Estimation of the marginal expected shortfall using extreme expectiles

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ERCIM' 2016, Seville

Outline of presentation

Expectiles vs Quantiles

Let Y be a random variable and τ in $(0,1)$.

- **The τth quantile of Y** [Koenker and Bassett (1978)] :

$$q_\tau = \arg\min_\theta \mathbb{E}\left\{|\tau - \mathbb{I}(Y \le \theta)| \cdot |Y - \theta|\right\}$$

- **The τth expectile of Y** [Newey and Powel (1987)] :

$$\xi_\tau = \arg\min_\theta \mathbb{E}\left\{|\tau - \mathbb{I}(Y \le \theta)| \cdot |Y - \theta|^2\right\}$$

  exists as soon as $\mathbb{E}|Y| < \infty$.

- **Special case** :

$$\tau = \frac{1}{2} \Longrightarrow \begin{cases} q_{0.5} & = & \text{median}(Y) \\ \xi_{0.5} & = & \mathbb{E}(Y) \end{cases}$$

In terms of interpretability

- q_τ determines the point below which $100\tau\%$ of the mass of Y lies :

$$\mathbb{E}\left\{\mathbb{I}(Y \leq q_\tau)\right\} = \mathbb{P}(Y \leq q_\tau) = \tau$$

- ξ_τ shares an interpretation similar to q_τ, replacing the number of observations by the distance :

$$\frac{\mathbb{E}\left\{|Y - \xi_\tau| \cdot \mathbb{I}(Y \leq \xi_\tau)\right\}}{\mathbb{E}\{|Y - \xi_\tau|\}} = \tau$$

that is, the average distance from the data below ξ_τ to ξ_τ itself is $100\tau\%$...

  - expectiles = quantiles for a transformation of F_Y [Jones (1994)]
  - expectiles = quantiles in case of a weighted symmetric distribution [Abdous and Remillard (1995)]
  - $\xi_\tau = q_{\tau'}$ for different levels τ and τ' [Yao and Tong (1996)]

Other merits

Advantages of least asymmetrically weighted squares (LAWS) estimation :

- Computing expedience (though efficient linear programming algorithms are available for quantiles)
- Efficiency of the LAWS estimator :
  - + Expectiles make more efficient use of the available data since they rely on the distance to observations
  - – Quantiles only utilize the information on whether an observation is below or above the predictor !
- More valuable tail information :
  - – Quantiles only depend on the frequency of tail realizations of Y and not on their values !
  - + Expectiles depend on both the tail realizations and their probability [Kuan, Yeh and Hsu (2009)]
- Inference on expectiles is much easier than inference on quantiles : Calculation of the variance without going via the density of the distribution

Basic properties

(i) Law invariance : a distribution is uniquely defined by its class of expectiles

(ii) Location and scale equivariance : the τth expectile of the linear transformation $\widetilde{Y} = a + bY$, where $a, b \in \mathbb{R}$, satisfies

$$\xi_{\widetilde{Y},\tau} = \begin{cases} a + b\,\xi_{Y,\tau} & \text{if} \quad b > 0 \\ a + b\,\xi_{Y,1-\tau} & \text{if} \quad b \leq 0 \end{cases}$$

(iii) Coherency : for any variables $Y, \tilde{Y} \in L^1$ and for all $\tau \geq \frac{1}{2}$,

- Translation invariance : $\xi_{Y+a,\tau} = \xi_{Y,\tau} + a$, for all $a \in \mathbb{R}$
- Positive homogeneity : $\xi_{bY,\tau} = b\xi_{Y,\tau}$, for all $b \geq 0$
- Monotonicity : if $Y \leq \tilde{Y}$ with probability 1, then $\xi_{Y,\tau} \leq \xi_{\tilde{Y},\tau}$
- Subadditivity : $\xi_{Y+\tilde{Y},\tau} \leq \xi_{Y,\tau} + \xi_{\tilde{Y},\tau}$

Expectiles as risk measures

Theoretical and numerical results, obtained recently, indicate that expectiles are perfectly reasonable alternatives to quantiles as risk measures :

- Taylor (2008) [Journal of Financial Econometrics]
- Kuan, Yeh and Hsu (2009) [Journal of Econometrics]
- Gneiting (2011) [JASA]
- Bellini (2012) [Statistics and Probability Letters]
- Bellini, Klar, Müller and Gianina (2014) [Insurance : Mathematics and Economics]
- Bellini and Di Bernardino (2015) [The European Journal of Finance]
- Ziegel (2016) [Mathematical Finance]
- Ehm, Gneiting, Jordan and Krüger (2016) [JRSS-B] •••

The estimation of expectiles did not, however, receive yet any attention from the perspective of extreme values !

Aim : We use tail expectiles to estimate an alternative measures to **Marginal Expected Shortfall** (MES)

Outline of presentation

1. The class of expectiles
2. Extremal expectiles
3. Estimation of high expectiles
4. Expectile-based MES
5. Extreme level selection
6. Application

Outline of presentation

High expectiles

- Let Y be the financial position and F_Y be its cdf with $\overline{F}_Y = 1 - F_Y$:
  - non-negative loss variable
  - real-valued variable (the negative of financial returns)
- In statistical finance and actuarial science, Pareto-type distributions describe quite well the tail structure of losses :

$$\overline{F}_Y(y) = 1 - \ell(y) \cdot y^{-1/\gamma}$$

  - $\ell(\lambda y)/\ell(y) \to 1$ as $y \to \infty$ for all $\lambda > 0$
  - $\gamma \in (0, 1)$ tunes the tail heaviness of F_Y
- Only Bellini, Klar, Müller and Gianina (2014), Mao, Ng and Hu (2015) and Mao and Yang (2015) have described what happens for large ξ_τ and how it is linked to extreme q_τ :

$$\frac{\overline{F}_Y(\xi_\tau)}{1-\tau} = \frac{\overline{F}_Y(\xi_\tau)}{\overline{F}_Y(q_\tau)} \sim \gamma^{-1} - 1 \quad \text{as} \quad \tau \to 1$$

  - $\xi_\tau > q_\tau$ when $\gamma > \frac{1}{2}$,
  - $\xi_\tau < q_\tau$ when $\gamma < \frac{1}{2}$, for all large τ

Second-order regular variation condition

Assume that **the tail quantile function** U of Y, defined by

$$U(t) = \inf\left\{ y \in \mathbb{R} \;\middle|\; \frac{1}{\overline{F}_Y(y)} \geq t \right\}, \quad \forall t > 1,$$

satisfies the second-order condition indexed by (γ, ρ, A), that is, there exist $\gamma > 0$, $\rho \leq 0$, and a function $A(\cdot)$ converging to 0 at infinity and having constant sign such that :

$\boldsymbol{\mathcal{C}_2(\gamma, \rho, A)}$ for all $x > 0$,

$$\lim_{t\to\infty} \frac{1}{A(t)}\left[\frac{U(tx)}{U(t)} - x^\gamma\right] = x^\gamma \frac{x^\rho - 1}{\rho}.$$

Here, $(x^\rho - 1)/\rho$ is to be understood as $\log x$ when $\rho = 0$.

Refined asymptotic connection

The precise **bias term** in the asymptotic approximation of (ξ_τ/q_τ) :

Proposition

Assume that condition $\mathcal{C}_2(\gamma,\rho,A)$ holds, with $0<\gamma<1$. If F_Y is strictly increasing, then

$$\frac{\xi_\tau}{q_\tau} = (\gamma^{-1}-1)^{-\gamma}(1+r(\tau))$$

with

$$\begin{aligned} r(\tau) &= \frac{\gamma(\gamma^{-1}-1)^{\gamma}\mathbb{E}(Y)}{q_\tau}(1+\mathrm{o}(1)) \\ &\quad + \left(\frac{(\gamma^{-1}-1)^{-\rho}}{1-\rho-\gamma} + \frac{(\gamma^{-1}-1)^{-\rho}-1}{\rho} + \mathrm{o}(1)\right) A((1-\tau)^{-1}) \end{aligned}$$

as $\tau \uparrow 1$

Other similar refinements can be found in Mao *et al.* (2015), Mao and Yang (2015) and Bellini and Di Bernardino (2015)

Outline of presentation

Conceptual results

- **Basic idea :**
  - We first estimate the intermediate expectiles of order $\tau_n \to 1$ such that $n(1-\tau_n) \to \infty$;
  - We then extrapolate these estimates to the very extreme level τ_n' which approaches 1 at an arbitrarily fast rate in the sense that $n(1-\tau_n') \to c$, for some constant c.
- **Two estimation methods are considered :**
  - The first (indirect) is based on the use of the asymptotic connection between expectiles and quantiles ;
  - The second relies directly on least asymmetrically weighted squares (LAWS) estimation.
- **Main results :** establish the asymptotic normality of the estimators
  - Intermediate expectiles ξ_{τ_n} : indirect + direct
  - Extreme expectiles $\xi_{\tau_n'}$: indirect + direct

Intermediate expectile estimation

(1) Estimation based on intermediate quantiles :

- Assume that the available data consists of an n-tuple $(Y_1, \ldots, Y_n)$ of independent copies of Y
- Denote by $Y_{1,n} \leq \cdots \leq Y_{n,n}$ their ascending order statistics
- Consider the intermediate expectile level $\tau_n \to 1$ such that $n(1-\tau_n) \to \infty$, as $n \to \infty$
- The asymptotic connection above entails that

$$\frac{\xi_{\tau_n}}{q_{\tau_n}} \sim \left(\gamma^{-1} - 1\right)^{-\gamma} \quad \text{as} \quad n \to \infty$$

- Define, for a suitable estimator $\widehat{\gamma}$ of γ,

$$\widehat{\xi}_{\tau_n} := \left(\widehat{\gamma}^{-1} - 1\right)^{-\widehat{\gamma}} \cdot \widehat{q}_{\tau_n}$$

  where

$$\widehat{q}_{\tau_n} := Y_{n-\lfloor n(1-\tau_n) \rfloor, n}$$

  with $\lfloor \cdot \rfloor$ being the floor function.

Intermediate expectile estimation (cont.)

(2) Asymmetric Least Squares (direct) estimation :

We consider estimating

$$\xi_{\tau_n} = \arg\min_{u \in \mathbb{R}} \mathbb{E}\left[\eta_{\tau_n}(Y - u)\right]$$

by

$$\widetilde{\xi}_{\tau_n} = \arg\min_{u \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} \eta_{\tau_n}(Y_i - u)$$

where $\eta_\tau(y) = |\tau - \mathbb{I}\{y \leq 0\}| y^2$ is the expectile check function

Extreme expectile estimation

- Consider the intermediate expectile level $\tau_n \to 1$ such that $n(1-\tau_n) \longrightarrow \infty$, as $n \to \infty$
- Consider the **extreme** expectile level $\tau'_n \to 1$ such that $n(1-\tau'_n) \longrightarrow c < \infty$, as $n \to \infty$
- The model assumption of Pareto-type tails

$$F_Y(y) = 1 - \ell(y) \cdot y^{-1/\gamma}$$

suggests that

$$\frac{\xi_{\tau'_n}}{\xi_{\tau_n}} \approx \left(\frac{1-\tau'_n}{1-\tau_n}\right)^{-\gamma}, \quad n \to \infty$$

- This approximation motivates the estimators

$$\widetilde{\xi}^{\star}_{\tau'_n} := \left(\frac{1-\tau'_n}{1-\tau_n}\right)^{-\widehat{\gamma}} \widetilde{\xi}_{\tau_n}$$

$$\hat{\xi}^{\star}_{\tau'_n} := \left(\frac{1-\tau'_n}{1-\tau_n}\right)^{-\widehat{\gamma}} \widehat{\xi}_{\tau_n} \equiv \left(\widehat{\gamma}^{-1} - 1\right)^{-\widehat{\gamma}} \hat{q}^{\star}_{\tau'_n}$$

Extreme expectile estimation (cont.)

$$\hat{\xi}^{\star}_{\tau'_n} := \left(\frac{1-\tau'_n}{1-\tau_n}\right)^{-\widehat{\gamma}} \widehat{\xi}_{\tau_n} \equiv \left(\widehat{\gamma}^{-1}-1\right)^{-\widehat{\gamma}} \hat{q}^{\star}_{\tau'_n}$$

Theorem

Assume that F_Y *is strictly increasing, that condition* $\mathcal{C}_2(\gamma, \rho, A)$ *holds with* $0 < \gamma < 1$ *and* $\rho < 0$*, and that* $\tau_n, \tau'_n \uparrow 1$ *with* $n(1-\tau_n) \to \infty$ *and* $n(1-\tau'_n) \to c < \infty$*. Assume further that*

$$\sqrt{n(1-\tau_n)}\left(\widehat{\gamma}-\gamma, \frac{\widehat{q}_{\tau_n}}{q_{\tau_n}}-1\right) \xrightarrow{d} (\Gamma, \Theta).$$

If $\sqrt{n(1-\tau_n)}q^{-1}_{\tau_n} \to \lambda_1 \in \mathbb{R}$, $\sqrt{n(1-\tau_n)}A((1-\tau_n)^{-1}) \to \lambda_2 \in \mathbb{R}$ *and* $\sqrt{n(1-\tau_n)}/\log[(1-\tau_n)/(1-\tau'_n)] \to \infty$*, then*

$$\frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\tau'_n)]}\left(\frac{\hat{\xi}^{\star}_{\tau'_n}}{\xi_{\tau'_n}}-1\right) \xrightarrow{d} \Gamma$$

$$\widetilde{\xi}^{\star}_{\tau_n'} := \left(\frac{1-\tau_n'}{1-\tau_n}\right)^{-\widehat{\gamma}} \widetilde{\xi}_{\tau_n}$$

Theorem

Assume that F_Y is strictly increasing, there is $\delta > 0$ such that $\mathbb{E}|Y_-|^{2+\delta} < \infty$, condition $\mathcal{C}_2(\gamma, \rho, A)$ holds with $0 < \gamma < 1/2$ and $\rho < 0$, and that τ_n, $\tau_n' \uparrow 1$ with $n(1-\tau_n) \to \infty$ and $n(1-\tau_n') \to c < \infty$. If in addition

$$\sqrt{n(1-\tau_n)}(\widehat{\gamma} - \gamma) \xrightarrow{d} \Gamma$$

and $\sqrt{n(1-\tau_n)}q_{\tau_n}^{-1} \to \lambda_1 \in \mathbb{R}$, $\sqrt{n(1-\tau_n)}A((1-\tau_n)^{-1}) \to \lambda_2 \in \mathbb{R}$ and $\sqrt{n(1-\tau_n)}/\log[(1-\tau_n)/(1-\tau_n')] \to \infty$, then

$$\frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\tau_n')]} \left(\frac{\widetilde{\xi}^{\star}_{\tau_n'}}{\xi_{\tau_n'}} - 1\right) \xrightarrow{d} \Gamma$$

Outline of presentation

Connection with systemic risk

- With the recent financial crisis and the rising interconnection between financial institutions, interest in the concept of systemic risk has grown ;
- Systemic risk : the propensity of a financial institution to be undercapitalized when the financial system as a whole is undercapitalized [Acharya *et al.* (2012), Brownlees and Engle (2012), Engle *et al.* (2015)] ;
- Econometric approaches have been proposed to measure the systemic risk of financial institutions ;
- An important step in constructing a systemic risk measure for a financial institution is to measure the contribution of the institution to a systemic crisis ;
- Systemic crisis : a major stock market decline that happens once or twice a decade ;
- *The total risk*, measured by the expected capital shortfall in the financial system during a systemic crisis, *can be decomposed* into **firm level contributions** ;
- **Firm level contributions** can be measured by the **Marginal expected shortfall = ?**

Marginal expected shortfall

- X : the loss return on the equity of **a financial firm**
- Y : the loss return on the equity of **the entire market**
- Marginal expected shortfall :

$$\mathrm{MES} = \mathbb{E}(X|Y > t)$$

where t is a high threshold reflecting a substantial market decline

- MES at probability level $(1 - \tau)$:

$$\mathrm{QMES}(\tau) = \mathbb{E}\{X|Y > q_{Y,\tau}\}$$

- The estimation of $\mathrm{QMES}(\tau)$
  - in Acharya *et al.* (2012) : relies on daily data from **only** 1 year and assumes a specific **linear** relationship between X and Y ;
  - in Brownlees and Engle (2012) and Engle *et al.* (2014) : a non-parametric kernel estimator was proposed ;

Cannot handle extreme events required for systemic risk measures, *i.e.*,

$$1 - \tau = \mathrm{O}(1/n)$$

Adapted extreme-value tools

Cai, Einmahl, de Haan & Zhou (2015) $\rightsquigarrow$ Adapted tools for the estimation of

$$\begin{aligned} \mathrm{QMES}(\tau_n') &= \mathbb{E}\left\{X | Y > q_{Y,\tau_n'}\right\} \\ &\approx \left(\frac{1-\tau_n'}{1-\tau_n}\right)^{-\gamma_X} \mathrm{QMES}(\tau_n) \end{aligned}$$

without recourse to any parametric structure on (X, Y) :

$$\widehat{\mathrm{QMES}}^{\star}(\tau_n') = \left(\frac{1-\tau_n'}{1-\tau_n}\right)^{-\widehat{\gamma}_X} \widehat{\mathrm{QMES}}(\tau_n)$$

where

$$\widehat{\mathrm{QMES}}(\tau_n) = \frac{1}{\lfloor n(1-\tau_n) \rfloor} \sum_{i=1}^{n} X_i \, \mathbb{I}\{X_i > 0, Y_i > \widehat{q}_{Y,\tau_n}\}.$$

Tail dependence condition $\boldsymbol{\mathcal{JC}_2(R, \beta, \kappa)}$: There exist $R(\cdot, \cdot)$, $\beta > \gamma_X$ and $\kappa < 0$ such that

$$\sup_{\substack{x \in (0,\infty) \\ y \in [1/2, 2]}} \left| \frac{t\mathbb{P}(\overline{F}_X(X) \leq x/t, \overline{F}_Y(Y) \leq y/t) - R(x,y)}{\min(x^{\beta}, 1)} \right| = \mathrm{O}(t^{\kappa}) \quad \text{as} \quad t \to \infty$$

Expectile-based MES

Daouia, Girard & Stupfler (2016) :

$$\begin{aligned}\mathrm{XMES}(\tau_n') &= \mathbb{E}\left\{X|Y > \xi_{Y,\tau_n'}\right\} \\ &\approx \left(\frac{1-\tau_n'}{1-\tau_n}\right)^{-\gamma_X} \mathrm{XMES}(\tau_n)\end{aligned}$$

with

$$\mathrm{XMES}(\tau_n) = \mathbb{E}\left\{X|Y > \xi_{Y,\tau_n}\right\}$$

(1) ALS type estimator :

$$\widetilde{\mathrm{XMES}}^{\star}(\tau_n') = \left(\frac{1-\tau_n'}{1-\tau_n}\right)^{-\widehat{\gamma}_X} \widetilde{\mathrm{XMES}}(\tau_n)$$

where

$$\widetilde{\mathrm{XMES}}(\tau_n) = \frac{\sum_{i=1}^n X_i\, \mathbb{I}\{X_i > 0, Y_i > \widetilde{\xi}_{Y,\tau_n}\}}{\sum_{i=1}^n \mathbb{I}\{Y_i > \widetilde{\xi}_{Y,\tau_n}\}}.$$

If $\sqrt{n(1-\tau_n)}\,(\widehat{\gamma}_X - \gamma_X) \xrightarrow{d} \Gamma$, then

$$\frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\tau_n')]}\left(\frac{\widetilde{\mathrm{XMES}}^{\star}(\tau_n')}{\mathrm{XMES}(\tau_n')} - 1\right) \xrightarrow{d} \Gamma$$

Expectile-based MES (cont.)

Suppose for all $(x, y) \in [0, \infty]^2$ such that at least x or y is finite, the limit

$$\lim_{t \to \infty} t\mathbb{P}(\overline{F}_X(X) \le x/t, \overline{F}_Y(Y) \le y/t) := R(x, y) \quad \text{exists.}$$

Under this tail dependence condition :

$$\lim_{\tau \uparrow 1} \frac{\text{XMES}(\tau)}{\text{QMES}(\tau)} = \left(\gamma_Y^{-1} - 1\right)^{-\gamma_X}$$

(2) Estimator based on tail QMES :

$$\widehat{\text{XMES}}^{\star}(\tau_n') = \left(\widehat{\gamma}_Y^{-1} - 1\right)^{-\widehat{\gamma}_X} \ \widehat{\text{QMES}}^{\star}(\tau_n').$$

If $\sqrt{n(1-\tau_n)}\,(\widehat{\gamma}_X - \gamma_X) \xrightarrow{d} \Gamma$, then

$$\frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\tau_n')]} \left(\frac{\widehat{\text{XMES}}^{\star}(\tau_n')}{\text{XMES}(\tau_n')} - 1 \right) \xrightarrow{d} \Gamma$$

Outline of presentation

Expectile level selection

- **Aim** : choice of $\tau_n' = ?$ in the instruments of risk protection

$$\xi_{\tau_n'}, \quad \text{XMES}(\tau_n')$$

- **In the case of**

$$q_{\alpha_n}, \quad \text{QMES}(\alpha_n)$$

  it is customary to choose

$$\alpha_n \to 1 \quad \text{with} \quad n(1-\alpha_n) \to c < \infty$$

  to allow for more 'prudent' risk management $\rightsquigarrow$ Typical interest in once-in-a-decade or twice-per-decade events

- **Idea** : select τ_n' so that each expectile-based risk measure has the same intuitive interpretation as its quantile-based analogue

$$\Updownarrow$$

choose $\tau_n' = \tau_n'(\alpha_n)$ s.t. $\xi_{\tau_n'} \equiv q_{\alpha_n}$ for a given relative frequency α_n

$$\Downarrow$$

$$\tau_n'(\alpha_n) = 1 - \frac{\mathbb{E}\left\{|Y - q_{\alpha_n}|\,\mathbb{I}\left(Y > q_{\alpha_n}\right)\right\}}{\mathbb{E}\,|Y - q_{\alpha_n}|}$$

How to estimate

$$\tau_n'(\alpha_n) = 1 - \frac{\mathbb{E}\left\{|Y - q_{\alpha_n}|\,\mathbb{I}\left(Y > q_{\alpha_n}\right)\right\}}{\mathbb{E}\,|Y - q_{\alpha_n}|} \quad ?$$

Proposition

Under the model assumption of Pareto-type tails with $0 < \gamma < 1$*,*

$$1 - \tau_n'(\alpha_n) \sim (1 - \alpha_n)\,\frac{\gamma}{1-\gamma}, \quad n \to \infty$$

$$\rightsquigarrow \quad \widehat{\tau}_n'(\alpha_n) = 1 - (1 - \alpha_n)\,\frac{\widehat{\gamma}_n}{1 - \widehat{\gamma}_n}$$

VaR estimation

With

$$\tau_n' = \widehat{\tau}_n'(\alpha_n) \equiv 1 - (1-\alpha_n)\frac{\widehat{\gamma}_n}{1-\widehat{\gamma}_n}$$

both extreme expectile estimators

$$\widetilde{\xi}^{\star}_{\tau_n'} = \left(\frac{1-\tau_n'}{1-\tau_n}\right)^{-\widehat{\gamma}_n} \widetilde{\xi}_{\tau_n}$$

$$\widehat{\xi}^{\star}_{\tau_n'} = \left(\frac{1-\tau_n'}{1-\tau_n}\right)^{-\widehat{\gamma}_n} \widehat{\xi}_{\tau_n} = \left(\widehat{\gamma}_n^{-1} - 1\right)^{-\widehat{\gamma}_n} \hat{q}^{\star}_{\tau_n'}$$

estimate the same VaR $\xi_{\tau_n'(\alpha_n)} \equiv q_{\alpha_n}$ as

$$\hat{q}^{\star}_{\alpha_n} := \left(\frac{1-\alpha_n}{1-\tau_n}\right)^{-\widehat{\gamma}} \widehat{q}_{\tau_n}$$

- $\widehat{\xi}^{\star}_{\widehat{\tau}_n'(\alpha_n)} \equiv \hat{q}^{\star}_{\alpha_n}$
- If $\sqrt{n(1-\tau_n)}(\widehat{\gamma}_n - \gamma) \xrightarrow{d} \Gamma$, then

$$\frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\alpha_n)]}\left(\frac{\widetilde{\xi}^{\star}_{\widehat{\tau}_n'(\alpha_n)}}{q_{\alpha_n}} - 1\right) \xrightarrow{d} \Gamma$$

MES estimation

With

$$\tau_n' = \widehat{\tau}_n'(\alpha_n) \equiv 1 - (1-\alpha_n)\,\frac{\widehat{\gamma}_n}{1-\widehat{\gamma}_n}$$

both estimators

$$\widetilde{\text{XMES}}^{\star}(\tau_n') = \left(\frac{1-\tau_n'}{1-\tau_n}\right)^{-\widehat{\gamma}_X} \widetilde{\text{XMES}}(\tau_n)$$

$$\widehat{\text{XMES}}^{\star}(\tau_n') = \left(\widehat{\gamma}_Y^{-1} - 1\right)^{-\widehat{\gamma}_X} \widehat{\text{QMES}}^{\star}(\tau_n')$$

estimate the same MES $\text{XMES}(\tau_n'(\alpha_n)) \equiv \text{QMES}(\alpha_n)$ as Cai *et al.* (2015)'s estimator

$$\widehat{\text{QMES}}^{\star}(\alpha_n) = \left(\frac{1-\alpha_n}{1-\tau_n}\right)^{-\widehat{\gamma}_X} \widehat{\text{QMES}}(\tau_n)$$

- $\widehat{\text{XMES}}^{\star}(\widehat{\tau}_n'(\alpha_n)) \equiv \widehat{\text{QMES}}^{\star}(\alpha_n)$
- If $\sqrt{n(1-\tau_n)}(\widehat{\gamma}_X - \gamma_X) \xrightarrow{d} \Gamma$, then

$$\frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\alpha_n)]}\left(\frac{\widetilde{\text{XMES}}^{\star}(\widehat{\tau}_n'(\alpha_n))}{\text{QMES}(\alpha_n)} - 1\right) \xrightarrow{d} \Gamma$$

Outline of presentation

MES of three large investment banks

- We consider the same investment banks as in Brownlees and Engle (2012) and Cai *et al.* (2015) :

  Goldman Sachs, Morgan Stanley, T. Rowe Price

- For the three banks, the dataset consists of the loss returns (X_i) on their equity prices at a **daily frequency** from July 3rd, 2000, to June 30th, 2010 **(ten years)**
- We follow the same set-up as in Cai *et al.* (2015) to extract, for the same time period, daily loss returns (Y_i) of **a value-weighted market index** aggregating three markets :
  - New York Stock Exchange
  - American Express stock exchange
  - National Association of Securities Dealers Automated Quotation system
- The interest is on $\widehat{\mathrm{QMES}}^{\star}(\alpha_n)$ and $\widetilde{\mathrm{XMES}}^{\star}(\widehat{\tau}_n'(\alpha_n))$ that estimate

$$\mathrm{QMES}(\alpha_n) \equiv \mathrm{XMES}(\tau_n'(\alpha_n)) \quad \text{with} \quad \alpha_n = 1 - 1/n$$

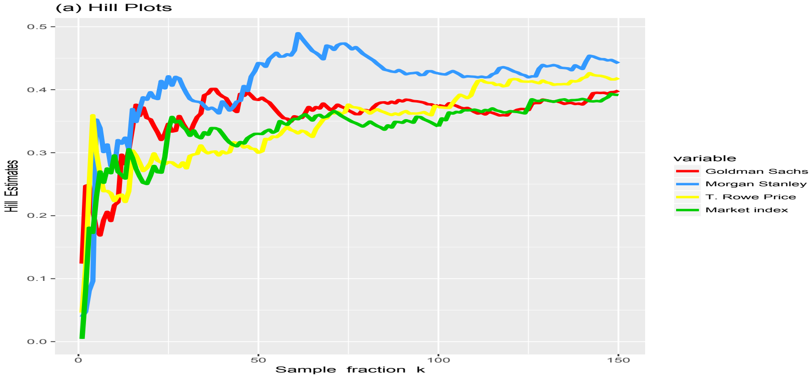
- They represent the average daily loss return for a **once-per-decade market crisis** $(n = 2513)$

Plots of the Hill estimates

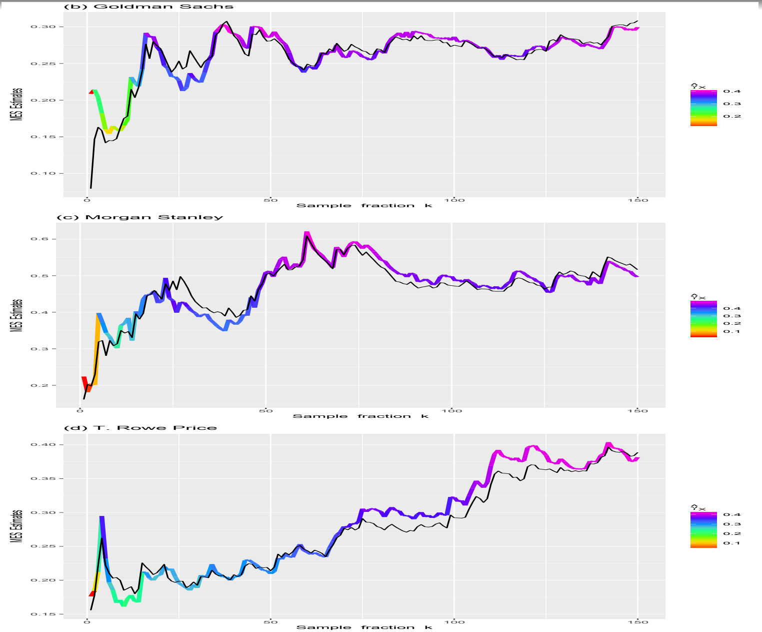
- $\widehat{\gamma}_Y$ based on daily loss returns of market index
- $\widehat{\gamma}_X$ based on daily loss returns of Goldman Sachs, Morgan Stanley, T. Rowe Price

$$\Downarrow$$

$$\gamma_X, \gamma_Y < 1/2$$



$\widehat{\text{QMES}}^{\star}$ (black) & $\widetilde{\text{XMES}}^{\star}$ (rainbow)



The final MES estimates

Bank	$\widetilde{\text{XMES}}^{\star}(\widehat{\tau}_n'(\alpha_n))$	$\widehat{\text{QMES}}^{\star}(\alpha_n)$
Goldman Sachs	0.286	0.280
Morgan Stanley	0.485	0.471
T. Rowe Price	0.297	0.279

The final estimates based on averaging the estimates from the first stable regions of the plots :

- The quantile-based estimates are less **conservative** than our ALS-based estimates, but not by much
- MES levels for **Morgan Stanley** are largely higher than those for **Goldman Sachs** and **T. Rowe Price**