A semiparametric family of bivariate copulas: dependence properties and estimation procedures

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Outline

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2. First sub-family, the case $\theta(1) = 0$.
3. Second sub-family, the case $\phi(1) = 0$.
4. Inference procedures.
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1. Definition and basic properties.

Definition. Let I be the unit interval. The family is defined for all $(u,v) \in I^2$ by,

$$C_{\theta,\phi}(u,v) = uv + \theta[\max(u,v)]\phi(u)\phi(v).$$

where ϕ and θ are differentiable $I \longrightarrow \mathbb{R}$ functions (vanishing at most on isolated points).

Theorem. $C_{\theta,\phi}$ is a copula if and only if ϕ and θ satisfy the following conditions:

- boundary conditions: $\phi(0) = 0$ and $(\phi\theta)(1) = 0$,
- θ is non increasing on I,
- $\phi'(u)(\theta\phi)'(v) \geq -1$ for all $0 \leq u \leq v \leq 1$.

Remark. The family can be split in two sub-families according to $\theta(1) = 0$ or $\phi(1) = 0$.

Measure of association.

Let (X, Y) a random pair with joint distribution $H(x, y) = C(F(x), G(y))$. Spearman's Rho: probability of concordance minus the probability of discordance of two random pairs with respective joint cumulative law $C(F, G)$ and FG.

$$\rho = 12 \int_0^1 \int_0^1 C(u, v) du dv - 3.$$

In the case of $C = C_{\theta,\phi}$, we have

$$\rho_{\theta,\phi} = 12 \left[\Phi^2(1)\theta(1) - \int_0^1 \Phi^2(t)\theta'(t) dt \right],$$

where $\Phi(t) = \int_0^t \phi(u) du$.

Remark.

- If $\theta(1) = 0$, then $\rho_{\theta,\phi} \geq 0$.
- If θ is a constant function, then $\rho_{\theta,\phi} = 12\theta\Phi^2(1)$.

Upper tail dependence.

The upper tail dependence coefficient is defined as

$$\lambda = \lim_{t\to 1} \mathbb{P}(F(X) > t | G(Y) > t) = \lim_{u \to 1} \frac{\bar{C}(u,u)}{1-u},$$

where $\bar{C}$ is the survival copula, *i.e.* $\bar{C}(u,v) = 1 - u - v + C(u,v)$.

In the case where $C = C_{\theta,\phi}$, we have

$$\lambda_{\theta,\phi} = -\phi^2(1)\theta'(1).$$

Remark.

- If $\phi(1) = 0$, then $\lambda_{\theta,\phi} = 0$.
- If θ is a constant function, then $\lambda_{\theta,\phi} = 0$.

2. First sub-family, the case $\theta(1) = 0$.

Examples.

- Fréchet upper bound. Choosing $\phi(x) = x$ and $\theta(x) = (1-x)/x$ yields $C_{\theta,\phi}(u,v) = M(u,v) = \min(u,v)$.
- Independent copula. $\theta(x) = 0$ yields $C_{\theta,\phi}(u,v) = \Pi(u,v) = uv$.
- Cuadras-Augé family: $\phi(x) = x$ and $\theta(x) = x^{-\alpha} - 1$, $0 \leq \alpha \leq 1$ yields

$$C_{\theta,\phi}(u,v) = \min(u,v)^{\alpha}(uv)^{1-\alpha} = M^{\alpha}(u,v)\Pi^{1-\alpha}(u,v),$$

which is the weighted geometric mean of M and Π.

Remark.

- $\theta(1) = 0$ and $\theta'(u) \leq 0$ imply $\theta(u) \geq 0$ for all $u \in I$.
- $0 \leq \rho_{\theta,\phi} \leq 1 \longrightarrow$ Modelling of positive dependences.
- Lower (0) and upper bounds (1) of $\rho_{\theta,\phi}$ and $\lambda_{\theta,\phi}$ are reached respectively by the Π and M copulas.

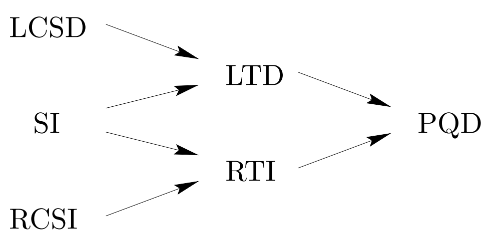
Dependence properties: definitions.

Assume X and Y are exchangeable. X and Y are

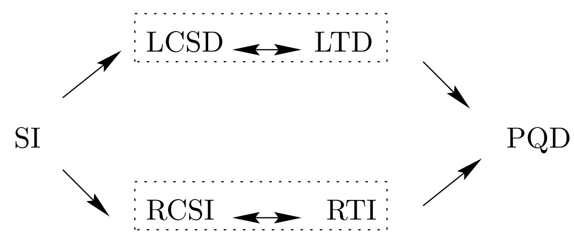
- Positively Quadrant Dependent (PQD) if $\mathbb{P}(X \leq x, Y \leq y) \geq \mathbb{P}(X \leq x)\mathbb{P}(Y \leq y)$ for all (x, y).
- Left Tail Decreasing (LTD) if $\mathbb{P}(Y \leq y | X \leq x)$ is non-increasing in x for all y.
- Right Tail Increasing (RTI) if $\mathbb{P}(Y > y | X > x)$ is nondecreasing in x for all y.
- Stochastically Increasing (SI) if $\mathbb{P}(Y > y | X = x)$ is nondecreasing in x for all y.
- Left Corner Set Decreasing (LCSD) if $\mathbb{P}(X \leq x, Y \leq y | X \leq x', Y \leq y')$ is non-increasing in x' and y' for all (x, y).
- Right Corner Set Increasing (RCSI) if $\mathbb{P}(X > x, Y > y | X > x', Y > y')$ is nondecreasing in x' and y' for all (x, y).

Theorem. X and Y are:

- PQD iff $\phi(u)$ has a constant sign on I.
- LTD or LCSD iff either $\{\phi(u)/u$ is non increasing and $\forall u \in I,\ \phi(u) \geq 0\}$ or $\{\phi(u)/u$ is non decreasing and $\forall u \in I,\ \phi(u) \leq 0\}$.
- RTI or RCSI iff $\phi(u)/(1-u)$ and $\theta(u)\phi(u)/(1-u)$ are monotone.
- SI iff either $\{\phi$ and $\theta\phi$ are concave and $\forall u \in I,\ \phi(u) \geq 0\}$ or $\{\phi$ and $\theta\phi$ are convex and $\forall u \in I,\ \phi(u) \leq 0\}$.



Implications in the general case



Implications in the sub-family

3. Second sub-family, the case $\phi(1) = 0$.

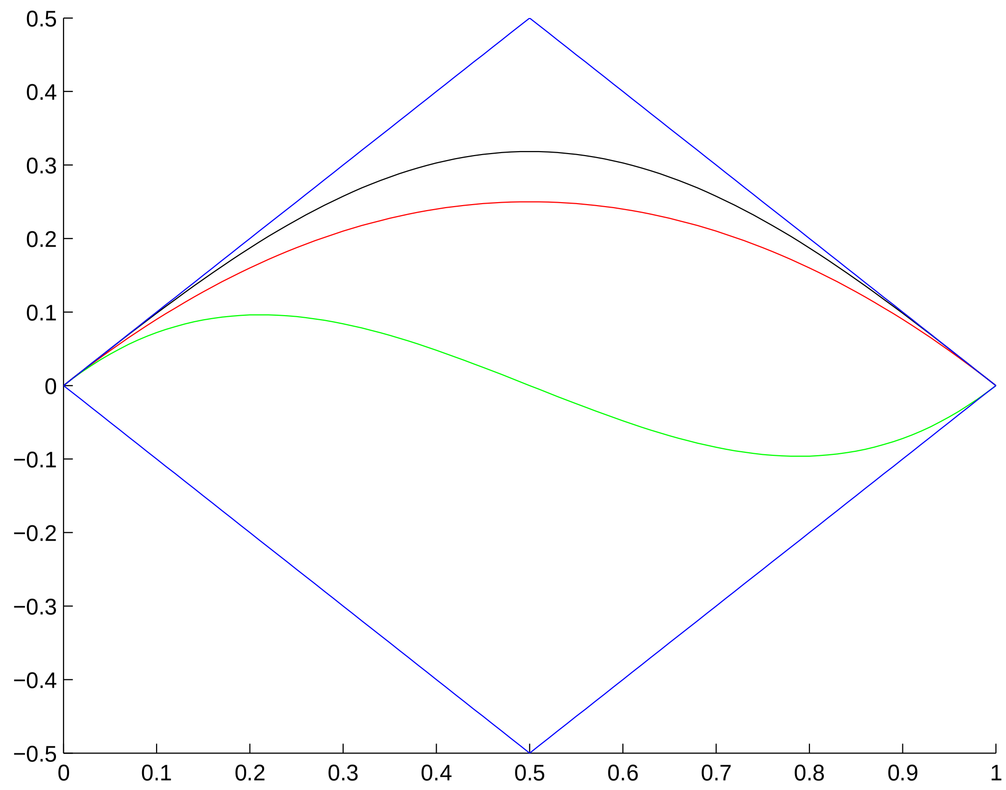
In this case, we restrict ourselves to a constant function θ, *i.e.* $\theta(x) = \theta \in [-1, 1]$.

Theorem. $C_{\theta,\phi}$ is a copula if and only if ϕ and θ satisfy the following conditions:

- boundary conditions: $\phi(0) = 0$ and $\phi(1) = 0$,
- $|\phi'(x)| \leq 1$ for all $x \in I$,
- $|\phi(x)| \leq \min(x, 1-x)$, for all $x \in I$.

Examples.

- $\phi(x) = \min(x, 1-x)$: upper bound of the above theorem,
- $\phi(x) = x(1-x)$: Farlie-Gumbel-Morgenstern family of copulas (Morgenstern, 1956), which contains all copulas with both horizontal and vertical quadratic sections (Quesada-Molina, Rodríguez-Lallena, 1995)
- $\phi(x) = x(1-x)(1-2x)$: symmetric copulas with cubic sections (Nelsen *et al*, 1997),
- $\phi(x) = \pi^{-1}\sin(\pi x)$.



Upper bound, Farlie-Gumbel-Morgenstern, cubic sections, sinus.

Measure of association. The Spearman's Rho can be rewritten as:

$$\rho_{\theta,\phi} = 12\theta \left(\int_I \phi(u) du \right)^2,$$

and it follows that $-3/4 \leq \rho_{\theta,\phi} \leq 3/4$ for all $\theta \in [-1, 1]$. Similar bounds hold for the Kendall's Tau: $-1/2 \leq \tau_{\theta,\phi} \leq 1/2$.

Upper tail dependence. $\rho_{\theta,\phi} = 0$.

Dependence properties. Similar to the previous family in the case $\theta > 0$.

Symmetry properties: definitions.

- X is symmetric about a if $(X-a)$ and $(a-X)$ are identically distributed (id).
- X and Y are exchangeable if (X,Y) and (Y,X) are id.
- (X,Y) is marginally symmetric about (a,b) if X and Y are symmetric about a and b respectively.
- (X,Y) is radially symmetric about (a,b) if $(X-a,Y-b)$ and $(a-X,b-Y)$ are id.
- (X,Y) is jointly symmetric about (a,b) if the pairs $(X-a,Y-b)$, $(a-X,b-Y)$, $(X-a,b-Y)$ and $(a-X,Y-b)$ are id.

Theorem. In the $C_{\theta,\phi}$ family:

- If X and Y are id then X and Y are exchangeable.

Besides, if (X,Y) is marginally symmetric about (a,b) then:

- (X,Y) is radially symmetric about (a,b) if and only if

$$\text{either } \forall u \in I,\ \phi(u)=\phi(1-u) \text{ or } \forall u \in I,\ \phi(u)=-\phi(1-u).$$

- (X,Y) is jointly symmetric about (a,b) if and only if $\forall u \in I,\ \phi(u)=-\phi(1-u)$.

4. Inference procedures.

Assumptions.

- We restrict ourselves to the second sub-family, with constant function θ:

$$C(u,v) = uv + \theta\phi(u)\phi(v).$$

$\longrightarrow$ Estimation of θ (scalar) and ϕ (univariate function).
$\longrightarrow$ Identifiability problem: (θ,ϕ) and $(\alpha\theta, \phi/\sqrt{\alpha})$ yield the same copula for all $\alpha > 0$.

- We focus on the PQD case: $\theta > 0$ and ϕ has a constant sign.

Under these assumptions, the family can be rewritten

$$C(u,v) = uv + \psi(u)\psi(v),$$

where $\psi(x) = \sqrt{\theta}|\phi(x)|$.
$\longrightarrow$ The estimation of C reduces to the estimation of ψ (positive univariate function).

Estimation of ψ

1) **Preprocessing:**

- $\{(x_i, y_i), i = 1, \dots, n\}$ a sample of (X, Y) from the cdf $H(x, y) = C(F(x), G(y))$.
- Rank transformations: $u_i = \text{rank}(x_i)/n$ and $v_i = \text{rank}(y_i)/n$.
  $\{(u_i, v_i), i = 1, \dots, n\}$ an approximate sample from the copula $C(u, v)$.
- Pseudo-observations $\{w_i = \max(u_i, v_i), i = 1, \dots, n\}$ from $C(w, w) = w^2 + \psi(w)$.

2) **Projection estimate:** linear combination of basis functions: $\{e_k,\ k \geq 1\}$

$$\widehat{\psi}(w) = \sum_{k \geq 1} a_k e_k(w),\ w \in I.$$

Choice of the set of functions:

- no orthogonality condition,
- boundary conditions $e_k(0) = e_k(1) = 0$ for all $k \geq 1$ so that $\widehat{\psi}(0) = \widehat{\psi}(1) = 0$.

3) **Optimization problem:** Define

- $w_{1,n} \leq \cdots \leq w_{n,n}$, the ordered pseudo-observations,
- M and M' two matrices $M_{i,k} = e_k(w_{i,n})$, $M'_{i,k} = e'_k(w_{i,n})$, $k \geq 1$, $i \in \{1, \ldots, n\}$,
- a and b two vectors $b_i = (i/(n+1) - w_{i,n}^2)^{1/2}$, a_i unknown, $i \in \{1, \ldots, n\}$.

Definition of the estimator.

- $\hat{\psi}(w_{i,n}) = C(w_{i,n}, w_{i,n}) - w_{i,n}^2 \simeq i/(n+1) - w_{i,n}^2$ for $i = 1, \ldots, n$ can be rewritten

$$\min_a \|Ma - b\|^2,$$

- $\hat{\psi}(w_{i,n}) \geq 0$ can be rewritten $Ma \geq 0$,
- $|\hat{\psi}(w_{i,n})| \leq 1$ can be rewritten $-1 \leq M'a \leq 1$.

$\longrightarrow$ Constrained least-square problem.

Estimation of the Spearman's rho

Recall that

$$\rho_{\theta,\phi} = 12\theta\left(\int_I \phi(u)du\right)^2 = 12\left(\int_I \psi(u)du\right)^2.$$

Replacing ψ by $\hat{\psi}$ yields the following semi-parametric estimator:

$$\hat{\rho}_{\text{SP}} = 12\left(\sum_{k\geq 1} a_k\beta_k\right)^2,$$

where we have introduced $\beta_k = \int_I e_k(u)du$.

Another solution: adapt the nonparametric estimator of the Kendall's Tau introduced in (Genest, Rivest, 1993) to obtain

$$\hat{\rho}_{\text{NP}} = \frac{6}{n(n-1)}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{1}\{u_j < u_i,\ v_j < v_i\} - \frac{3}{2},$$

Estimation of high probability regions

Definition. The α-quantile of the copula C is defined by

$$Q_\alpha = \inf\{\lambda(S) : \mathbb{P}(S) \geq \alpha,\ S \subset I^2\},\ 0 < \alpha \leq 1,$$

where λ is the Lebesgue measure on I^2.

Partitions. $\{I_k,\ k = 1, \ldots, N\}$ be the equidistant N-partition of I, $K_{k,\ell} = I_k \times I_\ell$ the associated $N \times N$ grid. Denote $\delta_{k,\ell} \in \{0, 1\}$, $(k, \ell) \in \{1, \ldots, N\}^2$.

Estimator: $\hat{Q}_\alpha = \bigcup_{k,\ell} K_{k,\ell} \mathbf{1}\{\delta_{k,\ell} = 1\}$.

Optimization problem. The $\delta_{k,\ell}$ are defined by

$$\min \sum_{k=1}^{N} \sum_{\ell=1}^{N} \delta_{k,\ell},$$

under the constraints $\delta_{k,\ell} \in \{0, 1\}$ and $\sum_{k=1}^{N} \sum_{\ell=1}^{N} \delta_{k,\ell} \widehat{P}(K_{k,\ell}) \geq \alpha$,

where $\widehat{P}(K_{k,\ell})$ is an estimation of the probability $P(K_{k,\ell})$.

Algorithm.

- First step: sort the $\widehat{P}(K_{k,\ell})$ in decreasing order to obtain the sequence $\tilde{P}_\tau$, $\tau = 1, \dots, N^2$.
- Second step: Computation of the number of subsets of the partition:

$$J = \min\left\{ j,\ \sum_{\tau=1}^{j} \tilde{P}_\tau \geq \alpha \right\}.$$

- Third step: selection of the J first subsets: $\delta_{k,\ell} = 1$ if $1 \leq \tau(k,\ell) \leq J$,

Estimation of $P(K_{k,\ell})$. Two solutions:

- Semi-parametric estimate based on $\widehat{\psi}$

$$\hat{P}_{\text{SP}}(K_{k,\ell}) = \frac{1}{N^2} + \left(\widehat{\psi}\left(\frac{k}{N}\right) - \widehat{\psi}\left(\frac{k-1}{N}\right)\right)\left(\widehat{\psi}\left(\frac{\ell}{N}\right) - \widehat{\psi}\left(\frac{\ell-1}{N}\right)\right).$$

- Nonparametric estimate

$$\widehat{P}_{\text{NP}}(K_{k,\ell}) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}\{(u_i, v_i) \in K_{k,\ell}\}.$$

5. Simulation results.

Numerical experiments on the family of copulas C_k generated by the set of functions

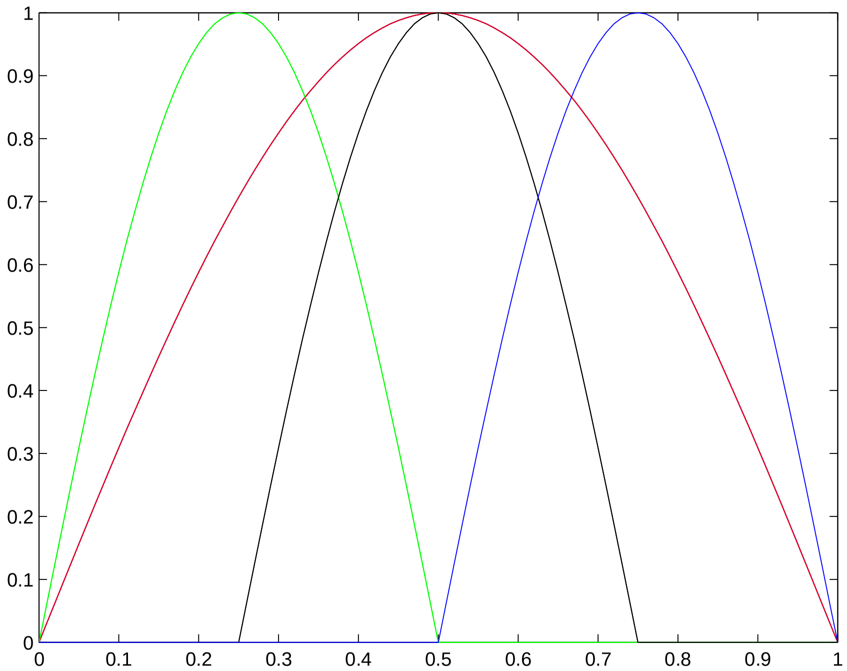
$$\forall k \geq 1, \quad \psi_k(x) = 1 - \left(x^k + (1-x)^k\right)^{1/k}, \; x \in I.$$

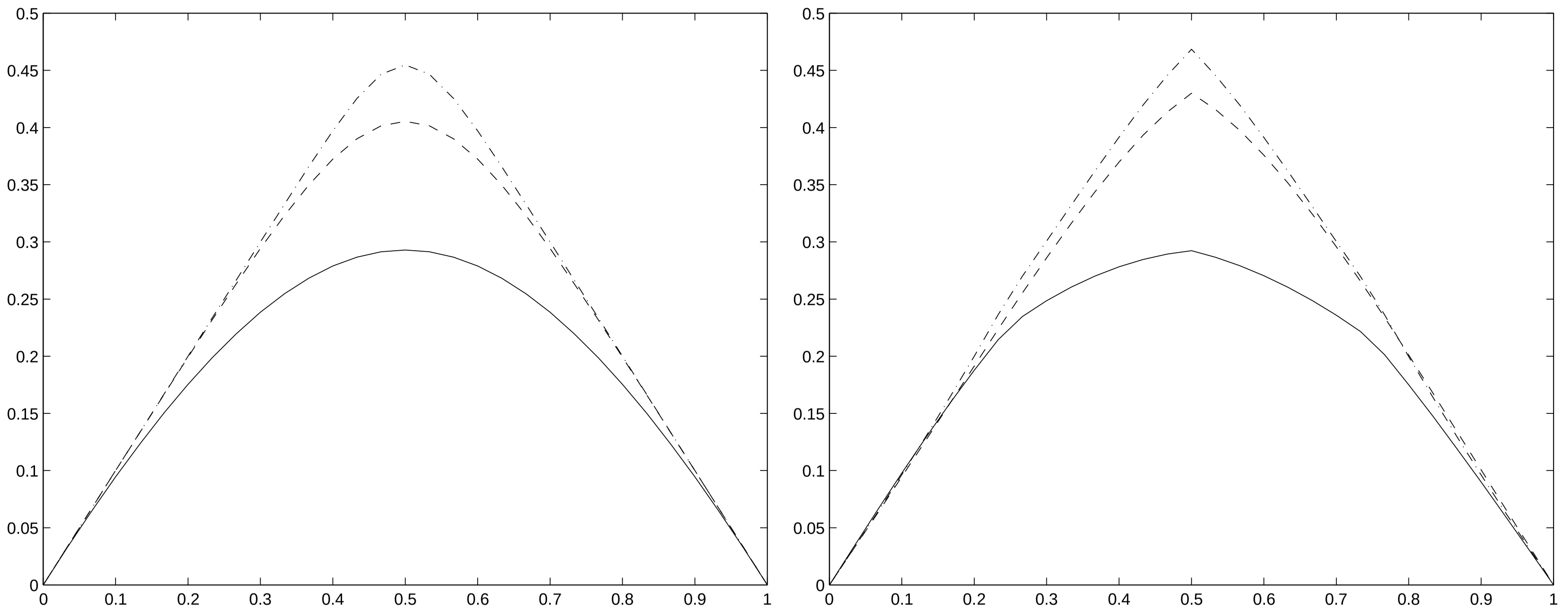
- When $k = 1$, C_1: uniform distribution on I^2. Spearman's Rho $\rho_1 = 0$.
- When $k \to \infty$, $\psi_k(x) \to \psi_\infty(x) = \min(x, 1-x)$ for all $x \in I$.
  C_∞: mixture of two uniform distributions on the squares $[0, 1/2]^2$ and $[1/2, 1]^2$ with mixing parameter 1/2. Spearman's Rho $\rho_\infty = 3/4$ (the maximum value in the sub-family).
- When $1 < k < \infty$, bivariate distribution "interpolating" between the two previous ones.

Chosen basis of functions:

$$e_{s,\ell}(x) = \sin\left(\frac{\pi}{2}(2^{s+1}x - \ell)\right) \mathbf{1}\{2^{s+1}x \in [\ell, \ell+2]\},$$

s is a scale parameter, ℓ is a location parameter.

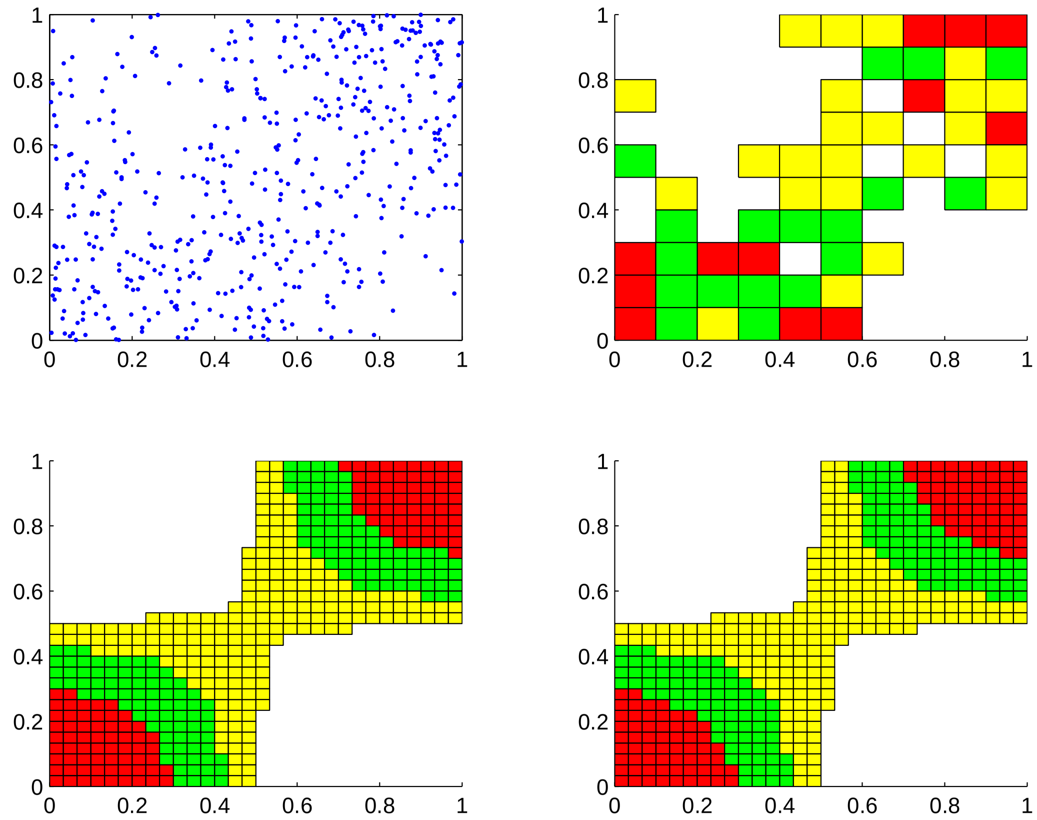




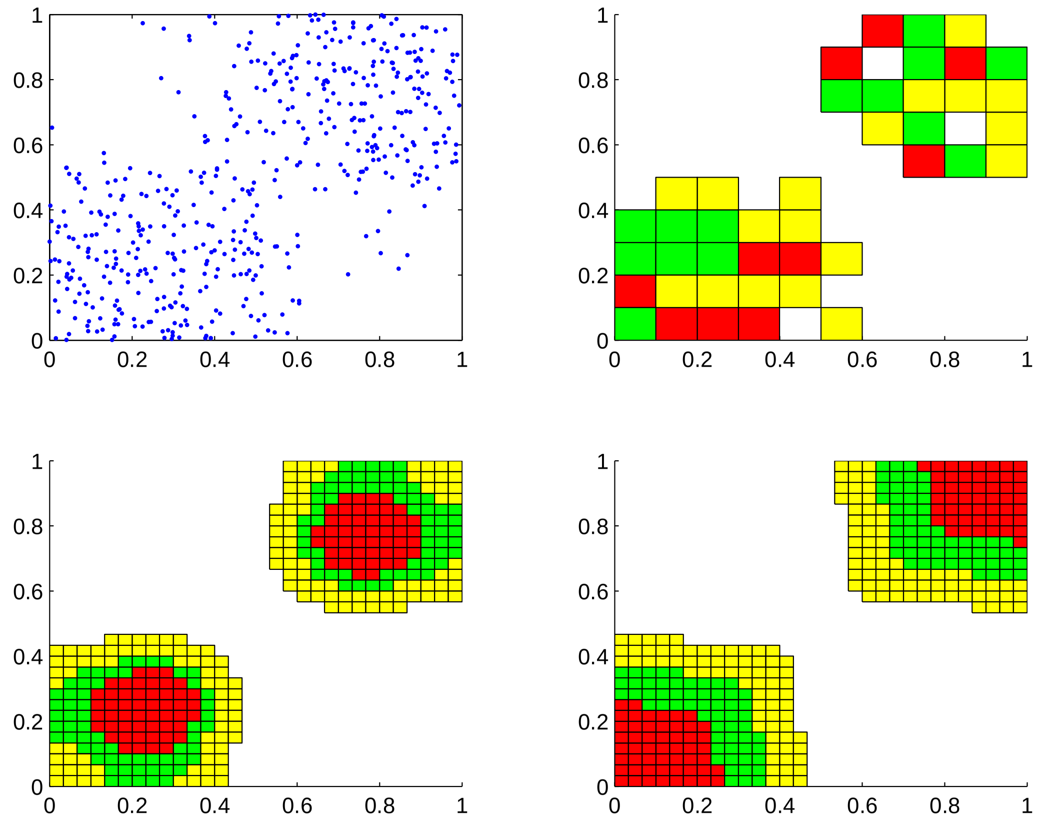
True functions $\psi_k(x)$, $k \in \{2, 4, 8\}$ – Estimated functions $\hat{\psi}_k(x)$, $k \in \{2, 4, 8\}$, $n = 100$.

k	$\rho_k \times 10^{-2}$	mean$(\hat{\rho}_{\text{SP}}) \times 10^{-2}$	mean$(\hat{\rho}_{\text{NP}}) \times 10^{-2}$
1	0	0.81	0.18
2	42.5	43.0	41.2
4	66.4	65.8	64.3
6	71.2	70.6	68.8
8	72.8	72.1	70.2

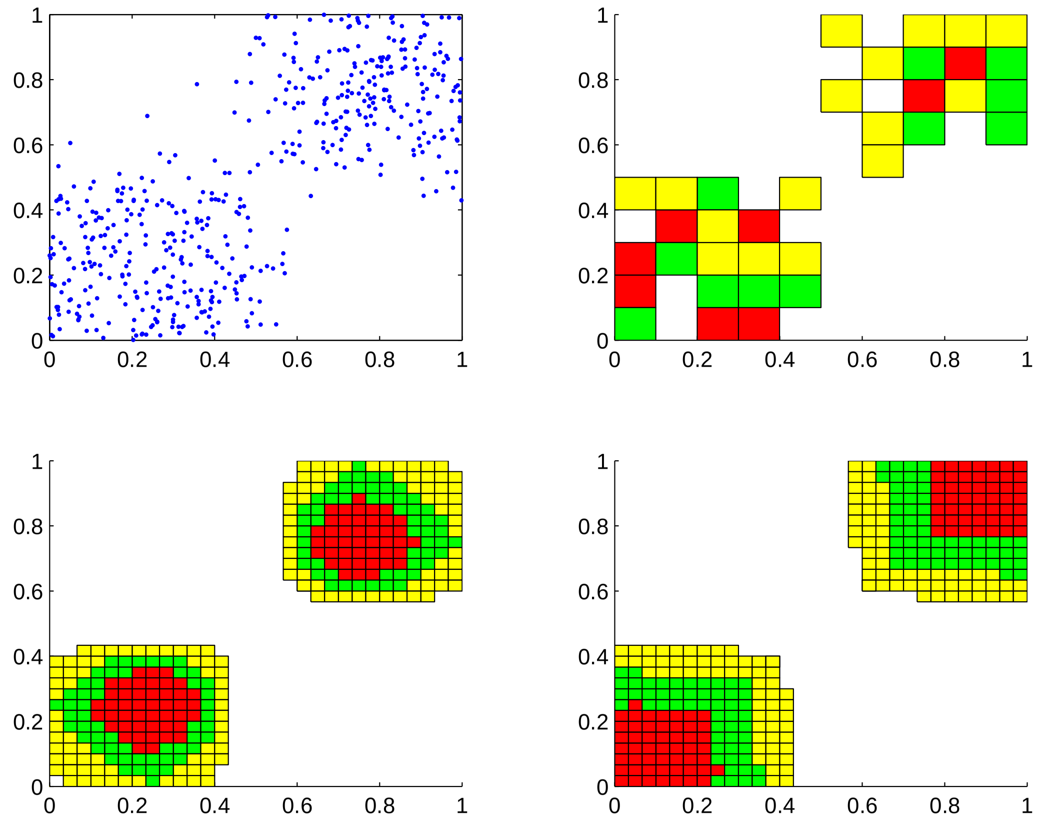
Estimation of the generating function and of the Spearman's Rho (ρ_k). The mean value of the estimates $\hat{\rho}_{\text{SP}}$ and $\hat{\rho}_{\text{NP}}$ are evaluated on 100 repetitions.



Estimation of high probability regions Q_α from C_2. Red: $\alpha = 0.25$, green: $\alpha = 0.5$, yellow: $\alpha = 0.75$. Top left: simulated sample, top right: nonparametric estimate, bottom left: semiparametric estimate, bottom right: semiparametric estimate with the true function ψ, $(n = 500)$.



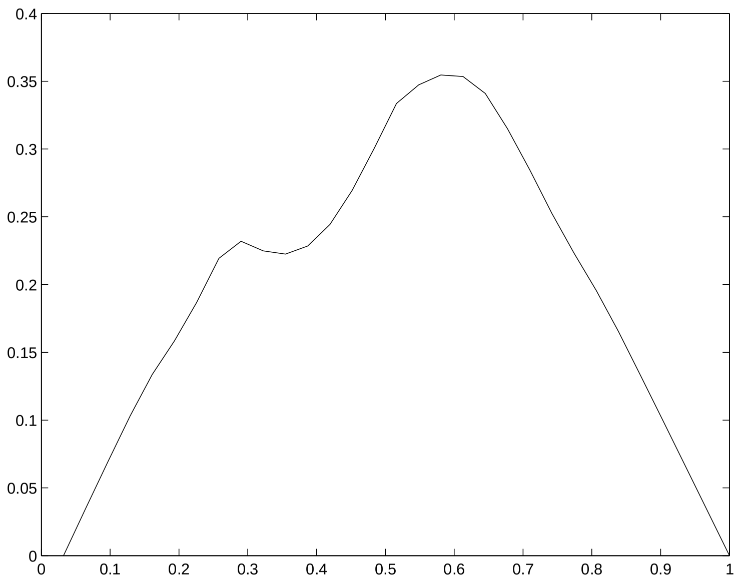
Estimation of high probability regions Q_α from C_4. Red: $\alpha = 0.25$, green: $\alpha = 0.5$, yellow: $\alpha = 0.75$. Top left: simulated sample, top right: nonparametric estimate, bottom left: semiparametric estimate, bottom right: semiparametric estimate with the true function ψ, $(n = 500)$.



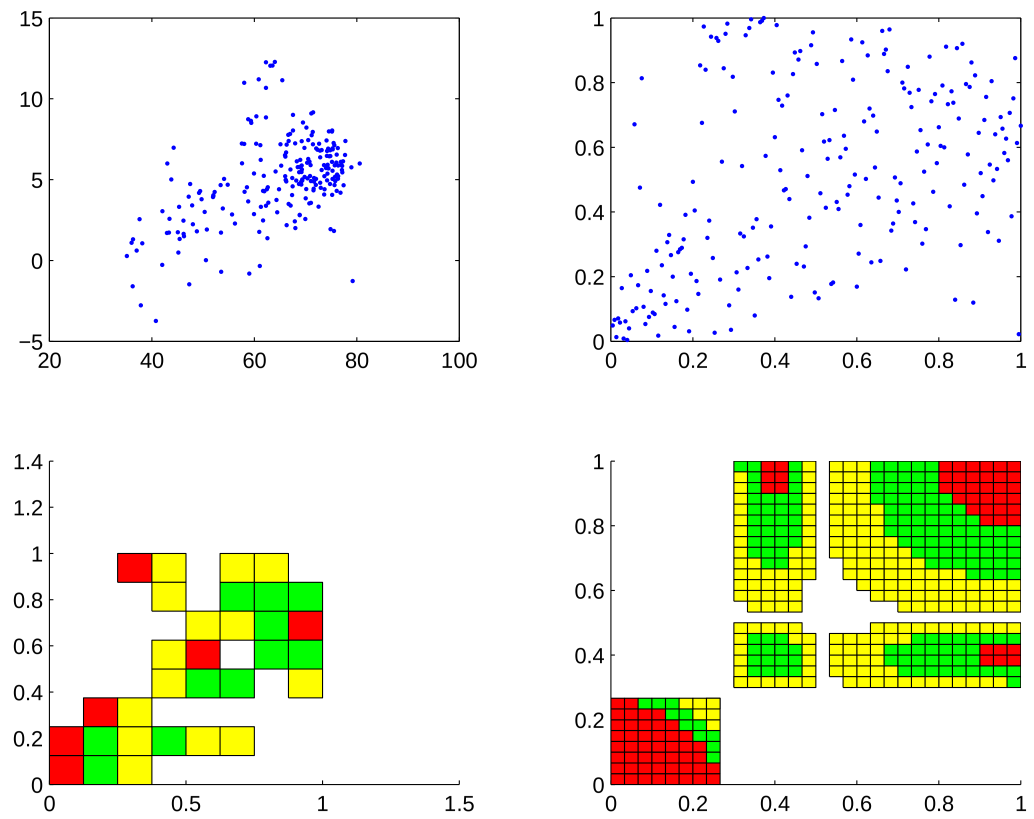
Estimation of high probability regions Q_α from C_8. Red: $\alpha = 0.25$, green: $\alpha = 0.5$, yellow: $\alpha = 0.75$. Top left: simulated sample, top right: nonparametric estimate, bottom left: semiparametric estimate, bottom right: semiparametric estimate with the true function ψ, $(n = 500)$.

6. Real data.

$n = 225$ countries, two variables: X, the life expectancy at birth (years) in 2002 of the total population and Y, the difference between the life expectancy at birth of women and men. `http://www.odci.gov/cia/publications/factbook/`.
According to the PQD test proposed in (Scaillet, 2004), these data are PQD.



$\hat{\rho}_{\text{NP}} = 52.4\%$
$\hat{\rho}_{\text{SP}} = 40.7\%$



Estimation of high probability regions Q_α from real data. Red: $\alpha = 0.25$, green: $\alpha = 0.5$, yellow: $\alpha = 0.75$. Top left: real data, top right: real data after rank transformation, bottom left: nonparametric estimate, bottom right: semiparametric estimate.

Further work.

- Goodness of fit test.
- Study of the sub-family $\phi(1) = 0$ without the assumption that θ is a constant function. (what is the lower bound of $\rho_{\theta,\phi}$?)
- Estimation of the function θ in the general case.

References.

- C. Amblard and S. Girard. A new bivariate extension of FGM copulas, *Metrika*, 70, 1–17, 2009.
- C. Amblard and S. Girard. Estimation procedures for a semiparametric family of bivariate copulas, *Journal of Computational and Graphical Statistics*, 14, 1–15, 2005.
- C. Amblard and S. Girard. Symmetry and dependence properties within a semiparametric family of bivariate copulas, *Nonparametric Statistics*, 14, 715–727, 2002.
- C. Amblard and S. Girard. A semiparametric family of symmetric bivariate copulas, *Comptes-Rendus de l'Académie des Sciences*, t. 333, Série I:129–132, 2001