Extreme L_p- quantiles as risk measures

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A simple way to assess the (environmental) risk is to compute a measure linked to the value X of the phenomenon of interest (rainfall height, wind speed, river flow, etc.):

- quantiles (= Value at Risk = return level),
- expectiles,
- tail conditional moments,
- spectral risk measures,
- distortion risk measures, etc.

Here, we focus on the first two measures: quantiles and expectiles.

We shall see how to estimate a simple extension of such measures.

Let X be a random variable with cumulative distribution function F. In the following, we assume that X has a right heavy tail.

Assumption (First order condition $\mathcal{C}_1(\gamma)$)

The survival function $\overline{F} := 1 - F$ is regularly varying with index $-1/\gamma < 0$:

$$\forall x > 0, \lim_{t \to +\infty} \frac{\overline{F}(tx)}{\overline{F}(t)} = x^{-1/\gamma}.$$

The next condition controls the rate of convergence in $\mathcal{C}_1(\gamma)$.

Assumption (Second order condition $\mathcal{C}_2(\gamma, \rho, A)$)

There exist $\gamma > 0$, $\rho < 0$ and a function A tending to zero at $+\infty$ with asymptotically constant sign such that:

$$\forall x > 0, \lim_{t \to \infty} \frac{1}{A(1/\overline{F}(t))} \left[\frac{\overline{F}(tx)}{\overline{F}(t)} - x^{-1/\gamma} \right] = x^{-1/\gamma} \frac{x^{\rho/\gamma} - 1}{\gamma\rho}.$$

It can be shown that $|A|$ is necessarily regularly varying with index ρ. The larger $|\rho|$ is, the smaller the approximation error $|A|$ is.

L^p-quantiles: Basic idea

Quantile of level τ: solution of the minimization problem

$$q(\tau) = \arg\min_{q\in\mathbb{R}} \mathbb{E}(|\tau - \mathbb{1}_{\{X\leq q\}}||X-q| - |\tau - \mathbb{1}_{\{X\leq 0\}}||X|).$$

Expectile of level τ: when $\mathbb{E}|X| < \infty$, solution of the minimization problem

$$\xi(\tau) = \arg\min_{q\in\mathbb{R}} \mathbb{E}(|\tau - \mathbb{1}_{\{X\leq q\}}||X-q|^2 - |\tau - \mathbb{1}_{\{X\leq 0\}}||X|^2).$$

Definition (L^p-quantile, Chen 1996)

Assume $\mathbb{E}|X|^{p-1} < \infty$. The L^p-quantile associated with X is the function $q(\cdot, p)$ defined on $(0,1)$ as

$$q(\tau, p) = \arg\min_{q\in\mathbb{R}} \mathbb{E}(|\tau - \mathbb{1}_{\{X\leq q\}}||X-q|^p - |\tau - \mathbb{1}_{\{X\leq 0\}}||X|^p).$$

Advantages and drawbacks

- Recall that classical quantiles ($p = 1$) and expectiles ($p = 2$) are particular cases of L^p-quantiles.
- When $p > 1$, the L^p-quantile $q(\tau, p)$ exists, is unique and verifies

$$\tau = \frac{\mathbb{E}(|X - q(\tau, p)|^{p-1} \mathbb{1}_{\{X \leq q(\tau,p)\}})}{\mathbb{E}(|X - q(\tau, p)|^{p-1})}.$$

  It can thus be interpreted in terms of (pseudo-)distance to X in L^{p-1}.

- The condition for the existence of L^p-quantiles is $\mathbb{E}|X|^{p-1} < \infty$. When $1 < p < 2$, it is a weaker condition than the existence condition of expectiles.
- When $p \neq 2$, the L^p-quantiles do not define a coherent risk measure (Bellini *et al.*, 2014) since they are not subadditive.

1) **Intermediate levels.** Introducing

$$\eta_\tau(x,p) = |\tau - \mathbb{1}_{\{x\le 0\}}||x|^p,$$

one has

$$q(\tau,p) = \arg\min_{q\in\mathbb{R}} \mathbb{E}(\eta_\tau(X-q,p) - \eta_\tau(X,p)).$$

The empirical estimator of $q(\tau,p)$ is obtained by minimizing the empirical counterpart of the previous criterion:

$$\widehat{q}_n(\tau,p) = \arg\min_{q\in\mathbb{R}} \frac{1}{n}\sum_{i=1}^{n} \eta_\tau(X_i - q, p).$$

According to Geyer (1996), since the empirical criterion is convex, the asymptotic behaviour of the minimizer directly depends on the asymptotic behaviour of the criterion itself.

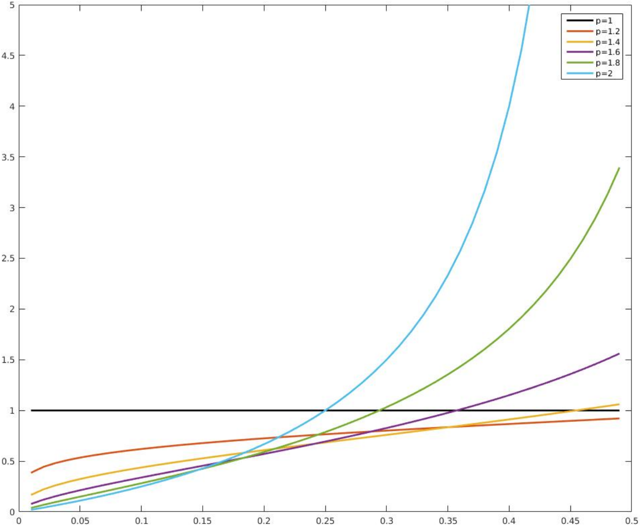
Theorem (Intermediate L^p−quantiles, Daouia, Girard & Stupfler, 2017)

Let $p > 1$. *Assume* $\mathcal{C}_2(\gamma, \rho, A)$ *holds with* $0 < \gamma < [2(p-1)]^{-1}$.

If $\tau_n \uparrow 1$ *such that* $n(1-\tau_n) \to \infty$ *and* $\sqrt{n(1-\tau_n)}A(1/(1-\tau_n)) \to \lambda \in \mathbb{R}$ *then,*

$$\sqrt{n(1-\tau_n)}\left(\frac{\widehat{q}_n(\tau_n, p)}{q(\tau_n, p)} - 1\right) \xrightarrow{d} \mathcal{N}\left(0, \gamma^2 V(\gamma, p)\right)$$

where $V(\gamma, p) = \dfrac{\Gamma(2p-1)\Gamma(\gamma^{-1} - 2p + 2)}{\Gamma(p)\Gamma(\gamma^{-1} - p + 1)}$ *and* $\Gamma(x)$ *is the Gamma function.*



Behaviour of variances $\gamma \in (0, 1/2) \mapsto V(\gamma, p)$ for some values of $p \in [1, 2]$.

2) **Arbitrary levels.** In order to remove the condition on τ_n, one has to show that Weissman's approximation is still valid for L^p-quantiles.

Proposition (Daouia, Girard & Stupfler, 2017)

Let $p > 1$. Assume $\mathcal{C}_1(\gamma)$ with $\gamma < 1/(p-1)$. Then,

$$\lim_{\tau \uparrow 1} \frac{q(\tau, p)}{q(\tau, 1)} = C(\gamma, p),$$

where $C(\gamma, p) = \left[\dfrac{\gamma}{B(p, \gamma^{-1} - p + 1)} \right]^{-\gamma}$ *and $B(x, y)$ is the Beta function.*

Extreme L^p-quantiles are asymptotically proportional to usual extreme quantiles, for all $p > 1$.

Weissman's approximation is thus still valid for L^p-quantiles:

$$q(\tau_n', p) \approx \left(\frac{1-\tau_n}{1-\tau_n'}\right)^{\gamma} q(\tau_n, p).$$

A Weissman type estimator can be derived for L^p-quantiles:

$$\widehat{q}_n^W(\tau_n'|\tau_n, p) = \left(\frac{1-\tau_n}{1-\tau_n'}\right)^{\widehat{\gamma}_n} \widehat{q}_n(\tau_n, p)$$

where $n(1-\tau_n) \to \infty$ and $n(1-\tau_n') \to c < \infty$.

Establishing the asymptotic behaviour of this estimator requires to investigate the error term in Weissman's approximation.

Theorem (Arbitrary extreme L^p-quantiles, Daouia, Girard & Stupfler, 2017)

Suppose $\mathcal{C}_2(\gamma, \rho, A)$ *holds with* $\gamma < [2(p-1)]^{-1}$.
Let τ_n, $\tau_n' \uparrow 1$ *such that* $n(1-\tau_n) \to \infty$, $n(1-\tau_n') \to c < \infty$ *and*

$$\begin{aligned} \sqrt{n(1-\tau_n)} \max\left(\frac{1}{q(\tau_n, 1)}, 1-\tau_n, A\left(\frac{1}{1-\tau_n}\right)\right) &= O(1) \\ \sqrt{n(1-\tau_n)}(\widehat{\gamma}_n - \gamma) &\xrightarrow{d} N. \end{aligned}$$

Then,

$$\frac{\sqrt{n(1-\tau_n)}}{\log([1-\tau_n]/[1-\tau_n'])}\left(\frac{\widehat{q}_n^W(\tau_n'|\tau_n, p)}{q(\tau_n', p)} - 1\right) \xrightarrow{d} N.$$

3) **Exploiting links between quantiles and L^p-quantiles**

– First, remark that from the property $q(\tau, p) \sim C(\gamma, p) q(\tau, 1)$, one can build other estimators of extreme L^p-quantiles:

- at intermediate levels: $\widetilde{q}_n(\tau_n, p) := C(\hat{\gamma}_n, p)\widehat{q}_n(\tau_n, 1)$ where $\hat{\gamma}_n$ is an estimator of the tail index and $\widehat{q}_n(\tau_n, 1) = X_{\lceil n\tau_n \rceil, n}$.
- at arbitrary levels: $\widetilde{q}_n^W(\tau_n' | \tau_n, p) := C(\hat{\gamma}_n, p)\widehat{q}_n^W(\tau_n' | \tau_n, 1)$ where $\widehat{q}_n^W(\tau_n' | \tau_n, 1)$ is Weissman's estimator.

Asymptotic normality results have been established under the condition $\gamma < (p-1)^{-1}$ instead of $\gamma < [2(p-1)]^{-1}$ for the previous theorem.

– Second, recall that the L^p-quantile $q(\tau_n, p)$ exists is unique and verifies

$$\tau_n = \frac{\mathbb{E}(|X - q(\tau_n, p)|^{p-1} \mathbb{1}_{\{X \leq q(\tau_n, p)\}})}{\mathbb{E}(|X - q(\tau_n, p)|^{p-1})}.$$

It is thus possible to find levels α_n and τ_n such that $q(\tau_n, p) = q(\alpha_n, 1)$ by imposing

$$\tau_n = \frac{\mathbb{E}(|X - q(\alpha_n, 1)|^{p-1} \mathbb{1}_{\{X \leq q(\alpha_n, 1)\}})}{\mathbb{E}(|X - q(\alpha_n, 1)|^{p-1})}.$$

Asymptotically, as $n \to \infty$, one can show that

$$\frac{1 - \tau_n}{1 - \alpha_n} \to \frac{1}{\gamma} B\left(p, \frac{1}{\gamma} - p + 1\right).$$

Starting from the two equations in red, one can estimate extreme quantiles from extreme L_p-quantiles.

Letting

$$\widehat{\tau}'_n(p, \alpha_n) := 1 - (1 - \alpha_n)\frac{1}{\widehat{\gamma}_n} B\left(p, \frac{1}{\widehat{\gamma}_n} - p + 1\right),$$

the extreme quantile $q(\alpha_n, 1)$ can be estimated by $q_n(\widehat{\tau}'_n(p, \alpha_n)|\tau_n, p)$ where $q_n(\cdot|\tau_n, p)$ is an estimator of the extreme L_p-quantile $q(\cdot, p)$ *i.e.* $q_n(\cdot|\tau_n, p) = \widehat{q}_n^W(\cdot|\tau_n, p)$ or $q_n(\cdot|\tau_n, p) = \widetilde{q}_n^W(\cdot|\tau_n, p)$.

Asymptotic normality results have been established for both estimators $\widehat{q}_n^W(\widehat{\tau}'_n(p, \alpha_n)|\tau_n, p)$ and $\widetilde{q}_n^W(\widehat{\tau}'_n(p, \alpha_n)|\tau_n, p)$.

Illustration on simulations

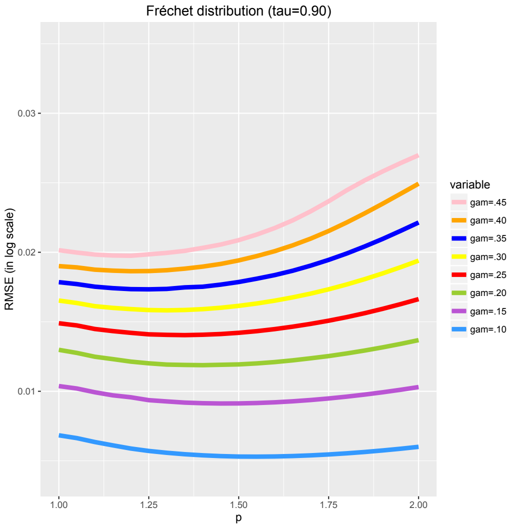
We focus on L^p-quantiles for $p \in (1,2)$ (alternative risk measure to expectiles with a weaker existence condition).

The accuracy is assessed by computing the relative mean-squared error (MSE) on 3000 replications of samples of size $n = 200$ from a Fréchet distribution:
$F(x) = e^{-x^{-1/\gamma}}, \ x > 0.$

1) Intermediate level

Which L^p-quantiles can be estimated accurately with $\widehat{q}_n(\tau_n, p)$?
We consider a L^p-quantile of level $\tau_n = 0.9$.
In the following, $\gamma \in \{0.1, 0.15, \dots, 0.45\}$ and $p \in \{1, 1.05, \dots, 2\}$.

Relative MSE - Fréchet distribution - Intermediate level



Horizontally: p, Vertically: relative MSE (in log scale) for different values of γ (small values: bottom curves, large values: top curves).

First conclusions:

- The estimation accuracy is getting lower when γ increases.
- When $\gamma \geq 0.2$, the estimation of expectiles ($p = 2$) is more difficult than the estimation of quantiles ($p = 1$).
- The value of p minimizing the relative MSE depends on the tail index γ. However, $p \in [1.2, 1.4]$ seems to be a good compromise.

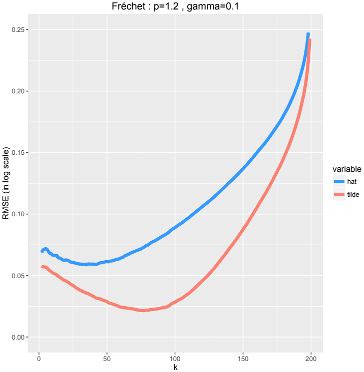
2) Extreme level

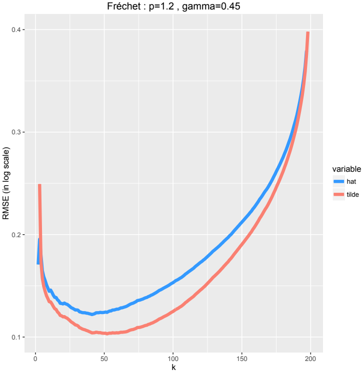
Comparison of:

- $\widehat{q}_n^W(\tau_n'|\tau_n, p)$ (based on empirical criterion + extrapolation),
- $\widetilde{q}_n^W(\tau_n'|\tau_n, p)$ (based on the extreme L_1- quantile).

We consider a L^p-quantile of level $\tau_n' = 1 - 1/n$.
In the following, $\tau_n = 1 - k/n$ where $k \in \{2, \ldots, n-1\}$, $\widehat{\gamma}_n$ is Hill's estimator, $\gamma \in \{0.1, 0.45\}$ and $p \in \{1.2, 1.5, 1.8\}$.

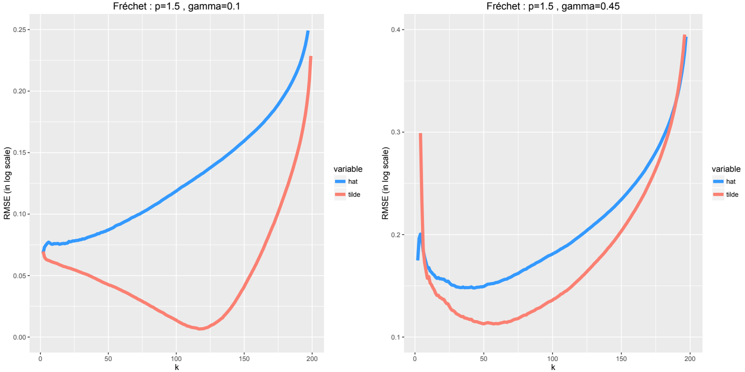
Relative MSE - Fréchet distribution - Extreme level





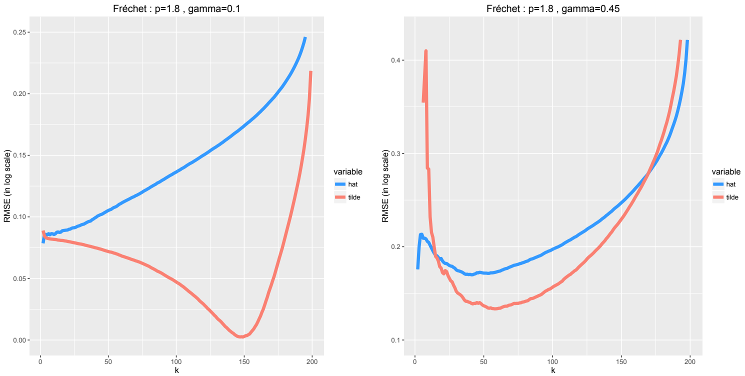
Horizontally: k, Vertically: relative MSE (in log scale) of $\widehat{q}_n^W(\tau_n'|\tau_n, p = 1.2)$ and $\widetilde{q}_n^W(\tau_n'|\tau_n, p = 1.2)$ as a function of $k \in \{2, \ldots, n-1\}$ (left: $\gamma = 0.1$, right: $\gamma = 0.45$).

Relative MSE - Fréchet distribution - Extreme level



Horizontally: k, Vertically: relative MSE (in log scale) of $\widehat{q}_n^W(\tau_n'|\tau_n, p = 1.5)$ and $\widetilde{q}_n^W(\tau_n'|\tau_n, p = 1.5)$ as a function of $k \in \{2, \dots, n-1\}$ (left: $\gamma = 0.1$, right: $\gamma = 0.45$).

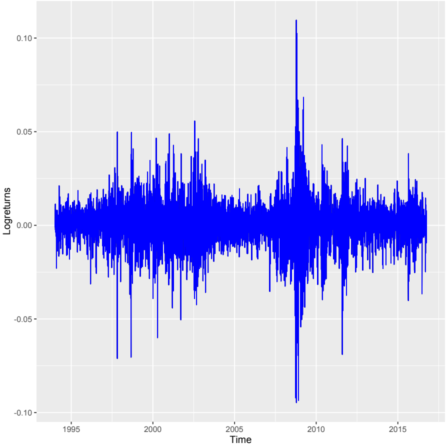
Relative MSE - Fréchet distribution - Extreme level



Horizontally: k, Vertically: relative MSE (in log scale) of $\widehat{q}_n^W(\tau_n'|\tau_n, p = 1.8)$ and $\widetilde{q}_n^W(\tau_n'|\tau_n, p = 1.8)$ as a function of $k \in \{2, \dots, n-1\}$ (left: $\gamma = 0.1$, right: $\gamma = 0.45$).

Illustration on real data

S&P500 index from Jan, 4th, 1994 to Sep, 30th, 2016 (5727 trading days).



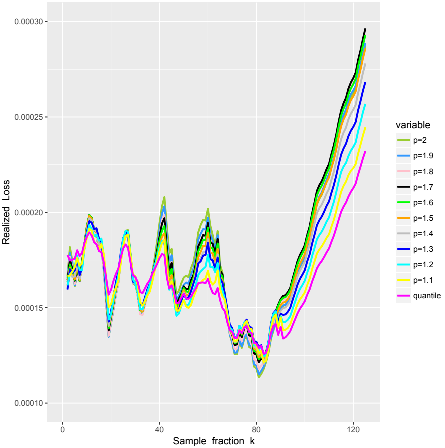
To reduce the potential serial dependence, we use lower frequency data by choosing weekly (Wednesday to Wednesday) returns in the same sample period (Cai *et al.*, 2015). This results in a sample $\{X_1, \ldots, X_{1176}\}$ of size 1176.

Computation of prediction errors

For $t = 1, \ldots, 656$

- Starting from $\{X_t, \ldots, X_{t+n-1}\}$ a training sample with $n = 520$,
- Our goal is to estimate $q(1/n, 1)$ which can be viewed as the weekly loss return for a once-per-decade financial crisis.
- Three estimators are computed:
  - $\widehat{q}_n^W(1/n|\tau_n, p = 1)$ (Weissman estimator for L_1- quantiles),
  - $\widehat{q}_n^W(\widehat{\tau}_n'(p, 1/n)|\tau_n, p)$ and $\widetilde{q}_n^W(\widehat{\tau}_n'(p, 1/n)|\tau_n, p)$ (based on estimators for L_p- quantiles).
- The associated prediction errors are computed with respect to X_{t+n}.

Prediction errors - Weekly loss returns



Horizontally: k, Vertically: Prediction error for $\widehat{q}_n^W(\widehat{\tau}_n'(p, 1/n)|\tau_n, p)$ (left), $\widetilde{q}_n^W(\widehat{\tau}_n'(p, 1/n)|\tau_n, p)$ (right) and $\widehat{q}_n^W(1/n|\tau_n, p = 1)$ (magenta) as a function of k.

Conclusion

- The extreme behaviour of L^p-quantile has been established.
- Classical quantiles as well as expectiles are particular cases of L^p-quantiles.
- In contrast to quantiles, L^p-quantiles take into account the whole tail of the distribution.
- The condition for existence of L^p-quantiles is weaker than for expectiles.
- It is possible to extrapolate to arbitrarily large levels.
- The theory has been extended to a mixing dependence framework and to real losses (non necessarily positive).

Some references

Bellini, F., Klar, B., Müller, A., Gianina, E.R. (2014). Generalized quantiles as risk measures, *Insurance: Mathematics and Economics* **54**: 41–48.

Cai, J., Einmahl, J., de Haan, L. and Zhou, C. (2015). Estimation of the marginal expected shortfall: the mean when a related variable is extreme, *JRSS B*, **77**: 417–442.

Chen, Z. (1996). Conditional L_p-quantiles and their application to testing of symmetry in non-parametric regression, *Statistics and Probability Letters* **29**: 107–115.

Daouia, A., Girard, S., Stupfler, G. (2017). Extreme M-quantiles as risk measures: From L_1 to L_p optimization, *Bernoulli*, to appear.

Daouia, A., Girard, S., Stupfler, G. (2017). Estimation of tail risk based on extreme expectiles, *JRSS-B*, to appear.

El Methni, J., Gardes, L., Girard, S. (2014). Nonparametric estimation of extreme risks from conditional heavy-tailed distributions, *Scandinavian Journal of Statistics* **41**: 988–1012.

Girard, S., Stupfler, G. (2015). Extreme geometric quantiles in a multivariate regular variation framework, *Extremes* **18**(4): 629–663.

Girard, S., Stupfler, G. (2017). Intriguing properties of extreme geometric quantiles, *REVSTAT: Statistical Journal*, **15**: 107–139.

Geyer, C.J. (1996). On the asymptotics of convex stochastic optimization, unpublished manuscript.

Koenker, R., Bassett, G.S. (1978). Regression quantiles, *Econometrica* **46**: 33–50.

Newey, W.K., Powell, J.L. (1987). Asymmetric least squares estimation and testing, *Econometrica* **55**: 819–847.

Weissman, I. (1978). Estimation of parameters and large quantiles based on the k largest observations, *Journal of the American Statistical Association* **73**: 812–815.