Estimation of the second order parameter for heavy-tailed distributions

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Extreme value theory

Let $X_1, \dots, X_n$ be independent copies of a real random variable X with survival function $\bar{F} = 1 - F$. The order statistics associated to this sample are denoted by : $X_{1,n} \leq \cdots \leq X_{n,n}$.

Fréchet Maximum domain of attraction

The cumulative distribution function F belongs to the Fréchet maximum domain of attraction if and only if

$$\bar{F}(x) = x^{-1/\gamma}\ell(x),$$

where $\gamma > 0$ is the extreme-value index and ℓ is a slowly varying function *i.e.*

$$\frac{\ell(\lambda x)}{\ell(x)} \to 1 \text{ as } x \to \infty \text{ for all } \lambda \geq 1.$$

This condition is equivalent to $\bar{F}$ is regularly varying with index $-1/\gamma$ (heavy-tailed distribution).

The asymptotic distribution of estimators of γ is obtained under a second order condition.

Extreme value theory

Second order condition

There exist a function $A(x) \to 0$ and a second order parameter $\rho \le 0$ such that, for all $\lambda > 0$,

$$\lim_{x\to\infty} \frac{1}{A(x)} \log\left(\frac{\ell(\lambda x)}{\ell(x)}\right) = K_\rho(\lambda) := \int_1^\lambda u^{\rho-1} du.$$

- $|A|$ is regularly varying with index ρ.
- If ρ is small, the rate of convergence of $\ell(\lambda x)/\ell(x)$ to one is high (and conversely).
- ρ controls the bias of the estimators of γ.
- ρ is of primordial importance in the adaptative choice of k which is the number of upper order statistics $X_{n-k,+1,n} \le \cdots \le X_{n,n}$ used in the estimation of γ.
- A third order condition is needed to deal with the asymptotic distribution of ρ estimators.

Extreme value theory

Third order condition

There exist functions $A(x) \to 0$ and $B(x) \to 0$, a second order parameter $\rho < 0$ and a third order parameter $\beta < 0$ such that, for every $\lambda > 0$,

$$\lim_{x\to\infty} \frac{\left(\log \ell(\lambda x) - \log \ell(x)\right)/A(x) - K_\rho(\lambda)}{B(x)} = L_{(\rho,\beta)}(\lambda)$$

with

$$L_{(\rho,\beta)}(\lambda) = \int_1^\lambda s^{\rho-1} \int_1^s u^{\beta-1} du ds,$$

and where the functions $|A|$ and $|B|$ are regularly varying with index ρ and β respectively.

Contributions

- A new class of estimators for the second order parameter ρ,
- Asymptotic properties,
- Links with existing estimators,
- New estimators.

Definition of the family of estimators for the second order parameter

Model

The two main ingredients of our approach are

- a random vector $T_n = T_n(X_1, \ldots, X_n) \in \mathbb{R}^d$
- a function $\psi : \mathbb{R}^d \to \mathbb{R}$

verifying the following assumptions :

- There exist a random variable ω_n such that

$$\omega_n^{-1}(T_n - \mathbb{I}) \xrightarrow{\mathbb{P}} f(\rho),$$

  where $\mathbb{I} = {}^t(1, \ldots, 1) \in \mathbb{R}^d$.

- Invariance properties

$$\psi(x + \lambda\mathbb{I}) = \psi(x) \text{ and } \psi(\lambda x) = \psi(x)$$

  for all $x \in \mathbb{R}^d$ and $\lambda \in \mathbb{R} \setminus \{0\}$.

Definition of the family of estimators for the second order parameter

Idea

- Invariance (and regularity) properties entail

$$\psi(T_n) = \psi(\omega_n^{-1}(T_n - \mathbb{I})) \xrightarrow{\mathbb{P}} \psi(f(\rho))$$

- Letting $Z_n := \psi(T_n)$ and $\varphi := \psi \circ f$, one obtains $Z_n \xrightarrow{\mathbb{P}} \varphi(\rho)$.
- Suppose there exist $J_0 \subseteq \mathbb{R}^-$ and $J \subset \mathbb{R}$ such that φ is a bijection $J_0 \to J$.

Definition

The family of estimators of the second order parameter is thus defined by :

$$\hat{\rho}_n = \begin{cases} \varphi^{-1}(Z_n) & \text{if } Z_n \in J, \\ 0 & \text{otherwise.} \end{cases}$$

Asymptotic properties

Theorem

Under the invariance (and regularity) conditions,

- If $\omega_n^{-1}(T_n - \mathbb{I}) \xrightarrow{\mathbb{P}} f(\rho)$ then $\hat{\rho}_n \xrightarrow{\mathbb{P}} \rho$.
- If, moreover, $v_n(\omega_n^{-1}(T_n - \mathbb{I}) - f(\rho)) \xrightarrow{d} \mathcal{N}_d(m(\rho), \gamma^2 \Sigma)$ where $v_n \to \infty$, $m \in \mathbb{R}^d$ and Σ is a regular $d \times d$ matrix then

$$v_n(\hat{\rho}_n - \rho) \xrightarrow{d} \mathcal{N}\left(\frac{{}^t m \nabla\psi(f(\rho))}{\varphi'(\rho)}, \gamma^2 \frac{{}^t\nabla\psi(f(\rho))\ \Sigma\ \nabla\psi(f(\rho))}{(\varphi'(\rho))^2} \right).$$

Link with existing estimators

Existing estimators

In the literature, at least two ways of estimating the second order parameter can be found :

- Estimators based on rescaled log-spacings $j(\log X_{n-j+1,n} - \log X_{n-j,n})$, $j = 1, \dots, k$
  Hall & Welsh (Annals of Statistics, 1985),
  Goegebeur *et al.* (JSPI, 2010),
  De Wet *et al.* (SPL, 2012), ...
- Estimators based on log-excesses, $(\log X_{n-j+1,n} - \log X_{n-k,n})$, $j = 1, \dots, k$
  Gomes *et al.* (Extremes, 2002),
  Fraga-Alves *et al.* (Portugaliae Mathematica, 2003),
  Ciuperca & Mercadier (Extremes, 2010), ...

Link with existing estimators based on rescaled log-spacings

1. Estimators based on rescaled log-spacings : $j(\log X_{n-j+1} - \log X_{n-j})$

$$R_k(\tau) = \frac{1}{k}\sum_{j=1}^{k} H_\tau\left(\frac{j}{k+1}\right) j(\log X_{n-j+1,n} - \log X_{n-j,n}),$$

- H_τ is a kernel function indexed by a parameter $\tau > 0$.
- This statistics is used for instance by Beirlant *et al.* (Extremes, 1999) to estimate the extreme-value index γ and by Hall & Welsh (Annals of Statistics, 1985), Goegebeur *et al.* (JSPI, 2010), De Wet *et al.* (SPL, 2012) to estimate the second order parameter ρ.
- They proved asymptotic normality of these estimators under a technical condition on the kernel, denoted by **(C1)** hereafter.

Links with existing estimators based on rescaled log-spacings

Statistics T_n

Suppose the third order condition and **(C1)** hold. If the sequence k satisfies

$$k \to \infty,\ n/k \to \infty,\ k^{1/2}A(n/k) \to \infty,$$

$$k^{1/2}A^2(n/k) \to \lambda_A \text{ and } k^{1/2}A(n/k)B(n/k) \to \lambda_B,$$

then the random vector

$$T_n := \left((R_k(\tau_i)/\gamma)^{\theta_i},\ i = 1, \dots, d \right),$$

properly normalised in asymptotically Gaussian. More precisely,
$\omega_n = A(n/k)/\gamma(1 + o_{\mathbb{P}}(1))$, $v_n = k^{1/2}A(n/k)$ and

$$f(\rho) = \left(\theta_i \int_0^1 H_{\tau_i}(u) u^{-\rho} du,\ i = 1, \dots, d \right).$$

Link with existing estimators based on rescaled log-spacings

Statistics T_n

- Let $d = 8$, T_n depends on 16 parameters $\theta_1, \ldots, \theta_8, \tau_1, \ldots, \tau_8$.
- Suppose $\theta_1 = \theta_2$, $\theta_3 = \theta_4$, $\theta_5 = \theta_6$ and $\theta_7 = \theta_8$.

Function ψ

The chosen function ψ is given by :

$$\psi(x_1, \ldots, x_8) = \frac{x_1 - x_2}{x_3 - x_4} \left(\frac{x_7 - x_8}{x_5 - x_6} \right)^{(\theta_1 - \theta_3)/(\theta_5 - \theta_7)}$$ and thus

$$Z_n = \frac{R_k^{\theta_1}(\tau_1) - R_k^{\theta_1}(\tau_2)}{R_k^{\theta_3}(\tau_3) - R_k^{\theta_3}(\tau_4)} \left(\frac{R_k^{\theta_7}(\tau_7) - R_k^{\theta_7}(\tau_8)}{R_k^{\theta_5}(\tau_5) - R_k^{\theta_5}(\tau_6)} \right)^{(\theta_1 - \theta_3)/(\theta_5 - \theta_7)}$$

Asymptotic normality

In this situation, the estimator of ρ is asymptotically Gaussian.

- The estimator still depends on 12 parameters.
- The estimator is not necessarily explicit, the inverse of φ has to be computed numerically.

Examples

- Let $H_\tau(u) = (\tau+1)u^\tau$,
- To simplify, we assume that $\tau_2 = \tau_3$, $\tau_4 = \tau_8$ and $\tau_6 = \tau_7$. There are 9 remaining parameters and φ is given in this case by :

$$\varphi(\rho) = \text{cste} \left[\frac{\tau_4 - \rho}{\tau_1 - \rho}\right] \left[\frac{\tau_5 - \rho}{\tau_4 - \rho}\right]^{(\theta_1-\theta_3)/(\theta_5-\theta_7)}$$

Three explicit estimators can be derived :

- $\theta_1 - \theta_3 = \theta_5 - \theta_7$, Goegebeur *et al.* (JSPI, 2010), 8 free parameters,
- $\theta_1 = \theta_3$, new estimator, 8 free parameters,

$$\hat{\rho} = \frac{\tau_1 Z_n - \text{cste}\ \tau_4}{Z_n - \text{cste}}$$

- $\tau_1 = \tau_5$, new estimator, 8 free parameters,

$$\hat{\rho} = \frac{\tau_4 Z_n^{1/(\delta-1)} - \text{cste}^{1/(\delta-1)}\ \tau_1}{Z_n^{1/(\delta-1)} - \text{cste}^{1/(\delta-1)}}$$

Link with existing estimators based on log-excesses

2. Estimators based on log-excesses : $(\log X_{n-j+1,n} - \log X_{n-k,n})$

$$S_k(\tau,\alpha) = \frac{1}{k}\sum_{j=1}^{k} G_{\tau,\alpha}\left(\frac{j}{k+1}\right)(\log X_{n-j+1,n} - \log X_{n-k,n})^{\alpha}, \ \alpha > 0,$$

- $G_{\tau,\alpha}$ is a positive function.
- This statistics is used for instance by Dekkers *et al.* (Annals of statistics, 1989), Gomes & Martins (JSPI, 2001), Segers (JSPI, 2001) to estimate the extreme-value index γ and by Hall & Welsh (Annals of Statistics, 1985), Peng (SPL, 1998), Fraga *et al.* (MMS, 2003), Ciuperca & Mercadier (Extremes, 2010), to estimate the second order parameter ρ.
- They proved the asymptotic normality under a technical condition on the function $G_{\tau,\alpha}$, denoted by **(C2)** hereafter.

Links with existing estimators based on log-excesses

Statistics T_n

Suppose the third order condition and **(C2)** hold. If the sequence k satisfies

$$k \to \infty,\ n/k \to \infty,\ k^{1/2}A(n/k) \to \infty,$$

$$k^{1/2}A^2(n/k) \to \lambda_A \text{ and } k^{1/2}A(n/k)B(n/k) \to \lambda_B,$$

then the random vector

$$T_n = \left(\left(\frac{S_k(\tau_i, \alpha_i)}{\gamma^{\alpha_i}} \right)^{\theta_i},\ i = 1, \dots, d \right)$$

properly normalised in asymptotically Gaussian. More precisely, $\omega_n = A(n/k)/\gamma(1 + o_{\mathbb{P}}(1))$, $v_n = k^{1/2}A(n/k)$ and

$$f(\rho) = \left(-\theta_i \alpha_i \int_0^1 G_{\tau_i,\alpha_i}(u)(\log(1/u))^{\alpha_i - 1} K_{-\rho}(u)du,\ i = 1, \dots, d \right),$$

Link with existing estimators based on log-excesses

Statistics T_n

- Let $d = 8$, T_n depends on 24 parameters $\theta_1, \dots, \theta_8, \tau_1, \dots, \tau_8, \alpha_1, \dots, \alpha_8$.
- Suppose $\theta_1\alpha_1 = \theta_2\alpha_2$, $\theta_3\alpha_3 = \theta_4\alpha_4$, $\theta_5\alpha_5 = \theta_6\alpha_6$ and $\theta_7\alpha_7 = \theta_8\alpha_8$.

Function ψ

The chosen function ψ is given by :

$$\psi(x_1, \dots, x_8) = \frac{x_1 - x_2}{x_3 - x_4} \left(\frac{x_7 - x_8}{x_5 - x_6} \right)^{(\theta_1\alpha_1 - \theta_3\alpha_3)/(\theta_5\alpha_5 - \theta_7\alpha_7)}$$ and thus

$$Z_n = \frac{S_k^{\theta_1}(\tau_1, \alpha_1) - S_k^{\theta_2}(\tau_2, \alpha_2)}{S_k^{\theta_3}(\tau_3, \alpha_3) - S_k^{\theta_4}(\tau_4, \alpha_4)} \left(\frac{S_k^{\theta_7}(\tau_7, \alpha_7) - S_k^{\theta_8}(\tau_8, \alpha_8)}{S_k^{\theta_5}(\tau_5, \alpha_5) - S_k^{\theta_6}(\tau_6, \alpha_6)} \right)^{(\theta_1\alpha_1 - \theta_3\alpha_3)/(\theta_5\alpha_5 - \theta_7\alpha_7)}$$

Asymptotic normality

In this situation, the estimator of ρ is asymptotically Gaussian.

- The estimator still depends on 20 parameters.
- The estimator is not necessarily explicit, the inverse of φ has to be computed numerically.

Link with existing estimators based on log-excesses

Examples

- Let $G_{\tau,\alpha}(u) = (1 - u^\tau) / \int_0^1 (1 - x^\tau)(-\log x)^\alpha dx$,
- To simplify, we assume that $\tau_2 = \tau_3 = \tau_5 = \tau_6 = \tau_7 = \tau_8 = \alpha_7 = 1$, $\alpha_6 = 3$ and $\alpha_8 = 2$. There are 11 remaining parameters.

Three explicit estimators can be recovered :

- $\theta_1 = \theta_3$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$, $\tau_1 = 2$ and $\tau_4 = 3$, Ciuperca & Mercadier (Extremes, 2010), no free parameter
- $\theta_1 = \theta_3$, $\alpha_1 = \alpha_3 = \alpha_4 = 1$ and $\tau_1 = \tau_4 = \alpha_2 = 2$, Ciuperca & Mercadier (Extremes, 2010), no free parameter
- $\theta_1 = \theta_3 = \theta_6 = \theta_8 = \alpha_2 = \alpha_4 = \alpha_5 = \tau_1 = \tau_4 = 1$, $\alpha_3 = \theta_4 = \theta_7 = 2$, $\theta_5 = 3$ and $\alpha_1 = 4$, Gomes *et al.* (Extremes, 2002), no free parameter

Link with existing estimators based on log-excesses

Examples

Three new estimators can be built :

- $\theta_1 - \theta_2 = 2\theta_5 - \theta_7$, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$, $\tau_1 = \alpha_5 = 2$ and $\tau_4 = 3$, 3 free parameters

$$\hat{\rho} = \frac{6Z_n + 4 \text{ cste}}{3Z_n + 4 \text{ cste}}$$

- $\theta_1 - \theta_2 = 2\theta_5 - \theta_7$, $\alpha_1 = \alpha_3 = \alpha_4 = 1$ and $\tau_1 = \tau_4 = \alpha_2 = \alpha_5 = 2$, 3 free parameters

$$\hat{\rho} = \frac{6Z_n - 4 \text{ cste}}{2Z_n - 1 \text{ cste}}$$

- $\tau_1 = \tau_4 = \alpha_1 = 1$, $\alpha_2 = \alpha_3 = \alpha_5 = 2$ and $\alpha_4 = 3$, 4 free parameters

$$\hat{\rho} = \frac{3Z_n^{1/(\delta+1)} - \text{cste}^{1/(\delta+1)}}{Z_n^{1/(\delta+1)} - \text{cste}^{1/(\delta+1)}}$$

If $\delta = 0$, we find back the estimator introduced in Fraga-Alves *et al.* (Portugaliae Mathematica, 2003)

Illustration on simulations

Estimators based on rescaled log-spacings

- $H_{\tau_i}(u) = (\tau_i + 1)u^{\tau_i}, \; i = 1, \dots, 8$
- $\tau_1, \dots, \tau_8$ and $\theta_1, \theta_2, \theta_3, \theta_5, \dots, \theta_8$ chosen as in Goegebeur *et al.* (JSPI, 2010) and De Wet *et al.* (SPL, 2012) : $\tau_1 = 1.25$, $\tau_2 = \tau_3 = 1.75$, $\tau_4 = \tau_8 = 2$, $\tau_5 = 1.5$, $\tau_6 = \tau_7 = 1.75$, $\theta_1 = \theta_2 = 0.01$, $\theta_5 = \theta_6 = 0.02$ and $\theta_7 = \theta_8 = 0.04$.
- $\theta_3 = \theta_4 = 0.01 + 0.02\delta$ for $\delta \geq 0$.

A simple expression if obtained for φ :

$$\varphi(\rho) = \text{cste} \left[\frac{2-\rho}{1.25-\rho}\right] \left[\frac{1.5-\rho}{2-\rho}\right]^{\delta}$$

Its inverse is explicit when $\delta = 0$ (new explicit estimator) or $\delta = 1$ (Goegebeur *et al.* (JSPI, 2010)). We shall also consider the case $\delta = 1.5$ which can be shown to be in some sense "optimal" when $\rho = 0$ (new implicit estimator) .

Illustration on simulations

Burr distribution

Survival distribution function :

$$1 - F(x) = (1 + x^{-\rho})^{1/\rho}$$

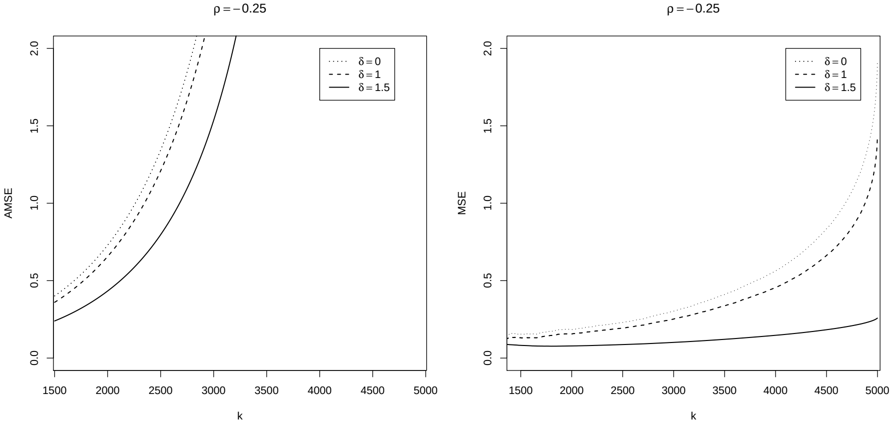
with $x \geq 0$ and $\rho < 0$.

- Extreme-value index $\gamma = 1$, second order parameter $\rho < 0$.
- The third order condition holds with $\beta = \rho$, $A(x) = \gamma x^{\rho}/(1 - x^{\rho})$ and $B(x) = \rho x^{\rho}/(1 - x^{\rho})$.

Experimental design

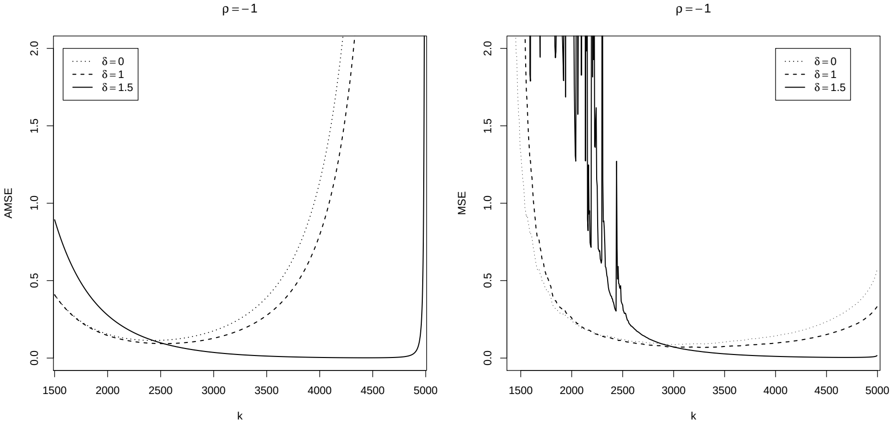
- Sample size $n = 5000$, 500 replications.
- Intermediate sequence $k = 1500, \ldots, 4995$.
- Second order parameter $\rho = -0.25$ and $\rho = -1$.

Asymptotic mean-squared error & empirical mean-squared error



Left : Asymptotic mean-squared error, Right : empirical mean-squared error.

Asymptotic mean-squared error & mean-squared error



Left : Asymptotic mean-squared error, Right : empirical mean-squared error.

Conclusion

+ General framework for building estimators of the second order parameter.
+ Asymptotic normality of the estimators is direclty derived from the asymptotic behavior of rescaled log-spacings or log-excesses.
+ Efficient tool for studying existing estimators or defining new ones.
– But ... How to compare in practice estimators depending on so many parameters ?

E. Deme, L. Gardes and S. Girard. On the estimation of the second order parameter for heavy-tailed distributions, *REVSTAT - Statistical Journal*, **11**, 277–299, 2013.