

Estimation of extreme expectiles from heavy tailed distributions

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Outline

- Quantiles, expectiles & expected-shortfall.
- Tail behaviour, application to inference:
 - Intermediate vs extreme levels,
 - Asymptotic results,
 - Illustration on simulations.
- Application on a real data example.

Quantiles

If X is a real-valued random variable, its univariate τ th **quantile**

$$q_\tau := \inf\{t \in \mathbb{R} \text{ s.t. } \mathbb{P}(X \leq t) \geq \tau\}$$

can be obtained by solving the optimisation problem (Koenker & Bassett, 1978)

$$q_\tau = \arg \min_{q \in \mathbb{R}} \mathbb{E}(\varphi_\tau(X - q) - \varphi_\tau(X))$$

where φ_τ is the “check function” defined by

$$\varphi_\tau(x) = (1 - \tau)|x|\mathbb{I}\{x < 0\} + \tau|x|\mathbb{I}\{x \geq 0\}.$$

Remarks:

- Subtracting $\mathbb{E}(\varphi_\tau(X))$ makes the cost function well-defined even when $\mathbb{E}|X| = \infty$.
- In particular, the **median** $q_{1/2}$ of X is obtained by minimising $\mathbb{E}|X - q|$ with respect to q .
- q_τ is also referred to as the **Value-at-Risk** (VaR) of level τ .

Expectiles

If X is a real-valued random variable, its univariate τ th **expectile** is defined by the optimisation problem (Newey & Powell, 1987)

$$\xi_\tau = \arg \min_{\theta \in \mathbb{R}} \mathbb{E}(\eta_\tau(X - \theta) - \eta_\tau(X))$$

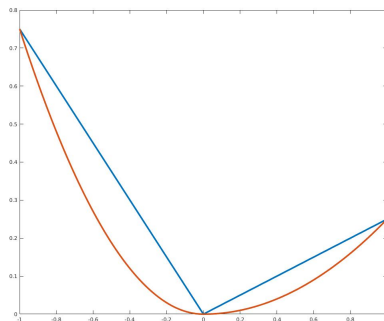
where η_τ is the function defined by

$$\eta_\tau(x) = (1 - \tau)x^2\mathbb{I}\{x < 0\} + \tau x^2\mathbb{I}\{x \geq 0\}.$$

Remarks:

- Subtracting $\mathbb{E}(\eta_\tau(X))$ makes the cost function well-defined provided that $\mathbb{E}|X| < \infty$.
- In particular, the **mean** $\xi_{1/2}$ of X is obtained by minimising $\mathbb{E}(X - \theta)^2$ with respect to θ .

Comparison of cost functions



Red: expectiles η_τ , blue: quantiles φ_τ with $\tau = 1/3$.

Expectiles vs quantiles

Theoretical point of view

- Both families of quantiles and expectiles are embedded in the more general class of **M-quantiles** (Breckling & Chambers (1988)) as the minimizers of an asymmetric convex loss function.
- The only M-quantiles that are **coherent risk measures** are the expectiles, for $\tau > 1/2$ (Bellini *et al.* (2014)).

Practical point of view

- Expectiles are more sensitive to the magnitude of extremes than quantiles are.
- Sample expectiles provide a class of smooth curves as functions of the level τ , which is not the case for sample quantiles.
- Expectiles do not have an intuitive interpretation as direct as quantiles.

Expected shortfall

- The (quantile-based) **expected shortfall**, also known under the names Conditional Value at Risk or Average Value at Risk, is defined as the average of the quantile function above a given confidence level τ :

$$\text{QES}(\tau) := \frac{1}{1-\tau} \int_{\tau}^1 q_{\alpha} d\alpha.$$

When X is continuous, $\text{QES}(\tau) = \mathbb{E}(X|X > q_{\tau})$.

- Similarly, one may define an alternative expectile-based expected-shortfall as

$$\text{XES}(\tau) := \frac{1}{1-\tau} \int_{\tau}^1 \xi_{\alpha} d\alpha.$$

Contributions

Let X_1, \dots, X_n be an i.i.d. sample from F . Our aim is to estimate expectiles ξ_{τ_n} and the associated expectile-based expected-shortfall $\text{XES}(\tau_n)$ when $\tau_n \rightarrow 1$ as $n \rightarrow \infty$ when F is an heavy-tailed distribution. Two situations are investigated:

- Intermediate levels, $n(1 - \tau_n) \rightarrow \infty$,
- Extreme levels, $n(1 - \tau_n) \rightarrow c \geq 0$ (extrapolation needed).

Inference (for intermediate levels)

We assume $\tau_n \rightarrow 1$ and $n(1 - \tau_n) \rightarrow \infty$ as $n \rightarrow \infty$ (intermediate level). Let $k = \lfloor n(1 - \tau_n) \rfloor$ be an intermediate sequence.

- Intermediate quantile (Thm 2.4.1, de Haan & Ferreira (2006)):

$$\hat{q}_{\tau_n} = X_{n-k,n},$$

- Intermediate quantile-based expected-shortfall (Elmethni *et al.*, 2014):

$$\widehat{\text{QES}}(\tau_n) = \frac{1}{k} \sum_{i=1}^n X_i \mathbb{I}(X_i > \hat{q}_{\tau_n}),$$

- Intermediate expectile:

$$\tilde{\xi}_{\tau_n} = \arg \min_{u \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^n \eta_{\tau_n}(X_i - u),$$

- Intermediate expectile-based expected-shortfall:

$$\widetilde{\text{XES}}(\tau_n) = \frac{1}{k} \sum_{i=1}^n X_i \mathbb{I}(X_i > \tilde{\xi}_{\tau_n}).$$

Heavy-tailed distributions

Definition. The cumulative distribution function F is said to be heavy-tailed if it belongs to Fréchet Maximum Domain of Attraction *i.e.*

$$F(x) = 1 - x^{-1/\gamma} \ell(x), \quad x > 0$$

where

- $\gamma > 0$ is the **extreme-value index** (or tail index),
- ℓ is a **slowly-varying function** *i.e.* such that $\ell(tx)/\ell(t) \rightarrow 1$ as $t \rightarrow \infty$ for all $x > 0$.

Consequences.

- $\gamma < 1$ implies $E|X| < \infty$ and thus the existence of expectiles.
- The survival function $\bar{F} := 1 - F$ is said to be **regularly-varying** with index $-1/\gamma$ *i.e.* $\bar{F}(tx)/\bar{F}(t) \rightarrow x^{-1/\gamma}$ as $t \rightarrow \infty$ for all $x > 0$.
- Equivalently, the tail quantile function $U := (1/\bar{F})^{\leftarrow}$ is regularly-varying with index γ .

Second order condition

- The regular-variation property is also referred to as a **first order condition**: $U(tx)/U(t) \rightarrow x^\gamma$ as $t \rightarrow \infty$ for all $x > 0$.
- The goal of the **second order condition** is to quantify the rate of convergence: there exist $\gamma > 0$, $\rho \leq 0$, and a function A converging to 0 at infinity such that for all $x > 0$,

$$\lim_{t \rightarrow \infty} \frac{1}{A(t)} \left[\frac{U(tx)}{U(t)} - x^\gamma \right] = x^\gamma \frac{x^\rho - 1}{\rho}.$$

This condition is denoted by $\mathcal{C}_2(\gamma, \rho, A)$. Note that $(x^\rho - 1)/\rho$ is to be understood as $\log x$ when $\rho = 0$.

Asymptotic distribution of $\tilde{\xi}_{\tau_n}$

From the continuity and the convexity of η_{τ_n} and a result of Geyer (1996):

Theorem 1

If F is heavy-tailed with $0 < \gamma < 1/2$ and $\tau_n \rightarrow 1$ is such that $n(1 - \tau_n) \rightarrow \infty$, then

$$\sqrt{n(1 - \tau_n)} \begin{pmatrix} \tilde{\xi}_{\tau_n} \\ \xi_{\tau_n} \end{pmatrix} - 1 \xrightarrow{d} \mathcal{N}(0, V_1(\gamma)) \quad \text{with} \quad V_1(\gamma) = \frac{2\gamma^3}{1 - 2\gamma}.$$

- No need for a second-order condition,
- Restriction on the extreme-value index.

First order expansions

Proposition 1

For all heavy-tailed distribution such that $0 < \gamma < 1$, when $\tau \rightarrow 1$, one has

$$\frac{XES(\tau)}{QES(\tau)} \sim \frac{\xi_\tau}{q_\tau} \sim (\gamma^{-1} - 1)^{-\gamma},$$

$$\frac{XES(\tau)}{\xi_\tau} \sim \frac{QES(\tau)}{q_\tau} \sim \frac{1}{1 - \gamma}.$$

- Second order approximations have been established under $\mathcal{C}_2(\gamma, \rho, A)$ (Daouia *et al.*, 2016).
- If $\gamma < 1/2$ then, asymptotically, $XES(\tau) < QES(\tau)$ and $\xi_\tau < q_\tau$.

Inference for heavy-tailed distributions

The order statistics are denoted by $X_{1,n} \leq \dots \leq X_{n,n}$.

- Hill estimator for the tail index (Hill, 1975)

$$\hat{\gamma}_H = \frac{1}{k} \sum_{i=1}^k \log \frac{X_{n-i+1,n}}{X_{n-k,n}},$$

- Weissman estimator for extreme quantiles (Weissman, 1978)

$$\hat{q}_{\tau'_n}^* = \hat{q}_{\tau_n} \left(\frac{1 - \tau_n}{1 - \tau'_n} \right)^{\hat{\gamma}_H},$$

- Estimator of the quantile-ES (Elmethni *et al.*, 2014)

$$\widehat{\text{QES}}^*(\tau'_n) = \widehat{\text{QES}}(\tau_n) \left(\frac{1 - \tau_n}{1 - \tau'_n} \right)^{\hat{\gamma}_H}.$$

An alternative estimator of the (intermediate) expectile

The property $\xi_\tau \sim q_\tau(\gamma^{-1} - 1)^{-\gamma}$ as $\tau \rightarrow 1$ suggests an estimator based on an intermediate quantile:

$$\hat{\xi}_{\tau_n} = X_{n-k, n}(\hat{\gamma}_H^{-1} - 1)^{-\hat{\gamma}_H}$$

Theorem 2

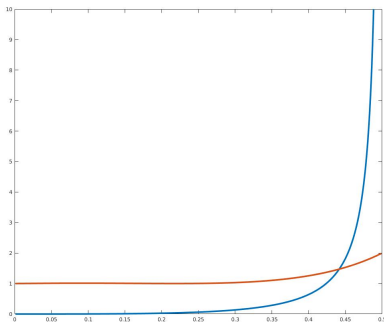
If F verifies $\mathcal{C}_2(\gamma, \rho, A)$ with $0 < \gamma < 1$ and $\tau_n \rightarrow 1$ is such that $n(1 - \tau_n) \rightarrow \infty$, $\sqrt{n(1 - \tau_n)}q_{\tau_n}^{-1} \rightarrow 0$ and $\sqrt{n(1 - \tau_n)}A((1 - \tau_n)^{-1}) \rightarrow 0$, then

$$\sqrt{n(1 - \tau_n)} \left(\frac{\hat{\xi}_{\tau_n}}{\xi_{\tau_n}} - 1 \right) \xrightarrow{d} \mathcal{N}(0, V_2(\gamma))$$

with $V_2(\gamma) = 1 + \left(\frac{\gamma}{1-\gamma} - \gamma \log \left(\frac{1}{\gamma} - 1 \right) \right)^2$.

- Need for a second-order condition,
- Bias conditions on τ_n .

Comparison of asymptotic variances



Horizontally: $\gamma \in (0, 1/2)$, Vertically: asymptotic variances $V_1(\gamma)$ in blue and $V_2(\gamma)$ in red.

Estimation of extreme expectiles

Let $\tau'_n \rightarrow 1$ and $n(1 - \tau'_n) \rightarrow c \geq 0$ as $n \rightarrow \infty$ (extreme level).

The property $\xi_\tau \sim q_\tau(\gamma^{-1} - 1)^{-\gamma}$ as $\tau \rightarrow 1$ also entails $\xi_{\tau'}/\xi_\tau \sim q_{\tau'}/q_\tau$ as both $\tau \rightarrow 1$ and $\tau' \rightarrow 1$. Thus, the same extrapolation factor can be applied for expectiles and quantiles leading to two possible estimators for extreme expectiles:

$$\tilde{\xi}_{\tau'_n}^* = \tilde{\xi}_{\tau_n} \left(\frac{1 - \tau_n}{1 - \tau'_n} \right)^{\hat{\gamma}_H}$$

and

$$\hat{\xi}_{\tau'_n}^* = \hat{\xi}_{\tau_n} \left(\frac{1 - \tau_n}{1 - \tau'_n} \right)^{\hat{\gamma}_H} = X_{n-k,n} (\hat{\gamma}_H^{-1} - 1)^{-\hat{\gamma}_H} \left(\frac{1 - \tau_n}{1 - \tau'_n} \right)^{\hat{\gamma}_H} = \hat{q}_{\tau'_n}^* (\hat{\gamma}_H^{-1} - 1)^{-\hat{\gamma}_H}.$$

In the following slide, we focus on the first estimator.

Asymptotic distribution of $\tilde{\xi}_{\tau'_n}^*$

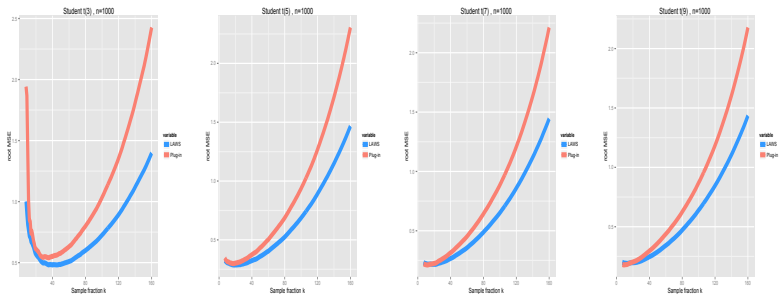
Theorem 3

If F verifies $\mathcal{C}_2(\gamma, \rho, A)$ with $0 < \gamma < 1/2$, $\rho < 0$ and $\tau_n \rightarrow 1$, $\tau'_n \rightarrow 1$ are such that $n(1 - \tau_n) \rightarrow \infty$, $n(1 - \tau'_n) \rightarrow c \geq 0$, $\sqrt{n(1 - \tau_n)}q_{\tau_n}^{-1} \rightarrow 0$ and $\sqrt{n(1 - \tau_n)}A((1 - \tau_n)^{-1}) \rightarrow 0$, then

$$\frac{\sqrt{n(1 - \tau_n)}}{\log[(1 - \tau_n)/(1 - \tau'_n)]} \left(\frac{\tilde{\xi}_{\tau'_n}^*}{\xi_{\tau'_n}} - 1 \right) \xrightarrow{d} \mathcal{N}(0, \gamma^2).$$

A similar asymptotic result is available for $\hat{\xi}_{\tau'_n}^*$ (Daouia *et al.*, 2016).

Numerical illustration



Horizontally: k , vertically: root MSE estimates (over 10,000 replications) for the t_3 , t_5 , t_7 and t_9 -distributions, with sample size $n = 1000$. Red: $\hat{\xi}_{T'_n}^*$, blue: $\tilde{\xi}_{T'_n}^*$.

Estimation of the extreme expectile-based expected-shortfall

The property $XES(\tau) \sim \xi_\tau/(1-\gamma)$ as $\tau \rightarrow 1$ suggests two possible estimators for the extreme expectile-based expected-shortfall:

$$\widetilde{XES}^*(\tau'_n) = \tilde{\xi}_{\tau'_n}^*/(1 - \hat{\gamma}_H) \quad \text{and} \quad \widehat{XES}^*(\tau'_n) = \hat{\xi}_{\tau'_n}^*/(1 - \hat{\gamma}_H).$$

In the following theorem, we focus on the first estimator.

Theorem 4

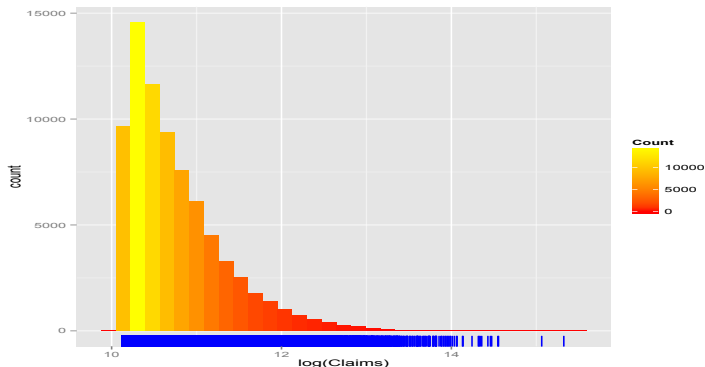
Under the assumptions of Theorem 3,

$$\frac{\sqrt{n(1-\tau_n)}}{\log[(1-\tau_n)/(1-\tau'_n)]} \left(\frac{\widetilde{XES}^*(\tau'_n)}{XES(\tau'_n)} - 1 \right) \xrightarrow{d} \mathcal{N}(0, \gamma^2).$$

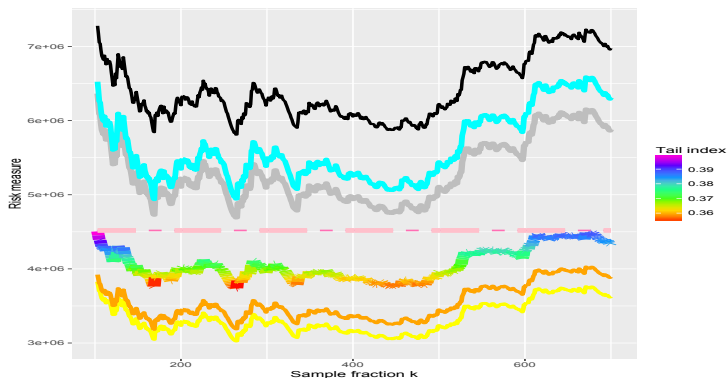
A similar asymptotic result is available for $\widehat{XES}^*(\tau'_n)$ (Daouia *et al.*, 2016).

Illustration on real data

The Society of Actuaries Group Medical Insurance Large Claims Database records all the claim amounts exceeding 25,000 USD over the period 1991-92. As in Beirlant *et al.* (2004), we only deal here with the $n = 75,789$ claims for 1991. Moreover, we focus on the extreme level $\tau'_n = 1 - 10^{-5}$.



Results



Horizontally: k , vertically: expectiles $\hat{\xi}_{\tau'_n}^*$ in yellow and $\tilde{\xi}_{\tau'_n}^*$ in orange, expectile-based expected-shortfall $\widehat{XES}^*(\tau'_n)$ in gray and $\widetilde{XES}^*(\tau'_n)$ in cyan, quantile $\hat{q}_{\tau'_n}^*$ as a rainbow curve, $\widehat{QES}^*(\tau'_n)$ in black, sample maximum $Y_{n,n}$ as an horizontal pink line. The estimated sample fraction is $\hat{k} = 486$ (Beirlant *et al.* (2004)).

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