Three different approaches to frontier estimation

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Joint work with A. Guillou (Université de Strasbourg, France), A. Iouditski (Université de Grenoble, France), L. Menneteau (Université de Montpellier, France), A. Nazin (Institute of Control Sciences, Moscow, Russia) and G. Stupfler (Université Aix-Marseille, France).

Outline

- Very brief overview of the literature:
  - First frontier estimator Geffroy (ISUP, 1964)
  - Piecewise polynomial estimators Härdle, Park, Tsybakov (JMVA, 1995)
- Extreme-value estimators,
- Linear programming estimators,
- High order moments estimators.

Framework

Let $(X_i,\ Y_i)$, $1 \le i \le n$ be n independent copies of a random pair $(X,\ Y)$ such that their common distribution has a support

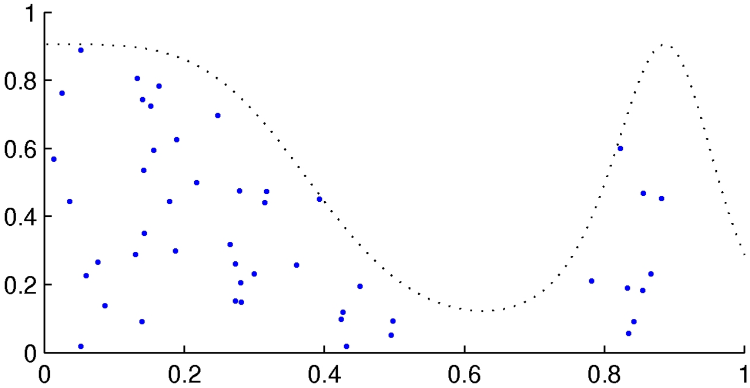
$$S := \{(x,\ y) \in \Omega \times \mathbb{R}\,;\, 0 \le y \le g(x)\}$$

where

- X has a density f_X on the compact subset $\Omega \subset \mathbb{R}^d$,
- $Y|X = x$ has a density $f(.|x)$ on $[0, g(x)]$,
- g is a positive function, $g(x) = \sup\{Y|X = x\}$.

We address the problem of the estimation of g, called the frontier of S.

Illustration $\Omega = [0, 1]$



Sharp/non-sharp boundaries

Härdle, Park, Tsybakov (JMVA, 1995) assumed that, for all $(x, y) \in S$,

- $f_X(x) \geq f_{\min} > 0$,
- $f(y|x) \geq c(g(x) - y)^\alpha$ where $c > 0$ and $\alpha \geq 0$.

Two cases arise:

- If $\alpha = 0$ then $f(y|x) \geq c > 0$ for all $y \in [0, g(x)]$, this is the situation of a "sharp boundary".
- If $\alpha > 0$ then we may have $f(y|x) \to 0$ as $y \to g(x)$, this is the situation of a "non-sharp boundary".

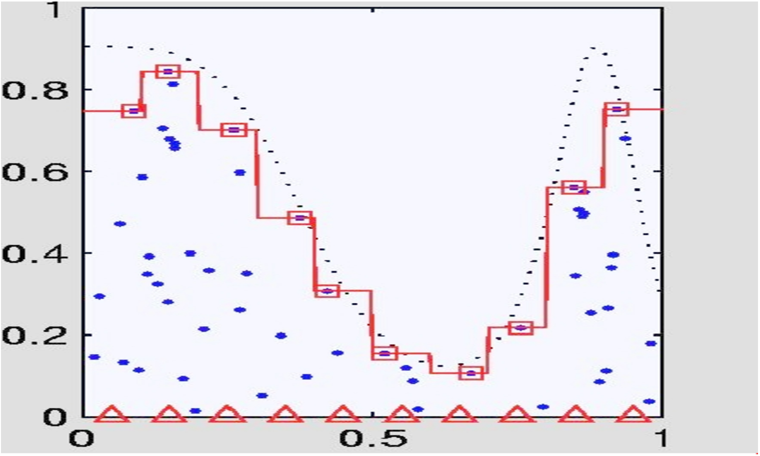
Geffroy's estimator

First frontier estimator Geffroy (ISUP, 1964), based on the extreme-values of the sample:

- Partition of $\Omega = [0,1]$ into equidistant k_n intervals $I_{n,r}$, $r = 1, \dots, k_n$,
- Maxima on each bin: $Y^*_{n,r} = \max\{Y_i : X_i \in I_{n,r}\}$,
- Piecewise constant estimator:

$$\hat{g}_n(x) = \sum_{r=1}^{k_n} \mathbb{I}\{x \in I_{n,r}\} Y^*_{n,r}.$$

Illustration: Geffroy's estimator



Geffroy's estimator

Asymptotic behaviour of the L_1- distance $\Delta_n := \int_0^1 |\hat{g}_n(x) - g(x)| dx$.

Theorem

Assume that g is $\gamma-$ Lipschitzian $\gamma \in (0, 1]$ and $\alpha = 0$ (sharp boundary). If some conditions on (k_n) hold, then $(n/k_n)(\Delta_n - \beta_n)$ converges in distribution to a Gumbel r.v. with c.d.f $\psi(z) = \exp(-\exp(-\theta z))$ where

$$\theta = \inf_{x \in [0,1]} f_X(x) f(g(x)|x),$$

and β_n is the solution of the equation

$$\int_0^1 \exp\left[\log k_n - \frac{n\beta_n}{k_n} f_X(x) f(g(x)|x)\right] dx = 1.$$

The rate of convergence (n/k_n) is (up to a logarithmic factor) $n^{\gamma/(1+\gamma)}$.

Piecewise polynomial estimators

Proposed in Härdle, Park, Tsybakov (JMVA, 1995) to deal with

- sharp or non-sharp boundaries ($\alpha \geq 0$),
- smoother frontiers, *i.e.* for $\gamma > 0$, it is assumed that the $\lfloor\gamma\rfloor$th derivative of the frontier g is $(\gamma - \lfloor\gamma\rfloor)-$ Lipschitzian.

The estimator requires a partition $I_{n,r}$, $r = 1, \ldots, k_n$ of $\Omega = [0,1]$. On the rth bin, the estimator is defined as the polynomial of degree $\lfloor\gamma\rfloor$ covering all the points and with smallest surface.

$$\hat{g}_n^\theta(x) = \sum_{r=1}^{k_n} \mathbb{I}\{x \in I_{n,r}\} P_{n,r}(x; \theta_{n,r}).$$

$$\theta_{n,r} = \arg\min_\theta \int_{I_{n,r}} P_{n,r}(x;\theta)dx \text{ s.t. } \quad P_{n,r}(X_i;\theta) \geq Y_i, \ \ X_i \in I_{n,r}.$$

Note that if $\gamma \in (0,1]$ then $\lfloor\gamma\rfloor = 0$ and we find back Geffroy's estimator.

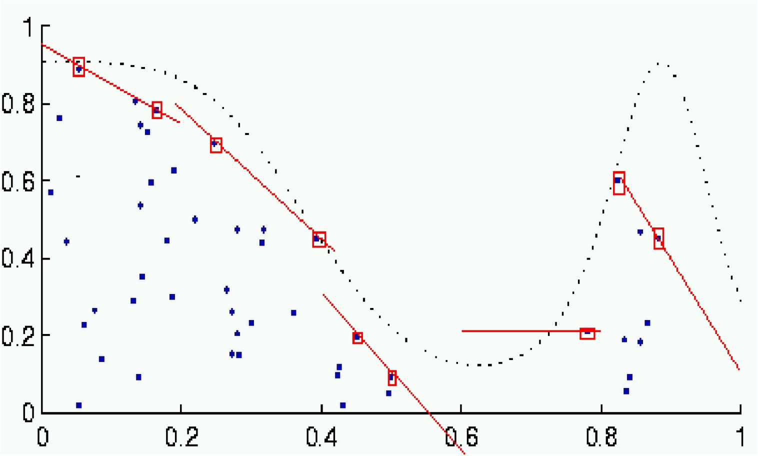
Piecewise polynomial estimators

Theorem

Under the above assumptions, and for a well chosen partition, piecewise polynomial estimators have the optimal rate of convergence for the L_1-*error, that is* $n^{\gamma/(1+(\alpha+1)\gamma)}$.

- In the case where $\alpha = 0$ (sharp boundary) and $\gamma \in (0, 1]$, Geffroy's estimator has the optimal rate of convergence.
- In practice, the estimators are biased downward and discontinuous. The choice of the partition (k_n) is also an issue.

Illustration: Piecewise linear estimator



Contributions

- Extreme-value estimator (smoothed, bias correction, sharp boundary, pointwise asymptotic normality)
- Linear programming estimator (smoothed, no partition of Ω, sharp boundary, strong L_1- consistency)
- High order moments estimator (smoothed, no partition of Ω, non-sharp boundary, pointwise asymptotic normality, strong $L_\infty-$ consistency)

Support $S = \{(x,\ y) \in \Omega \times \mathbb{R}\,;\ 0 \leq y \leq g(x)\}$ with $\Omega \subset \mathbb{R}^d$.

- **Geffroy's estimator.**

$$\hat{g}_n^{(0)}(x) = \sum_{r=1}^{k_n} \mathbb{I}\{x \in I_{n,r}\} Y^*_{n,r}.$$

  where $\{I_{n,r},\ \ r = 1, \dots, k_n\}$ is a partition of Ω and $Y^*_{n,r} = \max\{Y_i : X_i \in I_{n,r}\}$.

- **Bias correction.**
  Assume that $Y|X = x$ is uniformly distributed on $[0, g(x)]$ (sharp boundary).

$$\hat{g}_n^{(1)}(x) = \sum_{r=1}^{k_n} \mathbb{I}\{x \in I_{n,r}\} Y^*_{n,r}(1 + N^{-1}_{n,r}),$$

  where $N_{n,r}$ is the number of $X_i \in I_{n,r}$.

Extreme-value estimator

- **Smoothing**

$$\hat{g}_n^{(2)}(x) = \int_{\mathbb{R}^d} K_{h_n}(x-t)\hat{g}_n^{(1)}(t)dt$$

where $K_{h_n}(u) = h_n^{-d}K(u/h_n)$, K is $d-$ dimensional density with compact support and h_n is a smoothing parameter.

Nonparametric regression over the extreme-values of the sample:

$$\hat{g}_n^{(2)}(x) = \sum_{r=1}^{k_n} \int_{I_{n,r}} K_{h_n}(x-t)dt \; Y_{n,r}^*(1+N_{n,r}^{-1})$$

G & Menneteau (JSPI, 2005), Menneteau (ESAIM, 2008)

Theorem

Assume that g is $\gamma-$ Lipschitzian, $\gamma \in (0,1]$. Under some conditions on the (h_n) and (k_n) sequences, for all $(x_1,\dots,x_p) \subset \Omega$, the random vector

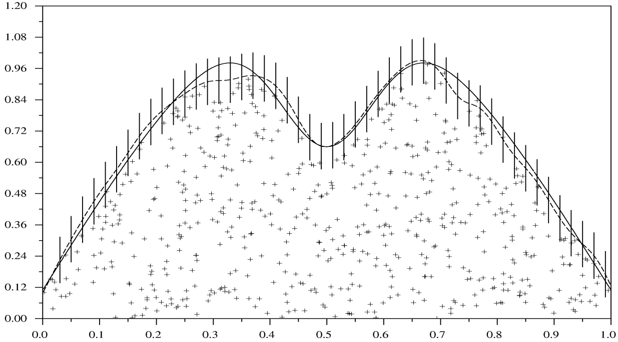
$$\left\{ nh_n^{d/2}k_n^{-1/2}\left(\hat{g}_n^{(2)}(x_j) - g(x_j)\right) : 1 \leq j \leq p \right\}$$

is asymptotically centred Gaussian with diagonal covariance matrix.

Extreme-value estimator

- Choosing $h_n \asymp n^{-1/(\gamma+d)}$ and $k_n \asymp n^{d/(\gamma+d)}$, the rate of convergence is $n^{\gamma/(d+\gamma)}$, up to logarithmic factors.
- Optimal L_1- rate of convergence for sharp boundaries ($\alpha = 0$) and $\gamma-$ Lipschitzian frontiers, $\gamma \in (0, 1]$.
- The rate of convergence of this extreme-value estimator is no more optimal for smoother frontier functions ($\gamma > 1$). The approximation of $g(x)$ by a constant value $Y^*_{n,r}$ for $x \in I_{n,r}$ is not precise enough.

Illustration: Extreme-value estimator



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- High order moments estimator (smoothed, no partition of Ω, non-sharp boundary, pointwise asymptotic normality, strong $L_\infty-$ consistency)

2. Linear programming estimator

Support $S = \{(x, y) \in [0,1] \times \mathbb{R}\,;\, 0 \le y \le g(x)\}$,
where g is $\gamma-$ Lipschitzian, $\gamma \in (0,1]$.
The estimator is a linear combination of kernel functions:

$$\hat{g}_n(x) = \sum_{i=1}^{n} \alpha_i K_{h_n}(x - X_i).$$

The coefficients $(\alpha_i)_{i=1,\dots,n}$ are obtained by **minimizing the surface** of the estimated support:

$$\min \int_{\mathbb{R}} \hat{g}_n(x)dx = \min \sum_{i=1}^{n} \alpha_i,$$

under the following **constraints**: for all $i = 1, \dots, n$

- $\hat{g}_n(X_i) \ge Y_i$ (the sample is below the estimated frontier)
- $\alpha_i \ge 0$ (the estimated frontier function is positive)
- $|\hat{g}_n'(X_i)| \le c_0 h_n^{\gamma-1}$ (Lipschitz constraint)

Linear Programming (LP) problem.

Remark 1. Assume that $Y|X = x$ is uniformly distributed on $[0, g(x)]$

- Joint distribution of the sample $\Sigma_n = (X_i, Y_i)_{i=1,\dots,n}$:

$$P(\Sigma_n \mid g) = \prod_{i=1}^{n} \frac{g(X_i)}{C_g} \cdot \frac{1}{g(X_i)} \mathbb{I}\{0 \le Y_i \le g(X_i)\},$$

with $C_g = \int_{\mathbb{R}} g(x)dx$.

- Log-likelihood. Since $C_{\hat{g}_n} = \sum_{i=1}^{n} \alpha_i$, we have

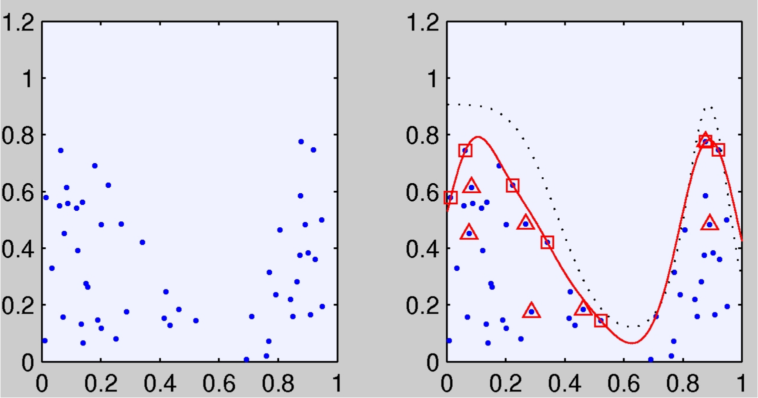
$$L(\alpha) = \log P(\Sigma_n \mid \hat{g}_n) = -n \log \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \log \mathbb{I}\{Y_i \le \hat{g}_n(X_i)\}.$$

The (LP) problem can be read as the maximization of the log-likelihood under the additional constraints $|\hat{g}_n'(X_i)| \le c_0 h_n^{\gamma-1}$, $i = 1, \dots, n$.

Illustration: Linear programming estimator

(LP): Linear optimisation problem under linear constraints.

- Efficient algorithms,
- The solution is sparse: only few $\alpha_i \neq 0$ (triangles), not the same points as $\hat{g}_n(X_i) = Y_i$ (squares).



Linear programming estimator

G, Iouditski & Nazin (ARC, 2005)

Theorem

Assume that g is $\gamma-$ Lipschitzian, $\gamma \in (0,1]$. Under some conditions on the (h_n) sequence (namely $h_n \asymp (\log n/n)^{1/(\gamma+1)}$)

$$\Delta_n := \int_0^1 |\hat{g}_n(x) - g(x)| dx = O\left(\left(\frac{\log n}{n}\right)^{\gamma/(1+\gamma)}\right),$$

almost surely.

- Optimal L_1- rate of convergence for sharp boundaries ($\alpha = 0$), $d = 1$ and $\gamma-$ Lipschitzian frontiers, $\gamma \in (0,1]$, up to the logarithmic factor.
- Extension to $\gamma > 1$ should be possible.

Linear programming estimator

Sketch of the proof: Lower bound.

Lemma

$$\hat{g}_n(x) \geq g(x) - O\left(h^{\gamma}\right) \; a.e.$$

Proof: There exists a.e. a point (X_i, Y_i) close to $(x, g(x))$ *i.e.* such that $|x - X_i| \leq c_1 h_n$ and $0 \leq g(X_i) - Y_i \leq c_2 h_n^{\gamma}$. Then,

$$\begin{aligned} g(x) - \hat{g}_n(x) &= [g(x) - g(X_i)] \\ &+ [g(X_i) - Y_i] \\ &+ [Y_i - \hat{g}_n(X_i)] \\ &+ [\hat{g}_n(X_i) - \hat{g}_n(x)]\,. \end{aligned}$$

The terms are respectively controlled:
i) $|g(x) - g(X_i)| \leq c_3 h_n^{\gamma}$: the frontier is $\gamma-$ Lipschitzian,
ii) $0 \leq g(X_i) - Y_i \leq c_2 h_n^{\gamma}$: choice of the point,
iii) $Y_i - \hat{g}_n(X_i) \leq 0$: the point is below the estimated frontier,
iv) $|\hat{g}_n(X_i) - \hat{g}_n(x)| \leq c_0 h_n^{\gamma-1} c_1 h_n$: Lipschitz constraint.

Sketch of the proof: Upper bound.

Lemma

There exist a solution $\tilde{g}_n$ to (LP) such that

$$\int_0^1 \tilde{g}_n(x)dx \leq \int_0^1 g(x)dx + c_4 h_n^{\gamma} \text{ a.e.}$$

Proof: The idea is to consider

$$\tilde{\alpha}_{i,n} = \int_{X_{i-1,n}}^{X_{i+1,n}} (g(x) + c_4 h_n^{\gamma})dx$$

and show that

$$\tilde{g}_n(x) = \sum_{i=1}^{n} \tilde{\alpha}_{i,n} K_{h_n}(x - X_{i,n})$$

satisfies the constraints of (LP).

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High order moments estimator

- Support $S = \{(x, y) \in \Omega \times \mathbb{R}\,;\, 0 \leq y \leq g(x)\}$ with $\Omega \subset \mathbb{R}^d$.
- Conditional survival function of Y given $X = x$

$$\overline{F}(y \,|\, x) = (1 - y/g(x))^{\alpha(x)+1}, \ \forall\, x \in \Omega, \ \forall\, y \in [0, g(x)],$$

  where $\alpha(x) \geq -1$ (sharp or non-sharp boundary).
- Conditional moments: $\forall\, p \geq 1$,

$$\mu_p(x) := \mathbb{E}(Y^p \,|\, X = x).$$

Then, for all $p \geq 1$ and $\theta > 1$,

$$\frac{1}{g(x)} = \frac{1}{(\theta - 1)p}\left[(\theta p + 1)\frac{\mu_{\theta p}(x)}{\mu_{\theta p+1}(x)} - (p+1)\frac{\mu_p(x)}{\mu_{p+1}(x)}\right].$$

High order moments estimator

1. Estimate $\mu_p(x)$ by a kernel estimator

$$\widehat{\mu}_p(x) := \sum_{i=1}^{n} Y_i^p \, K_{h_n}(x - X_i) \Bigg/ \sum_{i=1}^{n} K_{h_n}(x - X_i).$$

The bandwidth h_n selects the X_i's close to x.

2. To deal with the more general situation

$$\overline{F}(y \mid x) = (1 - y/g(x))^{\alpha(x)+1} \, \ell \left(x, (1 - y/g(x))^{-1}\right),$$

where $\ell(x, .)$ is a slowly-varying function at infinity, p is replaced with a sequence $p_n \to \infty$. The high power p_n gives more weight to the Y_i's close to $g(x)$.

High order moments estimator

We further assume a Hall model for the slowly-varying function: ℓ is supposed to be bounded on $\Omega \times [1, \infty)$ and

$$\ell(x, z) = C(x) + D(x)\, z^{-\beta(x)}\, (1 + \delta(x, z))$$

where all functions C, D and β are Lipschitzian. Moreover, for all $x \in \Omega$, $\delta(x, z) \to 0$ as $z \to \infty$.

Theorem

Let $x \in \Omega$ such that $f_X(x) > 0$. Then, under some conditions on the (h_n) and (p_n) sequences,

$$v_n(x) = n^{1/2}\, h_n^{d/2}\, p_n^{(1-\alpha(x))/2} \left(\frac{\hat{g}_n(x)}{g(x)} - 1 \right) \xrightarrow{d} \mathcal{N}\left(0, \frac{\|K\|_2^2\, V(\alpha(x), \theta)}{f_X(x)\, C(x)}\right)$$

G, Guillou & Stupfler (JMVA, 2013)

High order moments estimator

- In the case of $\gamma-$ Lipschizian frontier, choosing $h_n \asymp n^{-1/(\gamma(\alpha(x)+1)+d)}$ and $p_n \asymp n^{\gamma/(\gamma(\alpha(x)+1)+d)}$ yields a the rate of convergence is $n^{\gamma/(\gamma(\alpha(x)+1)+d)}$, up to logarithmic factors.
- Optimal L_1- rate of convergence for sharp/non-sharp boundaries $(\alpha(x) \geq 0)$ and $\gamma-$ Lipschitzian frontiers, $\gamma \in (0, 1]$.
- Compared to Härdle, Park, Tsybakov (JMVA, 1995), the case of "super-sharp" boundaries is also possible: $-1 < \alpha(x) < 0$. In this case, $f(y|x) \to \infty$ as $y \to g(x)$.

High order moments estimator

- The estimation of the conditional tail-index $\alpha(x)$ is possible with similar techniques:

$$\alpha_n(x) = (p_n + 1)\left(\hat{g}_n(x)\frac{\hat{\mu}_{p_n}(x)}{\hat{\mu}_{p_n+1}(x)} - 1\right)$$

- An uniform almost sure consistency result is also available G, Guillou & Stupfler (ESAIM, 2014):

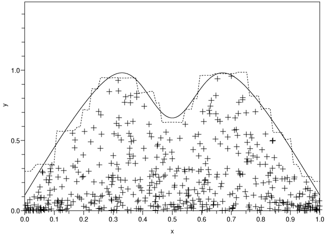
Theorem

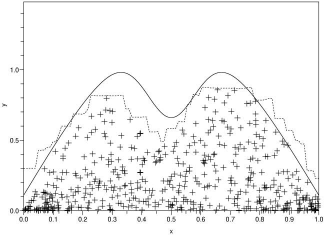
$$\sup_{x\in\Omega} |\hat{g}_n(x) - g(x)| = O\left(n^{-\gamma/(\gamma(\bar{\alpha}+1)+d)}\right),$$

where $\bar{\alpha} = \sup_{x\in\Omega} \alpha(x)$.

Illustration: High order moments estimator

$Y|X = x$ is beta distributed. Best (left) and worst (right) results obtained over 500 replications.





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Further work

- Arbitrary smoothness ($\gamma > 1$),
- Adaptive choice of the tuning parameters (bandwidth, ...).