A semiparametric family of bivariate copulas: dependence properties and estimation procedures

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Outline

1. Definition and basic properties.

2. First sub-family, the case $\phi(1) = 0$.

3. Second sub-family, the case $\theta(1) = 0$.

4. Inference procedures.

5. Simulation results.

6. Real data.
Definition. Let $I$ be the unit interval. The family is defined for all $(u,v) \in I^2$ by,

$$C_{\theta,\phi}(u,v) = uv + \theta \max(u,v) \phi(u) \phi(v).$$

where $\phi$ and $\theta$ are differentiable $I \to I$ functions (vanishing at most on isolated points).

Theorem. Let $I$ be the unit interval. The family is defined for all $(u,n) \in I \times \mathbb{R}$. If

$$0 = (1)\phi(\phi^{-1}(n)) \phi([\max(u,n)] \theta + vn = (u,n)\phi(\theta)$$

for all $u \in I$. Then the family is a copula if and only if $\phi$ and $\theta$ satisfy the following conditions:

- boundary conditions: $\phi(0) = 0$ and $(\phi(\theta))^2 = 0$,
- $\phi$ is non-increasing on $I$.
- $\theta$ is non-increasing on $I$.
- $\phi'$ is a copula if and only if $\phi$ and $\theta$ satisfy the following conditions:

Remark. The family can be split in two sub-families according to $\theta(1) = 0$ or $\phi(1) = 0$.
Let \((X,Y)\) a random pair with joint distribution \(H(x,y) = C(F(x),G(y))\).

Spearman's Rho: probability of concordance minus the probability of discordance of two random pairs with respective joint cumulative law \(C\) and \(FG\).

\[\rho = \frac{\int_0^1 \int_0^1 C(u,v) \, du \, dv - \int_0^1 \theta(1) \, du}{\int_0^1 \theta(1) \, du}\]

Remark.

- If \(\theta(1) = 0\), then \(\rho \geq 0\).
- If \(\theta\) is a constant function, then \(\rho = \theta \Phi(1)\).

Measure of association.
Upper tail dependence.

The upper tail dependence coefficient is defined as

\[ \lambda = \lim_{t \to 1} P(F(X) > t | G(Y) > t) = \lim_{u \to 1} \bar{C}(u, u) = 1 - u, \]

where \( \bar{C} \) is the survival copula, i.e., \( \bar{C}(u, v) = 1 - C(u, v) \).

In the case where \( C = \theta C_{\phi} \), we have

\[ \lambda_{\theta, \phi} = \phi' \theta' (1) \]

Remark.

- If \( \phi(1) = 0 \), then \( \lambda_{\theta, \phi} = 0 \).
- If \( \theta \) is a constant function, then \( \lambda_{\theta, \phi} = 0 \).

Remark.

\[ (I, \theta)(I, \phi) = \phi' \theta' \]

where \( C = C_{\phi} \phi' \theta' \).

The upper tail dependence coefficient is defined as

**Upper tail dependence.**
First sub-family, the case $\theta(1) = 0$. 

Examples.

- **Fréchet upper bound.** Choosing $\phi(x) = x$ and $\theta(x) = (1 - x)/x$ yields $C_{\theta,\phi}(u,v) = M(u,v) = \min(u,v)$.

- **Independent copula.** $\theta(x) = 0$ yields $C_{\theta,\phi}(u,v) = \Pi(u,v) = uv$.

- **Cuadras-Augé family:** $\phi(x) = x$ and $\theta(x) = x - \alpha - 1$, $0 \leq \alpha \leq 1$ yields $C_{\theta,\phi}(u,v) = \min(u,v) \alpha(uv) 1 - \alpha = M_{\alpha}(u,v) \Pi_{1 - \alpha}(u,v)$, which is the weighted geometric mean of $M$ and $\Pi$.

Remark.

- $\theta(1) = 0$ and $\theta'(u) \leq 0$ imply $\theta(u) \geq 0$ for all $u \in I$.

- $0 \leq \rho_{\theta,\phi} \leq 1 \rightarrow$ Modelling of positive dependences.

- Lower ($0$) and upper bounds ($1$) of $\rho_{\theta,\phi}$ and $\lambda_{\theta,\phi}$ are reached respectively by the $\Pi$ and $\Pi I_{\theta,\phi}$ copulas.

Which is the weighted geometric mean of $M$ and $\Pi$.

\[
(a',n)_{\Pi - 1} \Pi (a',n)_{\phi} I_{\theta,\phi} = (a',n)_{\Pi - 1} (a',n)_{\phi} \Pi = (a',n)_{\phi} \theta C
\]

Examples.

- **Fréchet-upper bound.** Choosing $\phi(x) = x$ and $x = (x)\theta$ yields $x/(x - 1) = (x)\theta$ and $x = (x)\phi$.

- **Fréchet-upper bound.** Choosing $\phi(x) = x$ and $x = (x)\phi$.

- **Independent copula.** $\phi(x) = x$ and $x = (x)\phi$.

- **Cuadras-Augé family:** $\phi(x) = x$ and $x = (x)\phi$.

- **Cuadras-Augé family:** $\phi(x) = x$ and $x = (x)\phi$.
Dependence properties: definitions.

Assume $X$ and $Y$ are exchangeable. $X$ and $Y$ are exchangeable.

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\[ (f, x) \quad \text{non-decreasing in } x \text{ and for all } f, \quad (f > x | f < X < x) \text{ if RI} \]

\[ (f < X, x < X | f < X < x) \text{ if RCSI} \]

\[ (f > X, x > X | f > X > x) \text{ if LCSD} \]

\[ (f > x | f < X < x) \text{ if non-decreasing in } x \text{ for all } f. \]

\[ (f < x | f < X < x) \text{ if non-decreasing in } x \text{ for all } f. \]

\[ (f > x | f > X < x) \text{ if non-decreasing in } x \text{ for all } f. \]

\[ (f < x | f < X < x) \text{ if non-decreasing in } x \text{ for all } f. \]

\[ (f > X, x > X | f > X > x) \text{ if RCSI} \]

Positive quadrant dependent (PQD) •

\[ (f > x, x > X | f > x, x > X) \text{ if RI} \]

\[ (f > x, x > X | f > x, x > X) \text{ if PDI} \]

\[ (f > x, x > X | f > x, x > X) \text{ if PDD} \]
Theorem. \( X \) and \( Y \) are:

- PQD iff \( \varphi(u) \) has a constant sign on \( I \).
- L TD or LCSD iff either \( \{ \frac{\varphi(u)}{u} \text{ is non increasing and } \forall u \in I, \varphi(u) \geq 0 \} \) or \( \{ \frac{\varphi(u)}{u} \text{ is non decreasing and } \forall u \in I, \varphi(u) \leq 0 \} \).
- R TI or RCSI iff \( \frac{\varphi(u)}{1 - u} \) and \( \theta(u) \frac{\varphi(u)}{1 - u} \) are monotone.
- SI iff either \( \{ \frac{\varphi(u)}{u} \text{ and } \theta \text{ are concave and } \forall u \in I, \varphi(u) \geq 0 \} \) or \( \{ \frac{\varphi(u)}{u} \text{ and } \theta \text{ are convex and } \forall u \in I, \varphi(u) \leq 0 \} \).

Implications in the sub-family

Implications in the general case

- \( \{ 0 \geq \frac{\varphi(n)}{n} \text{ and } A_n \in I \} \) or \( \{ 0 \leq (n-1) \frac{\varphi(n)}{n} \text{ and } (n-1) \frac{\varphi(n)}{n} \text{ and } \forall u \in I, n \frac{\varphi(n)}{n} \leq 0 \} \).

\( RTI \) or \( RCSI \) iff \( SI \) •

\( LTI \) or \( LCSI \) iff \( SI \) or \( RSI \) •

\( LCSI \) or \( LCSF \) iff \( SI \) or \( LCSI \) •

\( PD \) iff \( \{ \frac{\varphi(n)}{n} \text{ has a constant sign on } I \} \) •

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In this case, we restrict ourselves to a constant function \( \theta \) and only it.\( \theta \) satisfies the following conditions:

3. Second sub-family: the case \( \phi(1) = 0 \).

\[ C_{\theta, \phi} \text{ is a copula if and only if } \theta \text{ and } \phi \text{ satisfy the following conditions:} \]

- Boundary conditions: \( 0 = (1)\phi \) and \( 0 = (0)\phi \).
- For all \( x \in I \): \( x - \min(x, 1) \leq |(x)\phi| \).
- For all \( x \in I \): \( x \leq |(x)\phi| \).

\( \phi(x) = \min(x, 1 - x) \): upper bound of the above theorem.

\( \phi(x) = x(1 - x) \): Farlie-Gumbel-Morgenstern family of copulas (Morgenstern, 1966).

\( \phi(x) = x(1 - x)(1 - 2x) \): symmetric copulas with cubic sections (Quesada-Aliaga and Rodríguez-Lallena, 1992).

\( \phi(x) = \pi - 1 \sin(\pi x) \).

\( x(1 - x) \): symmetric copulas with cubic sections (Nelsen et al., 1997).

\( x(1 - x) \): copula with both horizontal and vertical quadratic sections (Quesada-Aliaga and Rodríguez-Lallena, 1992).

\( \phi(x) = \phi(1) = 0 \).

\( \theta(x) = (x)\theta \).

\( \theta(1) = 0 \).

In this case, we restrict ourselves to a constant function \( \theta \), i.e., \( \theta \) is a copula if and only if \( \theta \) satisfies the following conditions:

\( \phi(x) \) is a copula if and only if \( \phi \) satisfies the following conditions:

\( \theta(x) = (x)\theta \).

\( \theta(1) = 0 \).

Theorem 3. Second sub-family: the case \( \phi(1) = 0 \).

Examples:

- \( \phi(x) = \min(x, 1 - x) \): upper bound of the above theorem.
- \( \phi(x) = x(1 - x) \): Farlie-Gumbel-Morgenstern family of copulas (Morgenstern, 1966).
- \( \phi(x) = x(1 - x)(1 - 2x) \): symmetric copulas with cubic sections (Nelsen et al., 1997).
- \( \phi(x) = \pi - 1 \sin(\pi x) \).

In this case, we restrict ourselves to a constant function \( \theta \), i.e., \( \theta \) is a copula if and only if \( \theta \) satisfies the following conditions:

3. Second sub-family: the case \( \phi(1) = 0 \).

\( \theta(x) = (x)\theta \).

\( \theta(1) = 0 \).

\( \phi(x) \) is a copula if and only if \( \phi \) satisfies the following conditions:

- Boundary conditions: \( 0 = (1)\phi \) and \( 0 = (0)\phi \).
- For all \( x \in I \): \( x - \min(x, 1) \leq |(x)\phi| \).
- For all \( x \in I \): \( x \leq |(x)\phi| \).

\( \phi(x) = \min(x, 1 - x) \): upper bound of the above theorem.

\( \phi(x) = x(1 - x) \): Farlie-Gumbel-Morgenstern family of copulas (Morgenstern, 1966).

\( \phi(x) = x(1 - x)(1 - 2x) \): symmetric copulas with cubic sections (Nelsen et al., 1997).

\( \phi(x) = \pi - 1 \sin(\pi x) \).
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Upper bound, Farlie-Gumbel-Morgenstern, cubic sections, sinus.
Similar to the previous family in the case $\theta < 0$.

Dependence properties.

**Upper tail dependence.**

$$\rho_{\theta,\phi} = \phi \theta d$$

**Kendall's Tau:**

$$\tau_{\theta,\phi} = \frac{\phi \theta d}{1 - \frac{3}{4} \int_0^1 \phi \theta d}$$

and it follows that $-\frac{1}{2} \leq \tau_{\theta,\phi} \leq \frac{1}{2}$ for all $\theta \in [-1, 1]$. Similar bounds hold for the measure of association. The Spearman's Rho can be rewritten as:

$$\rho_{\theta,\phi} = \phi \theta d$$
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Symmetry properties: definitions.

- $X$ is symmetric about $a$ if $(X - a)$ and $(a - X)$ are identically distributed (id).
- $X$ and $Y$ are exchangeable if $(X, Y)$ and $(Y, X)$ are id.
- $(X, Y)$ is marginally symmetric about $(a, b)$ if $X$ and $Y$ are symmetric about $a$ and $b$ respectively.
- $(X, Y)$ is radially symmetric about $(a, b)$ if $(X - a, Y - b)$ and $(a - X, b - Y)$ are id.
- $(X, Y)$ is jointly symmetric about $(a, b)$ if the pairs $(X - a, Y - b)$, $(a - X, b - Y)$, $(X - a, b - Y)$, and $(a - X, Y - b)$ are id.

Theorem. In the $C_\theta, \phi$ family:

- If $X$ and $Y$ are id then $X$ and $Y$ are exchangeable.
- Besides, if $(X, Y)$ is marginally symmetric about $(a, b)$ then:
  - $(X, Y)$ is radially symmetric about $(a, b)$ if and only if
    
    $$\forall u \in I, \quad \phi(u) = \phi(1 - u) \quad \text{or} \quad \forall u \in I, \quad \phi(u) = -\phi(1 - u).$$
  - $(X, Y)$ is jointly symmetric about $(a, b)$ if and only if
    $$\forall u \in I, \quad \phi(u) = -\phi(1 - u).$$
We restrict ourselves to the second sub-family, with constant function $\theta$:

$$C(u,v) = uv + \theta \phi(u) \phi(v).$$

The estimation of $\theta$ (scalar) and $\phi$ (univariate function).

Under these assumptions, the family can be rewritten

$$C(a\phi(n) + \alpha n = (a,n).$$

We focus on the PQD case: $\theta > 0$ and $\phi$ has a constant sign.

Identification problem: $(\phi, \theta)$ yield the same copula for all $\alpha > 0$.

Estimation of $\theta$ (scalar) and $\phi$ (univariate function).

We restrict ourselves to the second sub-family, with constant function $\theta$.

Assumptions.

Inference procedures.
Estimation of $\psi$

Preprocessing:

1. $\{ (x_i, y_i), i = 1, ..., n \}$ a sample of $(X, Y)$ from the cdf $H(x, y) = C(F(x), G(y))$.
2. Rank transformations:
   - $u_i = \text{rank}(x_i)/n$ and $v_i = \text{rank}(y_i)/n$.
   - $\{ (u_i, v_i), i = 1, ..., n \}$ an approximate sample from the copula $C$.
3. Pseudo-observations $\{ w_i = \max(u_i, v_i), i = 1, ..., n \}$ from $C(w, w) = w^2 + \psi(w)$.

Projection estimate: linear combination of basis functions:

$\hat{\psi}(w) = \sum_{k \geq 1} a_k e_k(w), w \in I.$

Choice of the set of functions:

- No orthogonality condition.
- Boundary conditions $e_k(0) = e_k(1) = 0$ for all $k \geq 1$ so that $0 = (1) \varphi(0) = (1) \hat{\psi}$.

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Optimization problem:

1. \( \psi \) can be rewritten as \( \| q - M \| \) in

2. \( \psi \) can be rewritten

\( \frac{u_i}{m} - \frac{(1 + u)}{i} \approx \frac{u_i}{m} - (\frac{w_i}{m}) \varphi = (\frac{w_i}{m}) \varphi \)

3. Optimizeation problem: Define
Recall that
\[ \rho_{\theta, \phi} = \frac{1}{2} \left( \int \frac{I(1-u)}{u} \, du \right)^2 \]

Replacing \( \psi \) by \( \hat{\psi} \) yields the following semi-parametric estimator:
\[ \hat{\rho}_{SP} = \frac{1}{2} \left( \sum_{k \geq 1} a_k \beta_k \right)^2, \]
where we have introduced \( \beta_k = \int \frac{I(e_k(u))}{u} \, du \).

Another solution: adapt the nonparametric estimator of the Kendall's Tau introduced in (Genest, Rivest, 1993) to obtain
\[ \hat{\rho}_{NP} = \frac{6}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{1}(u_j < u_i, v_j < v_i) - \frac{3}{2}. \]

Rephrasing by \( \phi \) yields the following semi-parametric estimator:
\[ \hat{\rho}_{SP} = \int \left( \int \frac{I(z)}{z} \, dz \right) \theta \, d \theta = \phi \theta \, d \theta \]

Recall that
\[ \text{Estimation of the Spearman's Rho} \]

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Definition. The α-quantile of the copula $C$ is defined by

$$Q_\alpha = \inf \left\{ \lambda(S) : P(S) \geq \alpha, S \subset \mathbb{I}^2 \right\},$$

where $\lambda$ is the Lebesgue measure on $\mathbb{I}^2$.

Partitions. Let $\{I_k, k = 1, \ldots, N\}$ be the equidistant $N$-partition of $\mathbb{I}$, $I_{k,\ell} = I_k \times I_\ell$ the associated $N \times N$ grid. Denote $\delta_{k,\ell} \in \{0, 1\}$, $(k, \ell) \in \{1, \ldots, N\}^2$.

Estimator: $Q_\alpha$ are defined by

$$\hat{Q}_\alpha = \bigcup_{k,\ell} (I_k \cap I_\ell) \{I_k \cap I_\ell, \delta_{k,\ell} = 1\}.$$  

Optimization problem. The $\delta_{k,\ell}$ are defined by

$$\min_{\delta_{k,\ell} \in \{0, 1\}, \{(k, \ell) \in \{1, \ldots, N\}^2\}} \sum_{k=1}^N \sum_{\ell=1}^N \delta_{k,\ell},$$

under the constraints $\delta_{k,\ell} \in \{0, 1\}$ and

$$\sum_{k=1}^N \sum_{\ell=1}^N \delta_{k,\ell} \hat{P}(I_k \cap I_\ell) \geq \alpha,$$

where $\hat{P}(I_k \cap I_\ell)$ is an estimation of the probability $P(I_k \cap I_\ell)$.  

Estimation of high probability regions. The quantile $Q_\alpha$ is defined by

$$\{\alpha > 0, \exists S \subset \mathbb{I}^2 : Q_\alpha \geq (S) \| \{\alpha > 0, \exists S \subset \mathbb{I}^2 : Q_\alpha \geq (S) \| \} \cap = \alpha \mathbb{C}.$$
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Algorithm.

1. First step: sort the \( \hat{P}(k, \ell) \) in decreasing order to obtain the sequence \( \tilde{P}_\tau, \tau = 1, \ldots, N_2 \).
2. Second step: Computation of the number of subsets of the partition:

\[
J = \min \left\{ j, \sum_{\tau=1}^j \tilde{P}_\tau \geq \alpha \right\}
\]

3. Third step: selection of the \( J \) first subsets:

\[
\delta_{k, \ell} = 1 \text{ if } 1 \leq \tau(k, \ell) \leq J,
\]

Estimation of \( P(K_{k, \ell}) \):

- Two solutions:
  - Semi-parametric estimate based on \( \hat{\psi} \):
    \[
    \hat{P}_{SP}(K_{k, \ell}) = \frac{1}{N_2} \left( \frac{N}{I - j} \right) \hat{\phi} - \left( \frac{N}{j} \right) \phi \left( \frac{N}{I - y} \right) \hat{\phi} - \left( \frac{N}{y} \right) \phi + \frac{\alpha N}{I} = (\hat{\theta}_Y \phi, \hat{\theta}_Y \phi)
    \]
  - Nonparametric estimate \( \hat{P}_{NP}(K_{k, \ell}) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}(u_i, v_i) \in K_{k, \ell} \).

Theorem of Selection of the first subsets:

\[
\{ \psi \geq (\hat{\theta}_Y \phi, \hat{\theta}_Y \phi) \} \quad \text{min = } \psi
\]

- Second step: Computation of the number of subsets of the partition:

\[
\tilde{N}_2 \cdot I = \tilde{T} \quad \text{such that } \tilde{T} = \tilde{N}_2 \cdot I
\]

- First step: sort the sequence in decreasing order to obtain the sequence

\[
\{ \hat{P}(k, \ell) \}
\]
When $1 < k < \infty$, a bivariate distribution "interpolating" between the two previous ones.

$\forall k \geq 1$, $\psi_k(x) = 1 - (x^k + (1-x)^k)^{1/k}$, $x \in I$.

- When $k = 1$, $C_1$: uniform distribution on $I^2$. Spearman's Rho $\rho_1 = 0$.
- When $k \to \infty$, $\psi_k(x) \to \psi_\infty(x) = \min(x, 1-x)$ for all $x \in I$. $C_\infty$: mixture of two uniform distributions on the squares $[0, 1/2]^2$ and $[1/2, 1]^2$ with mixing parameter $1/2$. Spearman's Rho $\rho_\infty = \infty \leftrightarrow (x)_{1/2} \leftrightarrow (x)_{\infty}$ finite, the maximum value in the sub-family.
- When $1 < k < \infty$, bivariate distribution "interpolating" between the two previous ones.

### Simulation results

Numerical experiments on the family of copulas $C$ generated by the set of functions $\theta(x) = \gamma (x - 1) + \gamma x - 1 = (x)_{\gamma/1} - I \geq (x)_{\gamma/1}$ for all $x \in I$. $\forall \gamma \in \mathbb{R}$.
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\[
\begin{align*}
\{[z + \gamma, \gamma] \ni x_{1+z} \} & \left( \frac{z}{\gamma} \right) \sin \left( (x - x_{1+z}) \frac{z}{\gamma} \right) = (x)^{\gamma/s}
\end{align*}
\]

\[
\text{Chosen basis of functions:}
\]

\[
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\]
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True functions $\psi_k(x)$, $k \in \{2, 4, 8\}$ – Estimated functions $\hat{\psi}_k(x)$, $k \in \{2, 4, 8\}$, $n = 100.$
the estimates \( \hat{\rho}_{SP} \) and \( \hat{\rho}_{NP} \) are evaluated on 100 repetitions.

Estimation of the generating function and of the Spearman's Rho (\( \rho \)). The mean value of

<table>
<thead>
<tr>
<th>( \hat{\rho}_{SP} )</th>
<th>( \hat{\rho}_{NP} )</th>
<th>( \rho )</th>
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</thead>
<tbody>
<tr>
<td>70.2</td>
<td>72.1</td>
<td>8</td>
</tr>
<tr>
<td>68.8</td>
<td>70.6</td>
<td>6</td>
</tr>
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<td>64.3</td>
<td>67.8</td>
<td>4</td>
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<td>2</td>
</tr>
<tr>
<td>1.8</td>
<td>0.8</td>
<td>1</td>
</tr>
</tbody>
</table>

mean(\( \hat{\rho}_{SP} \)) \times 10^{-2} \times (\text{d}) \text{mean}(\( \hat{\rho}_{NP} \)) \times 10^{-2} \times (\text{d}) \text{mean}(\( \rho \)) \times 10^{-2} \times (\text{d})
Estimation of high probability regions $\phi^{-1}(u) \cap C^2$. Red: $\alpha = 0.25$, green: $\alpha = 0.75$. Yellow: $\alpha = 0.5$. Top left: simulated sample, top right: nonparametric estimate, bottom left: semiparametric estimate, bottom right: semiparametric estimate with the true function $\psi$. $n = 500$.
Estimation of high probability regions from \( C_u \). Red: \( \alpha = 0.25 \), green: \( \alpha = 0.75 \), yellow: \( \alpha = 0.5 \). Top left: simulated sample, top right: nonparametric estimate, bottom left: semiparametric estimate, bottom right: semiparametric estimate with the true function \( \psi \).
Estimation of high probability regions $\phi(u)$ from $C\alpha$. Red: $\alpha = 0.25$, green: $\alpha = 0.75$, yellow: $\alpha = 0.5$. Top left: simulated sample, top right: nonparametric estimate, bottom left: semiparametric estimate, bottom right: semiparametric estimate with the true function $\psi$. A semiparametric family of bivariate copulas.
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According to the PQD test proposed in (Scaillet, 2004), these data are PQD.

\[
\hat{\rho}_{NP} = 52.7\%
\]

\[
\hat{\rho}_{SP} = 40.7\%
\]

Real data.

\( n = 225 \) countries, two variables \( X \), the life expectancy at birth (years) in 2002 of the total population and \( Y \), the difference between the life expectancy at birth of women and men.
Estimation of high probability regions $Q$ from real data. Red: $\alpha = 0.25$, green: $\alpha = 0.5$, yellow: $\alpha = 0.75$. Top left: real data, top right: real data after rank transformation, bottom left: nonparametric estimate, bottom right: semiparametric estimate.
Further work:

- Estimation of the function in the general case.

(What is the lower bound of \( \phi' \)?

- Study of the sub-family \( \phi(1) = 0 \) without the assumption that \( \theta \) is a constant function.

- Goodness of fit test.

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References.


