_{\mathbf{f}^k_{i\to j}}$
- Transported deformations to any shape $S_k$ : $f^k_{i\to j} = T^G_{i\to k}(f_{i\to j})$ with reliability weights $w^k_{i\to j} = w^G_{i\to k}\,w^L_{i\to j}$
- $=$ the one obtained by weighted PCA on transported deformations

## Best metric for a given empirical manifold (again!)

- empirical distributions : $\mathcal{D}_{emp_i}$
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- $A = Id + \varepsilon\Delta$ where $\Delta$ = graph Laplacian (with transports)
- $g = A^{-1} g^0 \; = \; (Id + \varepsilon\Delta)^{-1} g^0 \; \simeq (Id - \varepsilon\Delta) g^0 \; \simeq \; \mathcal{N}_\varepsilon * g^0.$
- $g = (Id - \varepsilon\Delta) \, g^0$ coincides with the $\mathcal{D}^T_{emp}$
  and the inner products $(P_i)$ which suit $g = (g_i)$ the best (for K-L) are the ones we computed

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[NORDIA 2009 : *Learning Shape Metrics based on Deformations and Transport*]

# Future works

- learning functions defined on shape spaces / with values in shape spaces
- statistics on image patches through correspondences/transport

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- $\implies$ search over inner products = search over Gaussian distributions

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- similar result for kernel-smoothed distributions :
  $\mathcal{D}^{\mathcal{K}}_{emp}(\mathbf{f}) = \sum_j w_j \, \mathcal{K}(\mathbf{f}_j - \mathbf{f})$.

## Weighted PCA with $H^1$ norm

- PCA = find the best axes (to project data on this subspace)
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$$\inf_{\langle \mathbf{e}_n | \mathbf{e}_{n'} \rangle_{H^1_\alpha} = \delta_{n=n'}} \sum_{i,j} w^k_{i\to j} \left\| \mathbf{f}^k_{i\to j} - \sum_n \left\langle \mathbf{f}^k_{i\to j} \middle| \mathbf{e}_n \right\rangle_{H^1_\alpha} \mathbf{e}_n \right\|^2_{H^1_\alpha}$$

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- Change of variables: $\mathbf{d}_n = H^{1/2}\mathbf{e}_n$ : $\sup_{\langle \mathbf{d}_n | \mathbf{d}_{n'} \rangle_{L^2} = \delta_{n=n'}} \sum_n \mathbf{d}_n \, H^{1/2} F H^{1/2} \, \mathbf{d}_n$

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$$M_{(i,j),(i',j')} = \left\langle \sqrt{w^k_{i \to j}}\, \mathbf{f}^k_{i \to j} \middle| \sqrt{w^k_{i' \to j'}}\, \mathbf{f}^k_{i' \to j'} \right\rangle_{H^1_\alpha}$$

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- eigenvectors :

$$\mathbf{e}_n = \sum_{ij} \gamma_n^{(i,j)} \sqrt{w^k_{i\to j}}\, \mathbf{f}^k_{i\to j}$$