# Estimation methods for Weibull tail-distributions

Stéphane Girard

Inria Grenoble Rhône-Alpes & LJK (team MISTIS).

655, avenue de l'Europe, Montbonnot. 38334 Saint-Ismier Cedex, France

Stephane.Girard@inria.fr

**Abstract**

This note summarizes my contributions to the design of extreme-value methods dedicated to Weibull tail-distributions. In this framework, the survival function decreases to zero at an exponential rate. I focused on the estimation of extreme quantiles as well as on the estimation of the so-called Weibull tail-index.

## 1 Extreme-value analysis

Extreme value theory is a branch of statistics dealing with the extreme deviations from the bulk of probability distributions. More specifically, it focuses on the limiting distributions for the minimum or the maximum of a large collection of random observations from the same arbitrary (unknown) distribution. Let $x_1 < \cdots < x_n$ denote $n$ ordered observations from a random variable $X$ representing some quantity of interest. A $p_n$-quantile of $X$ is the value $q_{p_n}$ such that the probability that $X$ is greater than $q_{p_n}$ is $p_n$, i.e. $P(X > q_{p_n}) = p_n$. When $p_n < 1/n$, such a quantile is said to be extreme since it is usually greater than the maximum observation $x_n$. To estimate such extreme quantiles requires therefore specific methods to extrapolate information beyond the observed values of X. Those methods are based on Extreme value theory. This kind of issues appeared in hydrology. One objective was to assess risk for highly unusual events, such as 100-year floods, starting from flows measured over 50 years.

The decay of the survival function $P(X > x) = 1 - F(x)$, where $F$ denotes the cumulative distribution function associated to $X$, is driven by a real parameter called the extreme-value index $\gamma$. When this parameter is positive, the survival function is said to be heavy-tailed, when this parameter is negative, the survival function vanishes above its right end point. If this parameter is zero, then the survival function decreases to zero at an exponential rate. An important part of our work is dedicated to the study of such distributions. For instance, in reliability, the distributions of interest are included in a semi-parametric family whose tails are decreasing exponentially fast.

## 2 Weibull tail-distributions

Let us recall that a cumulative distribution function $F$ has a Weibull tail if it satisfies the following property: There exists $\theta > 0$ such that for all $\lambda > 0$,

$$\lim_{y \to \infty} \frac{\log(1 - F(\lambda y))}{\log(1 - F(y))} = \lambda^{1/\theta}.$$

These so-called Weibull tail-distributions encompass a variety of light-tailed distributions, such as Weibull, Gaussian, gamma and logistic distributions. The estimation of $\theta$, referred to as the Weibull tail-index, is investigated in [1, 2, 3], while the estimation of the associated extreme quantiles have been studied in [4]. Bias reduced estimators are proposed in [5, 6]. We also refer to [7] for a review on this topic and to [8] for the conditional case where a covariate is recorded simultaneously with the quantity of interest. Finally, some links are established between Weibull tail-distributions and heavy tailed distributions (associated with a positive extreme-value index) in [9, 10].

## References


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