Estimating an endpoint using high order moments

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The problem

Let $(X_1, \ldots, X_n)$ be n independent copies of a positive random variable X, with bounded support $[0, \theta]$:

$$\theta := \sup\{x > 0 \,|\, \mathbb{P}(X \leq x) < 1\}$$

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Some existing methods

1. The maximum estimator and its improvements: Quenouille (1949), Miller (1964), Robson and Whitlock (1964), Cooke (1979), de Haan (1981).
   Drawback: Huge loss of information.
2. The maximum likelihood estimator in the Hall class: Hall (1982), Li and Peng (2009). This method uses the r_n largest statistics of the sample, with $r_n \to \infty$, $r_n/n \to 0$ as $n \to \infty$.
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The estimator

Let Z be a random variable with survival function

$$\forall z \in [0, \theta], \quad \overline{G}(z) = \mathbb{P}(Z > z) = (1 - z/\theta)^{\alpha},$$

where $\theta, \alpha > 0$. We get

$$\forall p \geq 1, \quad M_p := \mathbb{E}(Z^p) = \alpha \, \theta^p \; B(p+1, \alpha)$$

where $B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt$ is the Beta function.

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Then, for all $n \in \mathbb{N} \setminus \{0\}$, every arbitrary sequence (p_n) and all $a > 0$

$$\frac{1}{\theta} = \frac{1}{ap_n}\left[((a+1)p_n+1)\frac{M_{(a+1)p_n}}{M_{(a+1)p_n+1}} - (p_n+1)\frac{M_{p_n}}{M_{p_n+1}}\right].$$

Our estimator is defined in two steps:

1. Replace M_{p_n} by the exact moment $\mu_{p_n} := \mathbb{E}(X^{p_n})$.
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where $p_n \to \infty$.

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Consistency

Theorem (Consistency)

Provided that $n\,\dfrac{\mu_{(a+1)p_n}}{\theta_{(a+1)p_n}} \to \infty$, *we have* $\widehat{\theta}_n \xrightarrow{\mathbb{P}} \theta$ *as* $n \to \infty$.

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Asymptotic normality

To examine the asymptotic normality of $\widehat{\theta}_n$, we need some more hypotheses:

(A_1) $\forall\, x \in [0,\, \theta],\ \overline{F}(x) = (1 - x/\theta)^{\alpha}\, L((1 - x/\theta)^{-1})$ where $\theta,\ \alpha > 0$ and L is a slowly varying function.

(A_1) is the classical model for $\overline{F}$ in the extreme value framework. Note that if (A_1) holds, then $\overline{F}$ has extreme value index $\gamma = -1/\alpha$.

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(A_2) L is *normalised*:

$$L(x) = \exp\left(\int_1^x \frac{\eta(t)}{t}\,dt\right),$$

where η goes to 0 at ∞, is continuously derivable in $(1, \infty)$, ultimately monotonic, non identically 0, and

$$\exists \nu \leq 0, \quad x\frac{\eta'(x)}{\eta(x)} \to \nu \quad \text{as } x \to \infty.$$

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Theorem (Asymptotic normality)

Assume that $n\, p_n^{-\alpha} L(p_n) \to \infty$ *and* (A_1), (A_2) *hold. Assume further that* $n\, p_n^{-\alpha} L(p_n) \eta^2(p_n) \to 0$. *Then*

$$v_n \left(\frac{\widehat{\theta}_n}{\theta} - 1 \right) \xrightarrow{d} \mathcal{N}\left(0,\ V(\alpha,\ a)\right) \quad \text{as } n \to \infty$$

with $v_n = \sqrt{n\, L(p_n)}\, p_n^{-\alpha/2+1}$ *and*

$$V(\alpha,\ a) = \frac{\alpha + 1}{a^2\, \Gamma(\alpha)} \left[2^{-\alpha-2} - 2\frac{(a+1)^{\alpha+1}}{(a+2)^{\alpha+2}} + 2^{-\alpha-2}(a+1)^{\alpha} \right].$$

Numerical study: first case

The performances of our estimator are examined on some finite sample situations.

First case: X has survival function

$$\forall\, x \in (0,\,1), \quad \overline{F}(x) = \left[1 + \left(\frac{1}{x} - 1\right)^{-\tau_1}\right]^{-\tau_2}$$

with $\tau_1,\, \tau_2 > 0$.

Namely, $X = 1 - \dfrac{1}{1+Y}$ where Y is Burr type XII distributed:

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In this example, $\theta = 1$, $\alpha = \tau_1 \tau_2$,

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and $\nu = -\min(\tau_1, 1)$.

In fact, L belongs to the Hall class (Hall 1982):

$$L(y) = C + D\, y^{-\beta}(1 + \delta(y)) \quad \text{for large enough } y$$

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Extreme value moment estimator:

Let $X_{1,n} \leq \ldots \leq X_{n,n}$ be the nth order statistics of the sample $(X_1, \ldots, X_n)$. For $j = 1, 2$, let

$$M_n^{(j)} = \frac{1}{k} \sum_{i=0}^{k-1} [\ln X_{n-i,n} - \ln X_{n-k,n}]^j$$

where $k = k(n) \to \infty$, $k/n \to 0$.

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\widehat{\gamma}_n^- &= 1 - \frac{1}{2}\left[1 - \frac{\left[M_n^{(1)}\right]^2}{M_n^{(2)}}\right]^{-1}, \\
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The *extreme value moment estimator* of the endpoint θ is (Dekkers *et al.*, 1989, Aarssen and de Haan, 1994)

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Methodology

Parameters:

- 1000 samples with size $n = 500$ of random variables with distribution function $\overline{F}$.
- Extreme value moment estimator: threshold k varying from 2 to $n - 2$.
- High order moments estimator: $p_n = n^{1/\alpha} / \ln \ln n$, and $a \in \{0.1, 0.2, \ldots, 25\}$.

Mean L^1-errors (over the 1000 samples) are computed for both estimators, for each value of k and a. The minimal mean L^1-errors obtained this way are then recorded.

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positif_evm_burr-eps-converted-to.

Figure 1: $x-$axis: threshold k. Dashed line: maximum estimator, solid line: extreme value moment estimator. Top left: $(\tau_1, \tau_2) = (1, 1)$, top right: $(\tau_1, \tau_2) = (5/6, 6/5)$, bottom left: $(\tau_1, \tau_2) = (2/3, 3/2)$, bottom right: $(\tau_1, \tau_2) = (1/2, 2)$.

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Figure 2: $x-$axis: parameter a. Dashed line: maximum estimator, solid line: high order moments estimator. Top left: $(\tau_1, \tau_2) = (1, 1)$, top right: $(\tau_1, \tau_2) = (5/6, 6/5)$, bottom left: $(\tau_1, \tau_2) = (2/3, 3/2)$, bottom right: $(\tau_1, \tau_2) = (1/2, 2)$.

Parameters	Extreme value moment estimator	$\widehat{\theta}_n$
$(\tau_1, \tau_2) = (1, 1)$ $\Rightarrow (\alpha, \nu) = (1, -1)$	$1.7 \cdot 10^{-3}$	$1.6 \cdot 10^{-3}$
$(\tau_1, \tau_2) = (5/6, 6/5)$ $\Rightarrow (\alpha, \nu) = (1, -5/6)$	$2.1 \cdot 10^{-3}$	$1.7 \cdot 10^{-3}$
$(\tau_1, \tau_2) = (2/3, 3/2)$ $\Rightarrow (\alpha, \nu) = (1, -2/3)$	$2.0 \cdot 10^{-3}$	$1.8 \cdot 10^{-3}$
$(\tau_1, \tau_2) = (1/2, 2)$ $\Rightarrow (\alpha, \nu) = (1, -1/2)$	$2.4 \cdot 10^{-3}$	$2.0 \cdot 10^{-3}$

Table 1: Minimum mean L^1-error associated to both estimators.

As $|\nu|$ decreases, performances of the estimators decrease. Note that the high order moments estimator outperforms the extreme value moment estimator in all four situations.

Numerical study: second case

Second case: X has survival function

$$\forall x \in (0,\,1), \quad \overline{F}(x) = \frac{1}{\Gamma(b)} \int_{-\ln(1-x)}^{\infty} (\lambda t)^{b-1}\, \lambda e^{-\lambda t}\, dt$$

with $b,\, \lambda > 0$.

Namely, $X = 1 - e^{-Y}$ where Y is Gamma$(b,\,\lambda)$ distributed.

Here $\theta = 1$, $\alpha = \lambda$,

$$L(y) = \frac{\lambda^{b-1}}{\Gamma(b)} \ln^{b-1}(y) \left[1 + (b-1) \int_{1}^{\infty} u^{b-2}\, e^{-\lambda(u-1)\ln y}\, du \right]$$

and $\nu = 0$.

Numerical study: second case

Second case: X has survival function

$$\forall x \in (0,1), \quad \overline{F}(x) = \frac{1}{\Gamma(b)} \int_{-\ln(1-x)}^{\infty} (\lambda t)^{b-1} \lambda e^{-\lambda t} \, dt$$

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positif_evm_loggamma-eps-converted

Figure 3: $x-$axis: threshold k. Dashed line: maximum estimator, solid line: extreme value moment estimator. Top left: $(b,\,\lambda)=(2,\,1)$, top right: $(b,\,\lambda)=(2,\,5/4)$, bottom left: $(b,\,\lambda)=(2,\,5/3)$, bottom right: $(b,\,\lambda)=(2,\,5/2)$.

positif_mom_loggamma-eps-converted

Figure 4: $x-$axis: parameter a. Dashed line: maximum estimator, solid line: high order moments estimator. Top left: $(b, \lambda) = (2, 1)$, top right: $(b, \lambda) = (2, 5/4)$, bottom left: $(b, \lambda) = (2, 5/3)$, bottom right: $(b, \lambda) = (2, 5/2)$.

Parameters	Extreme value moment estimator	$\widehat{\theta}_n$
$(b, \lambda) = (2, 1)$ $\Rightarrow (\alpha, \nu) = (1, 0)$	$1.9 \cdot 10^{-4}$	$1.9 \cdot 10^{-4}$
$(b, \lambda) = (2, 5/4)$ $\Rightarrow (\alpha, \nu) = (5/4, 0)$	$9.2 \cdot 10^{-4}$	$8.4 \cdot 10^{-4}$
$(b, \lambda) = (2, 5/3)$ $\Rightarrow (\alpha, \nu) = (5/3, 0)$	$4.5 \cdot 10^{-3}$	$3.9 \cdot 10^{-3}$
$(b, \lambda) = (2, 5/2)$ $\Rightarrow (\alpha, \nu) = (5/2, 0)$	$2.1 \cdot 10^{-2}$	$1.8 \cdot 10^{-2}$

Table 2: Minimum mean L^1-error associated to both estimators.

Both estimations worsen as α increases. Remark again that the high order moments estimator performs better than the extreme value moment estimator in all four situations.

Conclusion and forthcoming studies

The high order moments method yields satisfactory results, be them theoretical or practical. Contrary to most methods in endpoint estimation, the high order moments approach uses the whole given sample.

Future developments include:

1. Adapting this method to the standard Weibull domain of attraction.
2. Using the high order moments method to design an estimator of the extreme value index.

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References

Aarssen, K., de Haan, L. (1994). On the maximal life span of humans. *Mathematical Population Studies*, **4**:259–281.

Dekkers, A.L.M., Einmahl, J.H.J., de Haan, L. (1989). A moment estimator for the index of an extreme value distribution. *Annals of Statistics*, **17**:1833–1855.

Girard, S., Jacob, P. (2008). Frontier estimation via kernel regression on high power transformed data, *Journal of Multivariate Analysis*, **99**:403–420.

Hall, P. (1982). On estimating the endpoint of a distribution. *The Annals of Statistics*, **10**(2):556–568.

S. Girard, A. Guillou, G. Stupfler. (2012). Estimating an endpoint with high order moments in the Weibull domain of attraction. *Statistics and Probability Letters*, **82**, 2136–2144.

S. Girard, A. Guillou, G. Stupfler. (2012). Estimating an endpoint with high order moments. *Test*, **21**, 697–729.

S. Girard, A. Guillou, G. Stupfler. (2013). rontier estimation with kernel regression on high order moments. *Journal of Multivariate Analysis*, **116**, 172–189.

S. Girard, A. Guillou, G. Stupfler. (2014). Uniform strong consistency of a frontier estimator using kernel regression on high order moments. *ESAIM: Probability and Statistics*, **18**, 642–666.

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