Contributions to copula modeling

Stéphane Girard

INRIA Rhône-Alpes & LJK (team MISTIS).

655, avenue de l'Europe, Montbonnot. 38334 Saint-Ismier Cedex, France

Stephane.Girard@inria.fr

Abstract

This report summarizes my contributions to copulas modeling. Two main research topics are addressed: The construction of semiparametric family of copulas based on a set of orthonormal functions and a matrix and the design of efficient estimation procedures.

1 Introduction

A bivariate copula defined on the unit square $[0,1]^2$ is a bivariate cumulative distribution function (cdf) with univariate uniform margins. Sklar's Theorem [22] states that any bivariate distribution with cdf H and marginal cdf F and G can be written $H(x,y) = C(F(x), G(y))$, where C is a copula. This result justifies the use of copulas for building bivariate distributions.

2 Contributions

One of the most popular parametric family of copulas is the Farlie-Gumbel-Morgenstern (FGM) family [9, 12, 18] defined when $\theta \in [-1,1]$ by

$$C(u,v) = uv + \theta u(1-u)v(1-v). \tag{1}$$

A well-known limitation to this family is that it does not allow the modeling of large dependences since the associated Spearman's Rho is limited to $[-1/3, 1/3]$. A possible extension of the FGM family is to consider the semi-parametric family of symmetric copulas defined by

$$C(u,v) = uv + \theta\varphi(u)\varphi(v), \tag{2}$$

with $\theta \in [-1,1]$. It was first introduced in [19], and extensively studied in [1, 2, 3]. In particular, it can be shown that, for a properly chosen function φ, the range of Spearman's Rho is extended to $[-3/4, 3/4]$. In [4] an extension of (2) is proposed where θ is a univariate function. This modification allows the introduction of a singular component concentrated on the diagonal $v = u$ and extends the range of Spearman's Rho to $[-3/4, 1]$. We also refer to [6, 8] for another yet similar extensions.

In [5], a new extension of (2) is proposed where, roughly speaking, the single parameter θ is replaced by a matrix and the function φ is replaced by a set of functions. This new copula permits to reach values of Spearman's Rho arbitrarily close to 1 without singular component. Moreover, it also encompasses copulas based on partition of unity such as Bernstein copula [21] or checkerboard copula [13, 14]. Finally, it is also shown that projection of arbitrary densities of copulas onto tensor product bases can enter our framework.

Given a parametric family of copulas, the inference of the parameter vector commonly relies on likelihood-based methods. However, for some copula families, the likelihood may not exist, or may lead to slow or complex numerical optimization procedures. Therefore, it is desirable to consider alternative estimation strategies. A natural approach is to build the inference on bivariate dependence coefficients, where the weighted sum of the squared residuals between the dependence coefficients under the model and their empirical counterparts is minimized. This method has already been used in some applications but in a rather heuristic way. The asymptotic properties of the resulting estimator have not been investigated yet. In [15], we derive the consistency and asymptotic normality of the weighted least square estimator based on three standard dependence coefficients. Finally we illustrate how our results can be used to address three statistical questions.

While there exist various families of bivariate copulas, the construction of flexible and yet tractable copulas suitable for high-dimensional applications is much more challenging. This is even more true if one is concerned with the analysis of extreme values. In [17], we construct a class of one-factor copulas and a family of extreme-value copulas well suited for high-dimensional applications and exhibiting a good balance between tractability and flexibility. The inference for these copulas is performed by using a least-squares estimator based on dependence coefficients. The modeling capabilities of the copulas are illustrated on simulated and real datasets. This class of copula is extended in [7]. In [16], we propose a class of multivariate copulas based on products of transformed bivariate copulas. No constraints on the parameters refrain the applicability of the proposed class. Furthermore the analytical forms of the copulas within this class allow to naturally associate a graphical structure which helps to visualize the dependencies and to compute the likelihood efficiently even in high dimension.

Finally, the tail copula is widely used to describe the dependence in the tail of multivariate distributions. In some situations such as risk management, the dependence structure may be linked with some covariate. The tail copula thus depends on this covariate and is referred to as the conditional tail copula. The aim of [11] is to propose a nonparametric estimator of the conditional tail copula and to establish its asymptotic normality. Some illustrations are presented both on simulated and real datasets.

References

[1] Amblard, C. and Girard, S., 2001. A semiparametric family of symmetric bivariate copulas, *Comptes-Rendus de l'Académie des Sciences*, t. 333, Série I:129–132.

[2] Amblard, C. and Girard, S., 2002. Symmetry and dependence properties within a semiparametric family of bivariate copulas. *Journal of Nonparametric Statistics*, **14**, 715–727.

[3] Amblard, C. and Girard, S., 2005. Estimation procedures for a semiparametric family of bivariate copulas. *Journal of Computational and Graphical Statistics*, **14**, 1–15.

[4] Amblard, C. and Girard, S., 2009. A new extension of bivariate FGM copulas. *Metrika*, **70**, 1–17.

[5] Amblard, C., Girard, S. and Menneteau, L., 2014. Bivariate copulas defined from matrices. *submitted*, `http://hal.archives-ouvertes.fr/hal-00875303`

[6] Durante, F., 2006. A new class of symmetric bivariate copulas. *Journal of Nonparametric Statistics*, **18**, 499–510.

[7] Durante, F., Girard, S. and Mazo, G., 2015. Copulas based on Marshall-Olkin machinery, In U. Cherubini et al,editors, *Marshall-Olkin Distributions. Advances in Theory and Applications*, volume 141 of Springer Proceedings in Mathematics and Statistics, pages 15–31, Springer.

[8] Durante, F., Kolesárová, A., Mesiar, R. and Sempi, C., 2008. Semilinear copulas. *Fuzzy Sets and Systems*, **159**, 63–76.

[9] Farlie, D. G. J., 1960. The performance of some correlation coefficients for a general bivariate distribution. *Biometrika*, **47**, 307–323.

[10] Fischer, M. and Klein, I., 2007. Constructing symmetric generalized FGM copulas by means of certain univariate distributions, *Metrika*, **65**, 243–260.

[11] Gardes, L. and Girard, S., 2015. Nonparametric estimation of the conditional tail copula. *Journal of Multivariate Analysis*, to appear.

[12] Gumbel, E. J., 1960. Bivariate exponential distributions. *Journal of the American Statistical Association*, **55**, 698–707.

[13] Li X., Mikusiński, P., Sherwood, H. and Taylor, M. D., 1997. On approximation of copulas. In: Beneš V, Štěpàn J (eds) *Distributions with given marginals and moment problems.* Kluwer Academic Publishers, Dordrecht.

[14] Li X., Mikusiński, P. and Taylor, M. D., 1998. Strong approximation of copulas. *Journal of Mathematical Analysis and Applications*, **225**, 608–623.

[15] Mazo, G., Girard, S. and Forbes, F., 2014. Weighted least square inference based on dependence coefficients for multivariate copulas. *submitted*, `http://hal.archives-ouvertes.fr/hal-00979151`

[16] Mazo, G., Girard, S. and Forbes, F., 2014. A class of multivariate copulas based on products of bivariate copulas. *submitted*, `http://hal.archives-ouvertes.fr/hal-00910775`

[17] Mazo, G., Girard, S. and Forbes, F., 2015. A flexible and tractable class of one-factor copulas. *Statistics and Computing*, to appear.

[18] Morgenstern, D., 1956. Einfache Beispiele zweidimensionaler Verteilungen. *Mitteilungsblatt für Mathematische Statistik*, **8**, 234–235.

[19] Rodríguez-Lallena, J. A., 1992. *Estudio de la compabilidad y diseño de nuevas familias en la teoria de cópulas. Aplicaciones.* Tesis doctoral, Universidad de Granada.

[20] Rodríguez-Lallena, J. A. and Úbeda-Flores, M., 2004. A new class of bivariate copulas. *Statistics and Probability Letters*, **9**, 315–325.

[21] Sancetta, A. and Satchell, S., 2004. The Bernstein copula and its applications to modeling and approximations of multivariate distributions. *Econometric Theory*, **20**, 535–562,

[22] Sklar, A., 1959. Fonctions de répartition à n dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris*, **8**, 229–231.