On the asymptotic behaviour of extreme geometric quantiles

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Outline

Extreme multivariate quantiles?

- The natural order on $\mathbb{R}$ induces a universal definition of quantiles of underlying univariate distribution functions;
- This is not true in $\mathbb{R}^d$, $d \geq 2$, since no natural order exists in this case;
- Many definitions of multivariate quantiles have since been suggested in the literature:
  - ◊ Depth-based quantiles: Liu *et al.* (1999), Zuo and Serfling (2000);
  - ◊ Norm minimisation: Abdous and Theodorescu (1992), Chaudhuri (1996);
  - ◊ Generalised quantile processes: Einmahl and Mason (1992).

For a review, see *e.g.* Serfling (2002).

Furthermore, although extreme univariate quantiles are now used in many real-life applications (climatology, actuarial science, finance...), very few works actually study extreme multivariate quantiles:

- Chernozhukov (2005): extreme quantile estimation in a linear quantile regression model;
- Cai *et al.* (2011) and Einmahl *et al.* (2013): study of the extreme level sets of the underlying probability density function.

Goal of this talk: to introduce and study a possible notion of extreme multivariate quantile.

Geometric quantiles

If X is a real-valued random variable, the univariate p–th quantile $x_p := \inf\{t \in \mathbb{R} \,|\, \mathbb{P}(X \leq t) \geq p\}$ of X can be obtained by solving the optimisation problem

$$\arg\min_{q \in \mathbb{R}} \mathbb{E}(\phi(2p-1, X-q) - \phi(2p-1, X))$$

where $\phi(u,t) = |t| + ut$.

- When $|X|$ has a finite expectation, this problem becomes

  $$\arg\min_{q \in \mathbb{R}} \mathbb{E}|X-q| + (2p-1)\mathbb{E}(X-q).$$

  In particular, the median $x_{1/2}$ of X is obtained by minimising $\mathbb{E}|X-q|$ with respect to q;
- Subtracting $\phi(2p-1, X)$ makes the cost function well-defined even when $|X|$ has an infinite expectation.

In $\mathbb{R}^d$, $d \geq 2$, analogues of the absolute value $|\cdot|$ and product $\cdot$ are given by the Euclidean norm $\|\cdot\|$ and Euclidean inner product $\langle \cdot, \cdot \rangle$.

When X is a multivariate random vector, the geometric quantiles of X, introduced by Chaudhuri (1996), are thus obtained by adapting and solving the aforementioned problem in the multivariate context.

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When X is a multivariate random vector, the geometric quantiles of X, introduced by Chaudhuri (1996), are thus obtained by adapting and solving the aforementioned problem in the multivariate context.

Definition 1 (Chaudhuri 1996)

If $u \in \mathbb{R}^d$ is an arbitrary vector, a geometric u–th quantile of X, if it exists, is a solution of the optimisation problem

$$\arg\min_{q\in\mathbb{R}^d} \mathbb{E}(\phi(u, X - q) - \phi(u, X)) \qquad (P_u)$$

with $\phi(u, t) = \|t\| + \langle u, t\rangle$.

Such multivariate quantiles enjoy several interesting properties:

- For every u in the unit open ball B^d of $\mathbb{R}^d$, there exists a unique geometric u–th quantile whenever the distribution of X is not concentrated on a single straight line in $\mathbb{R}^d$ (Chaudhuri, 1996);
- They are equivariant under any orthogonal transformation (Chaudhuri, 1996);
- The geometric quantile function characterises the associated distribution (Koltchinskii, 1997).

They make reasonable candidates when trying to define multivariate quantiles. Our focus here is to define and study the properties of extreme geometric quantiles.

Asymptotic behaviour: a first step

From now on, we assume that the distribution of X is not concentrated on a single straight line in $\mathbb{R}^d$ and non-atomic. Then:

- For every $u \in B^d$, the u–th geometric quantile exists and is unique;
- For any $u \in \mathbb{R}^d$, if there is a solution $q(u) \in \mathbb{R}^d$ to problem (P_u), then the gradient of the cost function must be zero at $q(u)$, that is

$$u + \mathbb{E}\left(\frac{X - q(u)}{\|X - q(u)\|}\right) = 0.$$

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Proposition 1 (Chaudhuri 1996, Girard and S. 2014)

The optimisation problem (P_u) has a solution if and only if $u \in B^d$.

It follows from the previous result that:

- We cannot compute a geometric quantile with unit index vector, unlike in the univariate case if the distribution has a finite (left or right) endpoint;
- We may nevertheless study the asymptotics of a geometric quantile $q(v)$ when v approaches the unit sphere: such quantiles will be referred to as extreme geometric quantiles.

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Theorem 1 (Girard and S. 2014)

Let S^{d-1} be the unit sphere of $\mathbb{R}^d$.

(i) *It holds that $\|q(v)\| \to \infty$ as $\|v\| \to 1$.*

(ii) *Moreover, if $v \to u$ with $u \in S^{d-1}$ and $v \in B^d$ then*

$$\frac{q(v)}{\|q(v)\|} \to u.$$

Theorem 1 shows two properties of geometric quantiles:

- The magnitude of extreme geometric quantiles diverges to infinity.
  - ◊ Rather intriguing: it holds true even if the distribution of X has a compact support;
  - ◊ Related point: sample geometric quantiles do not necessarily lie within the convex hull of the sample, see Breckling *et al.* (2001).
- If $v \to u \in S^{d-1}$ then the extreme geometric quantile $q(v)$ has asymptotic direction u.

Our main results specify the convergences in Theorem 1 under further assumptions.

Asymptotic behaviour: when there are finite moments

Our first result is obtained in the case when $\|X\|$ satisfies certain moment conditions. It focuses on extreme geometric quantiles in the direction $u \in S^{d-1}$, *i.e.* having the form $q(\lambda u)$, with $\lambda \uparrow 1$.

Asymptotic behaviour: when there are finite moments

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Theorem 2 (Girard and S. 2014)

Let $u \in S^{d-1}$. *Define* $\Pi_u(x) = x - \langle x, u \rangle u$.

(i) *If* $\mathbb{E}\|X\| < \infty$ *then*

$$\|q(\lambda u)\| \left(\frac{q(\lambda u)}{\|q(\lambda u)\|} - u \right) \to \mathbb{E}(\Pi_u(X)) \quad \text{as} \quad \lambda \uparrow 1.$$

(ii) *If* $\mathbb{E}\|X\|^2 < \infty$ *and* Σ *denotes the covariance matrix of* X *then*

$$\|q(\lambda u)\|^2 (1 - \lambda) \to \frac{1}{2} \left(\operatorname{tr} \Sigma - u' \Sigma u \right) > 0 \quad \text{as} \quad \lambda \uparrow 1.$$

Consequences of Theorem 2

If $\|X\|$ has a finite second moment, then asymptotically:

- the asymptotic direction of an extreme geometric quantile in the direction u is exactly u;
- the magnitude of an extreme geometric quantile in the direction u is asymptotically determined by u and the covariance matrix Σ of X.

In particular, the extreme geometric quantiles of two probability distributions which admit the same finite covariance matrix are asymptotically equivalent.

$\Rightarrow$ no information can be recovered on the behaviour of X far from the origin basing solely on extreme geometric quantiles.

Asymptotic behaviour: in a multivariate regular variation framework

When the moment conditions in Theorem 2 are no longer satisfied, the asymptotic properties of extreme geometric quantiles can be studied in a multivariate regular variation framework:

(M_α) The random vector X has a probability density function f which is continuous on a neighborhood of infinity and such that:

- the function $y \mapsto \|y\|^d f(y)$ is bounded in any compact neighborhood of 0;
- there exist a positive function Q on $\mathbb{R}^d$ and a function V which is regularly varying at infinity with index $-\alpha < 0$, such that

$$\forall y \neq 0, \; \left| \frac{f(ty)}{t^{-d}V(t)} - Q(y) \right| \; \to \; 0$$

$$\text{and} \quad \sup_{w \in S^{d-1}} \left| \frac{f(tw)}{t^{-d}V(t)} - Q(w) \right| \; \to \; 0 \text{ as } t \to \infty.$$

This model is closely related to the one of Cai *et al.* (2011). If (M_α) holds, then:

- The function Q is a homogeneous continuous function of degree $-d-\alpha$ on $\mathbb{R}^d \setminus \{0\}$;
- We have that $f(y) = \|y\|^{-d} V(\|y\|) Q(y/\|y\|)(1+o(1))$ for large $\|y\|$ and thus $f(y)$ is roughly of order $\|y\|^{-d-\alpha}$;
- The expectation $\mathbb{E}\|X\|^\beta$ is finite if $\beta < \alpha$.

In particular, the case $\alpha > 2$ is covered by Theorem 2.

Theorem 3 (Girard and S. 2014)

Let $u \in S^{d-1}$.

(i) *If* (M_α) *holds with* $\alpha \in (0,1)$, *then*

$$\frac{1}{V(\|q(\lambda u)\|)}\left(\frac{q(\lambda u)}{\|q(\lambda u)\|} - u\right) \to \int_{\mathbb{R}^d} \frac{\Pi_u(y)}{\|y-u\|} Q(y)dy \quad \text{as} \quad \lambda \uparrow 1.$$

(ii) *If* (M_α) *holds with* $\alpha \in (0,2)$, *then*

$$\frac{1-\lambda}{V(\|q(\lambda u)\|)} \to \int_{\mathbb{R}^d} \left(1 + \frac{\langle y-u,\, u\rangle}{\|y-u\|}\right) Q(y)dy \quad \text{as} \quad \lambda \uparrow 1.$$

Comments on Theorem 3

Since V is regularly varying with index $-\alpha$, it follows that when $\alpha \in (0, 2)$, the magnitude of an extreme geometric quantile behaves roughly like $(1 - \lambda)^{-1/\alpha}$ as $\lambda \uparrow 1$.

$\Rightarrow$ In this case, the magnitude of an extreme geometric quantile features the behaviour of the distribution of X far from the origin.

However, Theorem 3 excludes the limit cases $\alpha = 1$ for the asymptotic direction and $\alpha = 2$ for the asymptotic magnitude.

To give an idea of what can be said when $\alpha = 1$ or $\alpha = 2$, we introduce the following sub-model of (M_α):

(M'_α) For all $x \neq 0$, $f(x) = (x'\Sigma^{-1}x)^{\alpha/2} Q(x) V((x'\Sigma^{-1}x)^{1/2})$ where

- Σ is a positive definite $d \times d$ symmetric matrix;
- $Q(x) = (x'\Sigma^{-1}x)^{(-d-\alpha)/2} \psi(x/(x'\Sigma^{-1}x)^{1/2})$ where ψ is positive and continuous on the ellipsoid $E_\Sigma^{d-1} = \{x \in \mathbb{R}^d \,|\, x'\Sigma^{-1}x = 1\}$;
- $V : t \mapsto t^{-\alpha} L(t)$ is a bounded function, with L being a slowly varying function at infinity which is continuous in a neighborhood of infinity and is such that

$$\int_0^\infty L(r) \frac{dr}{r^{1+\alpha}} < \infty \quad \text{and} \quad \mathcal{L}(t) := \int_1^t L(r) \frac{dr}{r} \to \infty \quad \text{as} \quad t \to \infty.$$

If (M'_α) holds, then:

- The expectation $\mathbb{E}\|X\|^\beta$ is finite if and only if $\beta < \alpha$;
- We may define a surface measure on the ellipsoid E_Σ^{d-1} by

$$\mu_\Sigma(C) = (\det \Sigma)^{1/2}\,\sigma\left(\Sigma^{-1/2}C\right)$$

where σ is the standard surface measure on S^{d-1}.

Theorem 4 (Girard and S. 2014)

Let $u \in S^{d-1}$.

(i) *If* (M_1') *holds then, as* $\lambda \to 1$:

$$\frac{\|q(\lambda u)\|}{\mathcal{L}(\|q(\lambda u)\|)}\left(\frac{q(\lambda u)}{\|q(\lambda u)\|} - u\right) \to \int_{E_\Sigma^{d-1}} \Pi_u(w)\psi(w)\mu_\Sigma(dw).$$

(ii) *If* (M_2') *holds then, as* $\lambda \to 1$:

$$\frac{\|q(\lambda u)\|^2}{\mathcal{L}(\|q(\lambda u)\|)}(1-\lambda) \to \frac{1}{2}\int_{E_\Sigma^{d-1}} \langle \Pi_u(w),\, w\rangle \psi(w)\mu_\Sigma(dw).$$

Comments on Theorem 4

A particular consequence is that if (M_2') holds then the magnitude of an extreme geometric quantile does again feature the behaviour of the distribution of X far from the origin, through the function L.

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Example

If $L(t) \propto (\log t)^\beta$ on $(1, \infty)$, where $\beta > -1$, then:

$$\|q(\lambda u)\| \propto (1-\lambda)^{-1/2} \left[\log\left(\frac{1}{1-\lambda}\right)\right]^{(\beta+1)/2} \quad \text{as} \quad \lambda \uparrow 1.$$

Thus, the slower f converges to 0 at infinity, the larger are the extreme geometric quantiles.

Consequences of our main results

For all $\alpha > 0$, we can write

$$\begin{aligned} \frac{q(\lambda u)}{\|q(\lambda u)\|} - u &\propto R_{1,\alpha}((1-\lambda)^{-1}) \\ \text{and } \|q(\lambda u)\| &\propto R_{2,\alpha}((1-\lambda)^{-1}) \text{ as } \lambda \uparrow 1, \end{aligned}$$

where $R_{1,\alpha}$ and $R_{2,\alpha}$ are regularly varying functions with respective indices $-\min(1,\alpha)/\min(2,\alpha)$ and $1/\min(2,\alpha)$.

$\Rightarrow$ Extreme geometric quantiles feature the behaviour of X far from the origin only when the distribution function of $\|X\|$ decays sufficiently slowly at infinity.

Numerical illustrations: Theorem 2

We choose $d = 2$ to make the display easier. The following two bivariate distributions are considered:

- the centred Gaussian bivariate distribution $\mathcal{N}(0, v_X, v_Y, v_{XY})$, whose probability density function is:

$$f(x, y) = \frac{1}{2\pi\sqrt{\det \Sigma}} \exp\left(-\frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix}' \Sigma^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \right)$$

$$\text{with } \Sigma = \begin{pmatrix} v_X & v_{XY} \\ v_{XY} & v_Y \end{pmatrix}.$$

- a double exponential distribution $\mathcal{E}(\lambda_-, \mu_-, \lambda_+, \mu_+)$, with λ_-, μ_-, λ_+, $\mu_+ > 0$, whose probability density function is:

$$f(x,y) = \begin{cases} \dfrac{\lambda_+\mu_+}{4} e^{-\lambda_+|x|-\mu_+|y|} & \text{if } \; xy > 0, \\ \dfrac{\lambda_-\mu_-}{4} e^{-\lambda_-|x|-\mu_-|y|} & \text{if } \; xy \leq 0. \end{cases}$$

In this case, X is centred and has covariance matrix:

$$\Sigma = \begin{pmatrix} \dfrac{1}{\lambda_-^2} + \dfrac{1}{\lambda_+^2} & \dfrac{1}{2}\left[\dfrac{1}{\lambda_+\mu_+} - \dfrac{1}{\lambda_-\mu_-}\right] \\ \dfrac{1}{2}\left[\dfrac{1}{\lambda_+\mu_+} - \dfrac{1}{\lambda_-\mu_-}\right] & \dfrac{1}{\mu_-^2} + \dfrac{1}{\mu_+^2} \end{pmatrix}.$$

Since both distributions have a finite covariance matrix Σ, Theorem 2 entails that their extreme geometric quantiles are asymptotically equal to:

$$q_{\mathrm{eq}}(\lambda u) := (1-\lambda)^{-1/2}\left[\frac{1}{2}\left(\operatorname{tr}\Sigma - u'\Sigma u\right)\right]^{1/2} u.$$

⇒ Goal: to show that for these two distributions, equal covariance matrices induce equivalent extreme geometric quantiles, and to assess the accuracy of the asymptotic equivalent.

We choose three different sets of parameters, in order that the related covariance matrices coincide:

- $\mathcal{N}(0, 1/2, 1/2, 0)$ and $\mathcal{E}(2, 2, 2, 2)$ with spherical covariance matrices;
- $\mathcal{N}(0, 1/8, 3/4, 0)$ and $\mathcal{E}(4, 2\sqrt{2/3}, 4, 2\sqrt{2/3})$ with diagonal but non-spherical covariance matrices;
- $\mathcal{N}(0, 1/2, 1/2, 1/6)$ and $\mathcal{E}(2\sqrt{3}, 2\sqrt{3}, 2\sqrt{3/5}, 2\sqrt{3/5})$ with full covariance matrices.

Any $u \in S^1$ can be written $u = u_\theta = (\cos\theta, \sin\theta)$, $\theta \in [0, 2\pi)$. We let $\lambda = 0.995$ and in each case, we compute:

- the true iso-quantile curve $\mathcal{C}q(\lambda) = \{q(\lambda u_\theta),\ \theta \in [0, 2\pi)\}$;
- its asymptotic equivalent $\mathcal{C}q_{\mathrm{eq}}(\lambda) = \{q_{\mathrm{eq}}(\lambda u_\theta),\ \theta \in [0, 2\pi)\}$.

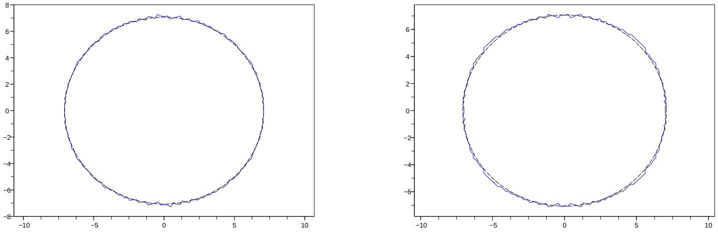

Figure 1: Case of the Gaussian $\mathcal{N}(0, 1/2, 1/2, 0)$ (left) and double exponential $\mathcal{E}(2, 2, 2, 2)$ (right) distributions for $\lambda = 0.995$. Iso-quantile curves $\mathcal{C}q(\lambda)$ (full blue line) and $\mathcal{C}q_{\text{eq}}(\lambda)$ (dashed black line).

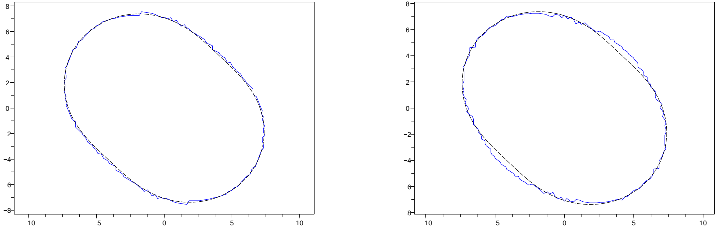

Figure 2: Case of the Gaussian $\mathcal{N}(0, 1/2, 1/2, 1/6)$ (left) and double exponential $\mathcal{E}(2\sqrt{3}, 2\sqrt{3}, 2\sqrt{3/5}, 2\sqrt{3/5})$ (right) distributions for $\lambda = 0.995$. Iso-quantile curves $\mathcal{C}q(\lambda)$ (full blue line) and $\mathcal{C}q_{\text{eq}}(\lambda)$ (dashed black line).

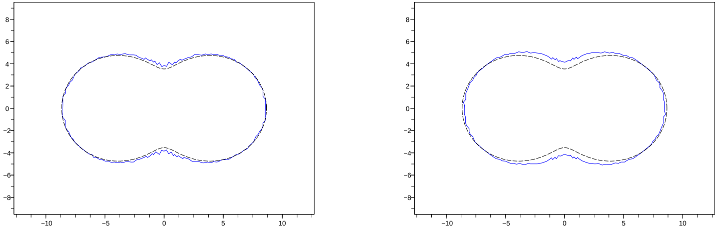

Figure 3: Case of the Gaussian $\mathcal{N}(0, 1/8, 3/4, 0)$ (left) and double exponential $\mathcal{E}(4, 2\sqrt{2/3}, 4, 2\sqrt{2/3})$ (right) distributions for $\lambda = 0.995$. Iso-quantile curves $\mathcal{C}q(\lambda)$ (full blue line) and $\mathcal{C}q_{\text{eq}}(\lambda)$ (dashed black line).

Numerical illustrations: Theorem 3

Here, we consider a bivariate Pareto$(\alpha, \sigma_1, \sigma_2)$ distribution, whose probability density function is:

$$f(x,y) = \frac{\alpha}{2\sigma_1\sigma_2\pi}\left(\frac{x^2}{\sigma_1^2}+\frac{y^2}{\sigma_2^2}\right)^{(-2-\alpha)/2} \mathbb{1}_{[1,\infty)}\left(\frac{x^2}{\sigma_1^2}+\frac{y^2}{\sigma_2^2}\right)$$

where α, σ_1^2 and $\sigma_2^2 > 0$. When $\alpha > 2$, this distribution has covariance matrix:

$$M = \frac{1}{2}\cdot\frac{\alpha}{\alpha-2}\Sigma, \quad \text{with} \quad \Sigma = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}.$$

Clearly, for any $\alpha > 0$, this distribution is part of the class (M'_α), with

$$\begin{aligned} Q(x) &= (x'\Sigma^{-1}x)^{(-2-\alpha)/2} \\ \text{and} \quad V(t) &= \frac{\alpha}{2\sigma_1\sigma_2\pi} t^{-\alpha} \mathbb{1}_{[1,\infty)}(t). \end{aligned}$$

Theorems 2 and 3 thus entail that the extreme geometric quantiles of this distribution are asymptotically equal to:

$$q_{\mathrm{eq}}(\lambda u) := (1-\lambda)^{-1/\alpha} I(\alpha, \sigma_1, \sigma_2) \;\; \text{if} \;\; \alpha < 2$$

where $I(\alpha, \sigma_1, \sigma_2)$ is a positive constant, and

$$q_{\mathrm{eq}}(\lambda u) := (1-\lambda)^{-1/2} \left[\frac{1}{2} \left(\operatorname{tr} M - u' M u \right) \right]^{1/2} u \;\; \text{if} \;\; \alpha > 2.$$

$\Rightarrow$ Goal: to examine if both these approximations are satisfactory on this heavy-tailed example.

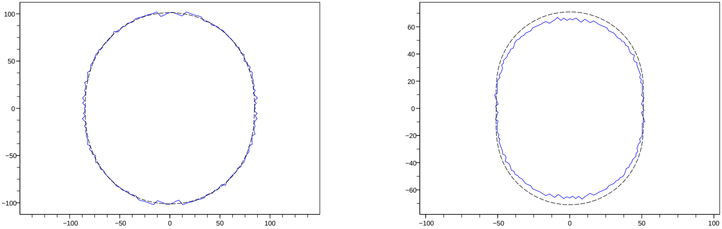

Figure 4: Case of the Pareto$(\alpha, 2, 1/2)$ model, with $\alpha = 1.3$ (left) and $\alpha = 1.5$ (right) for $\lambda = 0.995$. Iso-quantile curves $\mathcal{C}q(\lambda)$ (full blue line) and $\mathcal{C}q_{\rm eq}(\lambda)$ (black dashed line).

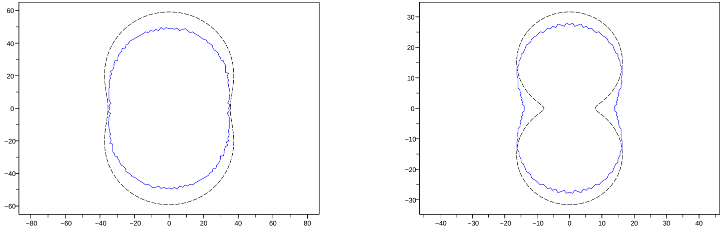

Figure 5: Case of the Pareto($\alpha, 2, 1/2$) model, with $\alpha = 1.7$ (left) and $\alpha = 2.5$ (right) for $\lambda = 0.995$. Iso-quantile curves $\mathcal{C}q(\lambda)$ (full blue line) and $\mathcal{C}q_{\rm eq}(\lambda)$ (black dashed line).

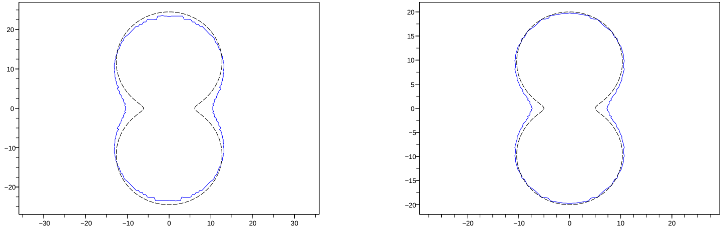

Figure 6: Case of the Pareto($\alpha, 2, 1/2$) model, with $\alpha = 3$ (left) and $\alpha = 4$ (right) for $\lambda = 0.995$. Iso-quantile curves $\mathcal{C}q(\lambda)$ (full blue line) and $\mathcal{C}q_{\text{eq}}(\lambda)$ (black dashed line).

Discussion

- Extreme geometric quantiles in the direction u have asymptotic direction u;
- They are asymptotically equal for two distributions which have the same finite covariance matrix, which is not satisfying from the extreme value perspective;
- They do however feature the behaviour of X far from the origin in a multivariate regular variation context when the tail of $\|X\|$ is sufficiently heavy.

Forthcoming studies on this topic include:

- In model (M_α), obtaining an estimator of α when $\alpha < 2$;
- Working on a modification of geometric quantiles which takes the behaviour of X far from the origin in all cases;
- Trying to obtain analogue results for depth-based quantiles or generalised quantile processes.

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Thanks for listening!