Estimation of the marginal expected shortfall using extreme expectiles

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Based on joint work with:

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Outline of presentation

1. The class of expectiles
2. Extremal expectiles
3. Estimation of high expectiles
4. Expectile-based MES
5. Extreme level selection
6. Application
Let \( Y \) be a random variable and \( \tau \) in \((0, 1)\).

- **The \( \tau \)th quantile of \( Y \) [Koenker and Bassett (1978)]**:
  \[
  q_\tau = \arg \min_\theta \mathbb{E} \{|\tau - I(Y \leq \theta)| \cdot |Y - \theta|\}
  \]

- **The \( \tau \)th expectile of \( Y \) [Newey and Powel (1987)]**:
  \[
  \xi_\tau = \arg \min_\theta \mathbb{E} \{|\tau - I(Y \leq \theta)| \cdot |Y - \theta|^2\}
  \]
  exists as soon as \( \mathbb{E}|Y| < \infty \).

- **Special case**:
  \[
  \tau = \frac{1}{2} \implies \begin{cases} 
  q_{0.5} = \text{median}(Y) \\
  \xi_{0.5} = \mathbb{E}(Y)
  \end{cases}
  \]
In terms of interpretability

- $q_\tau$ determines the point below which $100\tau\%$ of the mass of $Y$ lies:

$$
\mathbb{E}\{I(Y \leq q_\tau)\} = P(Y \leq q_\tau) = \tau
$$

- $\xi_\tau$ shares an interpretation similar to $q_\tau$, replacing the number of observations by the distance:

$$
\frac{\mathbb{E}\{|Y - \xi_\tau| \cdot I(Y \leq \xi_\tau)\}}{\mathbb{E}\{|Y - \xi_\tau|\}} = \tau
$$

that is, the average distance from the data below $\xi_\tau$ to $\xi_\tau$ itself is $100\tau\%$...

- expectiles = quantiles for a transformation of $F_Y$ [Jones (1994)]
- expectiles = quantiles in case of a weighted symmetric distribution [Abdous and Remillard (1995)]
- $\xi_\tau = q_{\tau'}$ for different levels $\tau$ and $\tau'$ [Yao and Tong (1996)]
Advantages of least asymmetrically weighted squares (LAWS) estimation:

- Computing expedience (though efficient linear programming algorithms are available for quantiles)

- Efficiency of the LAWS estimator:
  - Expectiles make more efficient use of the available data since they rely on the distance to observations
  - Quantiles only utilize the information on whether an observation is below or above the predictor!

- More valuable tail information:
  - Quantiles only depend on the frequency of tail realizations of \( Y \) and not on their values!
  - Expectiles depend on both the tail realizations and their probability [Kuan, Yeh and Hsu (2009)]

- Inference on expectiles is much easier than inference on quantiles: Calculation of the variance without going via the density of the distribution
(i) **Law invariance**: a distribution is uniquely defined by its class of expectiles

(ii) **Location and scale equivariance**: the $\tau$th expectile of the linear transformation $\tilde{Y} = a + bY$, where $a, b \in \mathbb{R}$, satisfies

$$
\xi_{\tilde{Y}, \tau} = \begin{cases} 
    a + b \xi_{Y, \tau} & \text{if } b > 0 \\
    a + b \xi_{Y, 1-\tau} & \text{if } b \leq 0
\end{cases}
$$

(iii) **Coherency**: for any variables $Y, \tilde{Y} \in L^1$ and for all $\tau \geq \frac{1}{2}$,

- **Translation invariance**: $\xi_{Y+a, \tau} = \xi_{Y, \tau} + a$, for all $a \in \mathbb{R}$
- **Positive homogeneity**: $\xi_{bY, \tau} = b\xi_{Y, \tau}$, for all $b \geq 0$
- **Monotonicity**: if $Y \leq \tilde{Y}$ with probability 1, then $\xi_{Y, \tau} \leq \xi_{\tilde{Y}, \tau}$
- **Subadditivity**: $\xi_{Y+\tilde{Y}, \tau} \leq \xi_{Y, \tau} + \xi_{\tilde{Y}, \tau}$
Theoretical and numerical results, obtained recently, indicate that expectiles are perfectly reasonable alternatives to quantiles as risk measures:

- Kuan, Yeh and Hsu (2009) [Journal of Econometrics]
- Gneiting (2011) [JASA]
- Bellini (2012) [Statistics and Probability Letters]
- Bellini, Klar, Müller and Gianina (2014) [Insurance : Mathematics and Economics]
- Bellini and Di Bernardino (2015) [The European Journal of Finance]
- Ziegel (2016) [Mathematical Finance]
- Ehm, Gneiting, Jordan and Krüger (2016) [JRSS-B]

The estimation of expectiles did not, however, receive yet any attention from the perspective of extreme values!

**Aim**: We use tail expectiles to estimate an alternative measures to Marginal Expected Shortfall (MES)
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Let $Y$ be the financial position and $F_Y$ be its cdf with $\overline{F}_Y = 1 - F_Y$ :
- non-negative loss variable
- real-valued variable (the negative of financial returns)

In statistical finance and actuarial science, Pareto-type distributions describe quite well the tail structure of losses:

$$F_Y(y) = 1 - \ell(y) \cdot y^{-1/\gamma}$$

- $\ell(\lambda y)/\ell(y) \to 1$ as $y \to \infty$ for all $\lambda > 0$
- $\gamma \in (0, 1)$ tunes the tail heaviness of $F_Y$

Only Bellini, Klar, Müller and Gianina (2014), Mao, Ng and Hu (2015) and Mao and Yang (2015) have described what happens for large $\xi_\tau$ and how it is linked to extreme $q_\tau$:

$$\frac{\overline{F}_Y(\xi_\tau)}{1 - \tau} = \frac{\overline{F}_Y(\xi_\tau)}{\overline{F}_Y(q_\tau)} \sim \gamma^{-1} - 1 \text{ as } \tau \to 1$$

- $\xi_\tau > q_\tau$ when $\gamma > \frac{1}{2}$,
- $\xi_\tau < q_\tau$ when $\gamma < \frac{1}{2}$, for all large $\tau$
Assume that the \textbf{tail quantile function} $U$ of $Y$, defined by

$$U(t) = \inf \left\{ y \in \mathbb{R} \left| \frac{1}{F_Y(y)} \geq t \right. \right\}, \quad \forall t > 1,$$

satisfies the second-order condition indexed by $(\gamma, \rho, A)$, that is, there exist $\gamma > 0$, $\rho \leq 0$, and a function $A(\cdot)$ converging to 0 at infinity and having constant sign such that:

$$C_2(\gamma, \rho, A) \quad \text{for all } x > 0,$$

$$\lim_{t \to \infty} \frac{1}{A(t)} \left[ \frac{U(tx)}{U(t)} - x^\gamma \right] = x^\gamma \frac{x^\rho - 1}{\rho}.$$ 

Here, $(x^\rho - 1)/\rho$ is to be understood as $\log x$ when $\rho = 0$. 
The precise bias term in the asymptotic approximation of \((\xi_\tau/q_\tau)\):

**Proposition**

Assume that condition \(C_2(\gamma, \rho, A)\) holds, with \(0 < \gamma < 1\). If \(F_Y\) is strictly increasing, then

\[
\frac{\xi_\tau}{q_\tau} = (\gamma^{-1} - 1)^{-\gamma}(1 + r(\tau))
\]

with

\[
r(\tau) = \frac{\gamma(\gamma^{-1} - 1)^{-\gamma}E(Y)}{q_\tau}(1 + o(1))
\]

\[
+ \left( \frac{(\gamma^{-1} - 1)^{-\rho}}{1 - \rho - \gamma} + \frac{(\gamma^{-1} - 1)^{-\rho} - 1}{\rho} + o(1) \right) A((1 - \tau)^{-1})
\]

as \(\tau \uparrow 1\)

Other similar refinements can be found in Mao *et al.* (2015), Mao and Yang (2015) and Bellini and Di Bernardino (2015)
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Conceptual results

- **Basic idea:**
  - We first estimate the intermediate expectiles of order $\tau_n \to 1$ such that $n(1 - \tau_n) \to \infty$;
  - We then extrapolate these estimates to the very extreme level $\tau'_n$ which approaches 1 at an arbitrarily fast rate in the sense that $n(1 - \tau'_n) \to c$, for some constant $c$.

- **Two estimation methods are considered:**
  - The first (indirect) is based on the use of the asymptotic connection between expectiles and quantiles;
  - The second relies directly on least asymmetrically weighted squares (LAWS) estimation.

- **Main results:** establish the asymptotic normality of the estimators
  - Intermediate expectiles $\xi_{\tau_n}$: indirect + direct
  - Extreme expectiles $\xi_{\tau'_n}$: indirect + direct
(1) **Estimation based on intermediate quantiles:**

- Assume that the available data consists of an \( n \)-tuple \((Y_1, \ldots, Y_n)\) of independent copies of \( Y \)
- Denote by \( Y_{1,n} \leq \cdots \leq Y_{n,n} \) their ascending order statistics
- Consider the intermediate expectile level \( \tau_n \to 1 \) such that \( n(1 - \tau_n) \to \infty \), as \( n \to \infty \)
- The asymptotic connection above entails that
  \[
  \frac{\xi_{\tau_n}}{q_{\tau_n}} \sim (\gamma^{-1} - 1)^{-\gamma} \quad \text{as} \quad n \to \infty
  \]

- Define, for a suitable estimator \( \hat{\gamma} \) of \( \gamma \),
  \[
  \hat{\xi}_{\tau_n} := (\hat{\gamma}^{-1} - 1)^{-\hat{\gamma}} \cdot \hat{q}_{\tau_n}
  \]

  where
  \[
  \hat{q}_{\tau_n} := Y_{n - \lfloor n(1-\tau_n) \rfloor, n}
  \]

  with \( \lfloor \cdot \rfloor \) being the floor function.
(2) Asymmetric Least Squares (direct) estimation:

We consider estimating

$$\hat{\xi}_{\tau_n} = \arg\min_{u \in \mathbb{R}} \mathbb{E} [\eta_{\tau_n} (Y - u)]$$

by

$$\tilde{\xi}_{\tau_n} = \arg\min_{u \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} \eta_{\tau_n} (Y_i - u)$$

where $$\eta_{\tau}(y) = |\tau - \mathbb{I}\{y \leq 0\}|y^2$$ is the expectile check function.
Consider the intermediate expectile level $\tau_n \to 1$ such that 
$n(1 - \tau_n) \to \infty$, as $n \to \infty$

Consider the **extreme** expectile level $\tau'_n \to 1$ such that 
$n(1 - \tau'_n) \to c < \infty$, as $n \to \infty$

The model assumption of Pareto-type tails 

$$F_Y(y) = 1 - \ell(y) \cdot y^{-1/\gamma}$$

suggests that 

$$\frac{\xi_{\tau'_n}}{\xi_{\tau_n}} \approx \left( \frac{1 - \tau'_n}{1 - \tau_n} \right)^{-\gamma}, \quad n \to \infty$$

This approximation motivates the estimators

$$\tilde{\xi}_{\tau'_n}^* := \left( \frac{1 - \tau'_n}{1 - \tau_n} \right)^{-\hat{\gamma}} \tilde{\xi}_{\tau_n}$$

$$\hat{\xi}_{\tau'_n}^* := \left( \frac{1 - \tau'_n}{1 - \tau_n} \right)^{-\hat{\gamma}} \hat{\xi}_{\tau_n} \equiv (\hat{\gamma}^{-1} - 1)^{-\hat{\gamma}} \hat{q}_{\tau'_n}^*$$
Extreme expectile estimation (cont.)

\[
\hat{\xi}_{\tau_n}^* := \left( \frac{1 - \tau'}{1 - \tau_n} \right)^{-\gamma} \hat{\xi}_{\tau_n} \equiv (\hat{\gamma}^{-1} - 1)^{-\gamma} \hat{q}_{\tau_n}^*
\]

**Theorem**

Assume that \(F_Y\) is strictly increasing, that condition \(C_2(\gamma, \rho, A)\) holds with \(0 < \gamma < 1\) and \(\rho < 0\), and that \(\tau_n, \tau_n' \uparrow 1\) with \(n(1 - \tau_n) \to \infty\) and \(n(1 - \tau_n') \to c < \infty\). Assume further that

\[
\sqrt{n(1 - \tau_n)} \left( \hat{\gamma} - \gamma, \frac{\hat{q}_{\tau_n} - 1}{q_{\tau_n}} \right) \xrightarrow{d} (\Gamma, \Theta).
\]

If \(\sqrt{n(1 - \tau_n)q_{\tau_n}^{-1}} \to \lambda_1 \in \mathbb{R},\ \sqrt{n(1 - \tau_n)}A((1 - \tau_n)^{-1}) \to \lambda_2 \in \mathbb{R}\) and
\(\sqrt{n(1 - \tau_n)}/ \log((1 - \tau_n)/(1 - \tau_n')) \to \infty\), then

\[
\frac{\sqrt{n(1 - \tau_n)}}{\log((1 - \tau_n)/(1 - \tau_n'))} \left( \frac{\hat{\xi}_{\tau_n}' - 1}{\hat{\xi}_{\tau_n}'} \right) \xrightarrow{d} \Gamma.
\]
Extreme expectile estimation (cont.)

\[ \tilde{\xi}_{\tau_n}^* := \left( \frac{1 - \tau_n'}{1 - \tau_n} \right)^{-\hat{\gamma}} \tilde{\xi}_{\tau_n} \]

Theorem

Assume that \( F_Y \) is strictly increasing, there is \( \delta > 0 \) such that \( \mathbb{E}|Y_-|^{2+\delta} < \infty \), condition \( C_2(\gamma, \rho, A) \) holds with \( 0 < \gamma < 1/2 \) and \( \rho < 0 \), and that \( \tau_n, \tau_n' \uparrow 1 \) with \( n(1 - \tau_n) \to \infty \) and \( n(1 - \tau_n') \to c < \infty \). If in addition

\[
\sqrt{n(1 - \tau_n)(\hat{\gamma} - \gamma)} \xrightarrow{d} \Gamma
\]

and \( \sqrt{n(1 - \tau_n)q^{-1}_{\tau_n}} \to \lambda_1 \in \mathbb{R} \), \( \sqrt{n(1 - \tau_n)}A((1 - \tau_n)^{-1}) \to \lambda_2 \in \mathbb{R} \) and

\[
\sqrt{n(1 - \tau_n)}/\log[(1 - \tau_n)/(1 - \tau_n')] \to \infty, \text{ then}
\]

\[
\sqrt{n(1 - \tau_n)} \frac{\tilde{\xi}_{\tau_n}^* - 1}{\log[(1 - \tau_n)/(1 - \tau_n')]} \xrightarrow{d} \Gamma
\]
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With the recent financial crisis and the rising interconnection between financial institutions, interest in the concept of systemic risk has grown;

Systemic risk: the propensity of a financial institution to be undercapitalized when the financial system as a whole is undercapitalized [Acharya et al. (2012), Brownlees and Engle (2012), Engle et al. (2015)];

Econometric approaches have been proposed to measure the systemic risk of financial institutions;

An important step in constructing a systemic risk measure for a financial institution is to measure the contribution of the institution to a systemic crisis;

Systemic crisis: a major stock market decline that happens once or twice a decade;

The total risk, measured by the expected capital shortfall in the financial system during a systemic crisis, can be decomposed into firm level contributions;

Firm level contributions can be measured by the Marginal expected shortfall = ?
Marginal expected shortfall

- $X$ : the loss return on the equity of a financial firm
- $Y$ : the loss return on the equity of the entire market
- Marginal expected shortfall :

$$\text{MES} = \mathbb{E}(X|Y > t)$$

where $t$ is a high threshold reflecting a substantial market decline

- MES at probability level $(1 - \tau)$ :

$$Q_{\text{MES}}(\tau) = \mathbb{E}\{X|Y > q_{Y,\tau}\}$$

- The estimation of $Q_{\text{MES}}(\tau)$
  - in Acharya et al. (2012) : relies on daily data from only 1 year and assumes a specific linear relationship between $X$ and $Y$;
  - in Brownlees and Engle (2012) and Engle et al. (2014) : a non-parametric kernel estimator was proposed;

  Cannot handle extreme events required for systemic risk measures, i.e.,

  $$1 - \tau = O(1/n)$$
Adapted extreme-value tools

Cai, Einmahl, de Haan & Zhou (2015) ⇔ Adapted tools for the estimation of

\[ \text{QMES}(\tau'_n) = \mathbb{E} \left\{ X | Y > q_{Y,\tau'_n} \right\} \]

\[ \approx \left( \frac{1 - \tau'_n}{1 - \tau_n} \right)^{-\gamma_X} \text{QMES}(\tau_n) \]

without recourse to any parametric structure on \((X, Y)\):

\[ \hat{\text{QMES}}^\star(\tau'_n) = \left( \frac{1 - \tau'_n}{1 - \tau_n} \right)^{-\gamma_X} \hat{\text{QMES}}(\tau_n) \]

where

\[ \hat{\text{QMES}}(\tau_n) = \frac{1}{n(1 - \tau_n)} \sum_{i=1}^{n} X_i \mathbb{1}\{X_i > 0, Y_i > \hat{q}_{Y,\tau_n}\}. \]

**Tail dependence condition**  \( J_{\mathcal{C}2}(R, \beta, \kappa) \): There exist \( R(\cdot, \cdot), \beta > \gamma_X \) and \( \kappa < 0 \) such that

\[
\sup_{\begin{subarray}{c} x \in (0, \infty) \\ y \in [1/2, 2] \end{subarray}} \left| \frac{t \mathbb{P}(F_X(X) \leq x/t, F_Y(Y) \leq y/t) - R(x, y)}{\min(x^\beta, 1)} \right| = O(t^\kappa) \quad \text{as} \quad t \to \infty
\]
Daouia, Girard & Stupfler (2016):

\[
\text{XMES}(\tau_n') = \mathbb{E} \{ X | Y > \xi_{Y,\tau_n'} \} \\
\approx \left( \frac{1 - \tau_n'}{1 - \tau_n} \right)^{-\gamma_X} \text{XMES}(\tau_n)
\]

with

\[
\text{XMES}(\tau_n) = \mathbb{E} \{ X | Y > \xi_{Y,\tau_n} \}
\]

(1) ALS type estimator:

\[
\text{XMES}^*(\tau_n') = \left( \frac{1 - \tau_n'}{1 - \tau_n} \right)^{-\tilde{\gamma}_X} \text{XMES}(\tau_n)
\]

where

\[
\text{XMES}(\tau_n) = \frac{\sum_{i=1}^{n} X_i \mathbb{I}\{X_i > 0, Y_i > \tilde{\xi}_{Y,\tau_n}\}}{\sum_{i=1}^{n} \mathbb{I}\{Y_i > \tilde{\xi}_{Y,\tau_n}\}}.
\]

If \( \sqrt{n(1 - \tau_n)} (\tilde{\gamma}_X - \gamma_X) \overset{d}{\rightarrow} \Gamma \), then

\[
\sqrt{n(1 - \tau_n) \log[(1 - \tau_n)/(1 - \tau_n')] \left( \frac{\text{XMES}^*(\tau_n')}{\text{XMES}(\tau_n')} - 1 \right) \overset{d}{\rightarrow} \Gamma
\]
Suppose for all \((x, y) \in [0, \infty]^2\) such that at least \(x\) or \(y\) is finite, the limit

\[
\lim_{t \to \infty} t \mathbb{P}(F_X(X) \leq x/t, F_Y(Y) \leq y/t) := R(x, y)
\]

exists.

Under this tail dependence condition:

\[
\lim_{\tau \uparrow 1} \frac{X_{\text{MES}}(\tau)}{Q_{\text{MES}}(\tau)} = (\gamma_Y^{-1} - 1)^{-\gamma_X}
\]

(2) Estimator based on tail QMES:

\[
\hat{X}_{\text{MES}}^*(\tau'_n) = (\hat{\gamma}_Y^{-1} - 1)^{-\hat{\gamma}_X} \hat{Q}_{\text{MES}}^*(\tau'_n).
\]

If \(\sqrt{n(1 - \tau_n)} (\hat{\gamma}_X - \gamma_X) \xrightarrow{d} \Gamma\), then

\[
\frac{\sqrt{n(1 - \tau_n)}}{\log[(1 - \tau_n)/(1 - \tau'_n)]} \left( \frac{\hat{X}_{\text{MES}}^*(\tau'_n)}{X_{\text{MES}}(\tau'_n)} - 1 \right) \xrightarrow{d} \Gamma
\]
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Expectile level selection

- **Aim**: choice of $\tau'_n = ?$ in the instruments of risk protection

  \[ \xi_{\tau'_n}, \quad \text{XMES}(\tau'_n) \]

- **In the case of**

  \[ q_{\alpha_n}, \quad \text{QMES}(\alpha_n) \]

  it is customary to choose

  \[ \alpha_n \to 1 \quad \text{with} \quad n(1 - \alpha_n) \to c < \infty \]

  to allow for more ‘prudent’ risk management $\rightsquigarrow$ Typical interest in once-in-a-decade or twice-per-decade events

- **Idea**: select $\tau'_n$ so that each expectile-based risk measure has the same intuitive interpretation as its quantile-based analogue

  \[ \uparrow \]

  choose $\tau'_n = \tau'_n(\alpha_n)$ s.t. $\xi_{\tau'_n} \equiv q_{\alpha_n}$ for a given relative frequency $\alpha_n$

  \[ \downarrow \]

  \[ \tau'_n(\alpha_n) = 1 - \frac{\mathbb{E} \{ |Y - q_{\alpha_n}| \mathbb{I}(Y > q_{\alpha_n}) \}}{\mathbb{E} |Y - q_{\alpha_n}|} \]
How to estimate

\[
\tau'_n(\alpha_n) = 1 - \frac{\mathbb{E}\{|Y - q_{\alpha_n}| \mathbb{I}(Y > q_{\alpha_n})\}}{\mathbb{E}|Y - q_{\alpha_n}|}
\]

Proposition

Under the model assumption of Pareto-type tails with \(0 < \gamma < 1\),

\[
1 - \tau'_n(\alpha_n) \sim (1 - \alpha_n) \frac{\gamma}{1 - \gamma}, \quad n \to \infty
\]

\[
\Rightarrow \hat{\tau}'_n(\alpha_n) = 1 - (1 - \alpha_n) \frac{\hat{\gamma}_n}{1 - \hat{\gamma}_n}
\]
VaR estimation

With

$$\tau_n' = \tilde{\tau}_n'(\alpha_n) \equiv 1 - (1 - \alpha_n) \frac{\tilde{\gamma}_n}{1 - \tilde{\gamma}_n}$$

both extreme expectile estimators

$$\hat{\xi}_{\tau_n'} = \left( \frac{1 - \tau_n'}{1 - \tau_n} \right)^{-\gamma} \hat{\xi}_{\tau_n}$$

$$\hat{\xi}_{\tau_n'} = \left( \frac{1 - \tau_n'}{1 - \tau_n} \right)^{-\gamma} \hat{\xi}_{\tau_n} = (\hat{\gamma}^{-1} - 1)^{-\gamma} \hat{q}_{\tau_n'}$$

estimate the same VaR $\xi_{\tau_n'(\alpha_n)} \equiv q_{\alpha_n}$ as

$$\hat{q}_{\alpha_n} := \left( \frac{1 - \alpha_n}{1 - \tau_n} \right)^{-\gamma} \hat{q}_{\tau_n}$$

- $\hat{\xi}_{\tau_n'(\alpha_n)} \equiv \hat{q}_{\alpha_n}$
- If $\sqrt{n(1 - \tau_n)(\hat{\gamma}_n - \gamma)} \xrightarrow{d} \Gamma$, then

$$\sqrt{n(1 - \tau_n)} \left( \frac{\hat{\xi}_{\tau_n'(\alpha_n)}}{q_{\alpha_n}} - 1 \right) \xrightarrow{d} \Gamma$$
MES estimation

With

\[ \tau'_n = \hat{\tau}'_n(\alpha_n) \equiv 1 - (1 - \alpha_n) \frac{\hat{\gamma}_n}{1 - \hat{\gamma}_n} \]

both estimators

\[ \hat{\text{XMES}}^*(\tau'_n) = \left( \frac{1 - \tau'_n}{1 - \tau_n} \right)^{-\hat{\gamma}_X} \hat{\text{XMES}}(\tau_n) \]

\[ \hat{\text{XMES}}^*(\tau'_n) = (\hat{\gamma}_Y - 1)^{-\hat{\gamma}_X} \hat{\text{QMES}}^*(\tau'_n) \]

estimate the same MES \( \hat{\text{XMES}}(\tau'_n(\alpha_n)) \equiv \hat{\text{QMES}}(\alpha_n) \) as Cai et al. (2015)'s estimator

\[ \hat{\text{QMES}}^*(\alpha_n) = \left( \frac{1 - \alpha_n}{1 - \tau_n} \right)^{-\hat{\gamma}_X} \hat{\text{QMES}}(\tau_n) \]

- \( \hat{\text{XMES}}^*(\hat{\tau}'_n(\alpha_n)) \equiv \hat{\text{QMES}}^*(\alpha_n) \)
- If \( \sqrt{n(1 - \tau_n)(\hat{\gamma}_X - \gamma_X)} \xrightarrow{d} \Gamma \), then

\[ \frac{\sqrt{n(1 - \tau_n)}}{\log[(1 - \tau_n)/(1 - \alpha_n)]} \left( \frac{\hat{\text{XMES}}^*(\hat{\tau}'_n(\alpha_n))}{\hat{\text{QMES}}(\alpha_n)} - 1 \right) \xrightarrow{d} \Gamma \]
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For the three banks, the dataset consists of the loss returns ($X_i$) on their equity prices at a daily frequency from July 3rd, 2000, to June 30th, 2010 (ten years).

We follow the same set-up as in Cai et al. (2015) to extract, for the same time period, daily loss returns ($Y_i$) of a value-weighted market index aggregating three markets:

- New York Stock Exchange
- American Express stock exchange
- National Association of Securities Dealers Automated Quotation system

The interest is on $\hat{Q}_{\text{MES}}(\alpha_n)$ and $\hat{X}_{\text{MES}}(\tau'_n(\alpha_n))$ that estimate $Q_{\text{MES}}(\alpha_n) \equiv X_{\text{MES}}(\tau'_n(\alpha_n))$ with $\alpha_n = 1 - 1/n$.

They represent the average daily loss return for a once-per-decade market crisis ($n = 2513$).
Plots of the Hill estimates

- $\hat{\gamma}_Y$ based on daily loss returns of market index
- $\hat{\gamma}_X$ based on daily loss returns of Goldman Sachs, Morgan Stanley, T. Rowe Price

$\Downarrow$

$\gamma_X, \gamma_Y < 1/2$
QMES* (black) & XMES* (rainbow)
The final MES estimates

<table>
<thead>
<tr>
<th>Bank</th>
<th>( \text{( \tilde{\text{X}}_{\text{M}} )}(\tau_n'(\alpha_n)) )</th>
<th>( \text{( \tilde{\text{Q}}_{\text{M}} )}(\alpha_n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Goldman Sachs</td>
<td>0.286</td>
<td>0.280</td>
</tr>
<tr>
<td>Morgan Stanley</td>
<td>0.485</td>
<td>0.471</td>
</tr>
<tr>
<td>T. Rowe Price</td>
<td>0.297</td>
<td>0.279</td>
</tr>
</tbody>
</table>

The final estimates based on averaging the estimates from the first stable regions of the plots:

- The quantile-based estimates are less conservative than our ALS-based estimates, but not by much.
- MES levels for **Morgan Stanley** are largely higher than those for **Goldman Sachs** and **T. Rowe Price**.