Contributions to the Chair "Stress Test"

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1 Ongoing works

1.1 Estimation of extreme quantile from a mixture of Pareto distributions (with E. Gobet)

We focus on the situation where the data are drawn from a continuous mixture a Pareto-type distributions with tail-index $\gamma \in [\gamma_1, \gamma_2]$ where $\gamma_1 > 0$. We show that the mixture is still heavy-tailed with tail-index γ_2 and that, in this case, the Hill estimator of the tail-index is strongly biased. A bias correction is introduced thanks to a sharp expansion of the survival distribution of the mixture in the tail. An estimator of extreme quantiles is then derived. Our current work consists in investigating the behavior of this new bias-corrected estimator (i) from the asymptotic point of view, (ii) on simulated data, and (iii) on financial data.

1.2 Neural networks, fractional Brownian motion and extreme-value theory (with M. Allouche and E. Gobet)

The main objective of the PhD thesis of M. Allouche is to build a generative neural networks model based on extreme synthetic data simulated from effective but computationally expensive Monte-Carlo methods. More specifically, the goal is to approximate a quantile $q(u)$ for all level $u \in [0, 1)$ although extreme quantiles (observed as $u \to 1$) are our main region of interest. We moreover focus on extreme quantiles from heavy-tailed distributions *i.e.* belonging to the Fréchet maximum domain of attraction. In such a case, $q(u) \to \infty$ at a power rate as $u \to 1$ and thus, approximating such a function with linear combinations of Relu functions is a challenging task for a neural network. To overcome this problem, a nonlinear transformation of the quantile function to another tail function is proposed. The uniform approximation rate of this new function by the neural network is then established as a function of the number of units. Some illustrations are provided both on simulated and real financial data. We also proposed generative models based on Relu neural networks for classical and fractional Brownian motions. Some bounds are provided on the network construction, in terms of number of hidden layers and total number of neurons. The results are submitted for publication.

2 Results

2.1 Publications

- Extreme conditional expectile estimation in heavy-tailed heteroscedastic regression models `girard:stupfler:2021`

  Expectiles define a least squares analogue of quantiles. They have been the focus of a substantial quantity of research in the context of actuarial and financial risk assessment over the last 10 years. Unlike quantiles, expectiles induce coherent risk measures and are calculated using tail expectations rather than merely tail probabilities ; contrary to the popular quantile-based Expected Shortfall, they define elicitable risk measures. The behaviour and estimation of extreme expectiles using independent and identically distributed heavy-tailed observations has been investigated in a recent series of papers. The case of extreme conditional expectile estimation has, however, not been addressed so far in the literature. We build here a general theory for the estimation of extreme conditional expectiles in heteroscedastic regression models with heavy-tailed noise; an important feature of our approach is that it is intended to cope with covariates having a large but fixed dimension. We demonstrate how our results can be applied to a wide class of important examples, among which linear heteroscedastic models, heteroscedastic single-index models and autoregressive time series models. Our estimators are showcased on a numerical simulation study, as well as on real sets of actuarial and financial data.

- Nonparametric extreme conditional expectile estimation `girard:stupfler:2021bis`

  Expectiles and quantiles can both be defined as the solution of minimization problems. Contrary to quantiles though, expectiles are determined by tail expectations rather than tail probabilities, and define a coherent risk measure. For these two reasons in particular, expectiles have recently started to be considered as serious candidates to become standard tools in actuarial and financial risk management. However, expectiles and their sample versions do not benefit from a simple explicit form, making their analysis significantly harder than that of quantiles and order statistics. This difficulty is compounded when one wishes to integrate auxiliary information about the phenomenon of interest through a finite-dimensional covariate, in which case the problem becomes the estimation of conditional expectiles. In this paper, we exploit the fact that the expectiles of a distribution F are in fact the quantiles of another distribution E explicitly linked to F, in order to construct nonparametric kernel estimators of extreme conditional expectiles.

- Functional estimation of extreme conditional expectiles `girard:stupfler:2021ter`

  Quantiles and expectiles can be interpreted as solutions of convex minimization problems. Unlike quantiles, expectiles are determined by tail expectations rather than tail probabilities, and define a coherent risk measure. For these reasons, among others, they have recently been the subject of renewed attention in actuarial and financial risk management. Here, we focus on the challenging problem of estimating extreme expectiles, whose order converges to one as the sample size increases, given a functional covariate. We construct a functional kernel estimator of extreme conditional expectiles by writing expectiles as quantiles of a different distribution. The asymptotic properties of the estimators are studied in the context of conditional heavy-tailed distributions. We also provide and analyse different ways of estimating the functional tail index, as a way to extrapolate our estimates to the very far conditional tails. A numerical illustration of the finite-sample performance of our estimators is provided on simulated and real datasets.

- On the estimation of the variability in the distribution tail `gardes:girard:2021`

We propose a new measure of variability in the tail of a distribution by applying a Box-Cox transformation of parameter $p \geq 0$ to the tail-Gini functional. It is shown that the so-called Box-Cox Tail Gini Variability measure is a valid variability measure whose condition of existence may be as weak as necessary thanks to the tuning parameter p. The tail behaviour of the measure is investigated under a general extreme-value condition on the distribution tail. We then show how to estimate the Box-Cox Tail Gini Variability measure within the range of the data. These methods provide us with basic estimators that are then extrapolated using the extreme-value assumption to estimate the variability in the very far tails. The finite sample behavior of the estimators is illustrated both on simulated and real data.

- ExpectHill estimation, extreme risk and heavy tails `daouia:girard:2021`

  Risk measures of a financial position are, from an empirical point of view, mainly based on quantiles. Replacing quantiles with their least squares analogues, called expectiles, has recently received increasing attention. The novel expectile-based risk measures satisfy all coherence requirements. We revisit their extreme value estimation for heavy-tailed distributions. First, we estimate the underlying tail index via weighted combinations of top order statistics and asymmetric least squares estimates. The resulting expectHill estimators are then used as the basis for estimating tail expectiles and Expected Shortfall. The asymptotic theory of the proposed estimators is provided, along with numerical simulations and applications to actuarial and financial data.

- Estimation of extreme quantiles from heavy-tailed distributions in a location-dispersion regression models `ahmad:deme:2020`

  We consider a location-dispersion regression model for heavy-tailed distributions when the multidimensional covariate is deterministic. In a first step, nonparametric estimators of the regression and dispersion functions are introduced. This permits, in a second step, to derive an estimator of the conditional extreme-value index computed on the residuals. Finally, a plug-in estimator of extreme conditional quantiles is built using these two preliminary steps. It is shown that the resulting semi-parametric estimator is asymptotically Gaussian and may benefit from the same rate of convergence as in the unconditional situation. Its finite sample properties are illustrated both on simulated and real tsunami data.

2.2 Submitted manuscripts

- On automatic bias reduction for extreme expectile estimation `girard:stupfler:2021four`

  Expectiles induce a law-invariant risk measure that has recently gained popularity in actuarial and financial risk management applications. Unlike quantiles or the quantile-based Expected Shortfall, the expectile risk measure is coherent and elicitable. The estimation of extreme expectiles in the heavy-tailed framework, which is reasonable for extreme financial or actuarial risk management, is not without difficulties; currently available estimators of extreme expectiles are typically biased and hence may show poor finite-sample performance even in fairly large samples. We focus here on the construction of bias-reduced extreme expectile estimators for heavy-tailed distributions. The rationale for our construction hinges on a careful investigation of the asymptotic proportionality relationship between extreme expectiles and their quantile counterparts, as well as of the extrapolation formula motivated by the heavy-tailed context. We accurately quantify and estimate the bias incurred by the use of these relationships when constructing extreme expectile estimators. This motivates the introduction of a class of bias-reduced estimators whose asymptotic properties are rigorously shown, and whose finite-sample properties are assessed on a simulation study and three samples of real data from economics, insurance and finance.

- Extreme L^p-quantile kernel regression `girard:stupfler:2021five`

Quantiles are recognized tools for risk management and can be seen as minimizers of an L^1-loss function, but do not define coherent risk measures in general. Expectiles, meanwhile, are minimizers of an L^2-loss function and define coherent risk measures; they have started to be considered as good alternatives to quantiles in insurance and finance. Quantiles and expectiles belong to the wider family of L^p-quantiles. We propose here to construct kernel estimators of extreme conditional Lp-quantiles. We study their asymptotic properties in the context of conditional heavy-tailed distributions and we show through a simulation study that taking $p \in (1, 2)$ may allow to recover extreme conditional quantiles and expectiles accurately. Our estimators are also showcased on a real insurance data set.

- Extreme Partial Least-Squares regression `bousebata:enjolras:2021`

  We propose a new approach, called Extreme-PLS, for dimension reduction in regression and adapted to distribution tails. The objective is to find linear combinations of predictors that best explain the extreme values of the response variable in a non-linear inverse regression model. The asymptotic normality of the Extreme-PLS estimator is established in the single-index framework and under mild assumptions. The performance of the method is assessed on simulated data. A statistical analysis of French farm income data, considering extreme cereal yields, is provided as an illustration.

- Generative model for fBm with deep ReLU neural networks `allouche:girard:2021`

  We provide a large probability bound on the uniform approximation of fractional Brownian motion B^H with Hurst parameter H, by a deep-feedforward ReLU neural network fed with a NN-dimensional Gaussian vector, with bounds on the network construction (number of hidden layers and total number of neurons). Essentially, up to log terms, achieving an uniform error of $\mathcal{O}(N^{-H})$ is possible with $\log(N)$ hidden layers and $\mathcal{O}(N)$ parameters. Our analysis relies, in the standard Brownian motion case ($H = 1/2$), on the Levy construction of B^H and in the general fractional Brownian motion case ($H \neq 1/2$), on the Lemarié-Meyer wavelet representation of B^H. This work gives theoretical support on new generative models based on neural networks for simulating continuous-time processes.

- Tail-GAN: Simulation of extreme events with ReLU neural networks `allouche:girard:2021bis`

  Feedforward neural networks based on Rectified linear units (ReLU) cannot efficiently approximate quantile functions which are not bounded, especially in the case of heavy-tailed distributions. We thus propose a new parametrization for the generator of a Generative adversarial network (GAN) adapted to this framework, basing on extreme-value theory. We provide an analysis of the uniform error between the extreme quantile and its GAN approximation. It appears that the rate of convergence of the error is mainly driven by the second-order parameter of the data distribution. The above results are illustrated on simulated data and real financial data.

2.3 Conferences

Some of the above works have been presented at international conferences as invited talks: `girard:gobet:2021,usseglio:girard:2020,girard:ahmad:2020` or in a contributed session: `bousebata:enjolas:2020`

3 PhD theses supervision

One PhD thesis is founded by the Chair:

- Michael Allouche (co-advised with Emmanuel Gobet, Ecole Polytechnique) "*Simulation of extreme events with IA generative models and applications to risk management in banking*", started on April 2020.

Two PhD theses are connected to the Chair topics:

- Meryem Bousebata (co-advised with Geoffroy Enjolras, CERAG, Université Grenoble Alpes) "*Bayesian estimation of extreme risk measures: Implication for the insurance of natural disasters*", started on October 2018.
- Aboubacrène Ag Ahmad (co-advised with Aliou Diop, Université Gaston-Berger, Sénégal) "*A new location-scale model for heavy-tailed distributions*", defended on September, 16th, 2020.

4 Editorial activities

- Associate Editor of *Journal of Multivariate Analysis* since 2016.
- Associate Editor of *Revstat* since 2019.
- Member of the Advisory Board of *Dependence Modeling* since 2015.

`ahmad:deme:2020`
A. Ahmad, E. Deme, A. Diop, S. Girard, and A. Usseglio-Carleve.
Estimation of extreme quantiles from heavy-tailed distributions in a location-dispersion regression model.
Electronic Journal of Statistics, 14:4421–4456, 2020.

`allouche:girard:2021bis`
M. Allouche, S. Girard, and E. Gobet.
EV-GAN: Simulation of extreme events with ReLU neural networks.
`https://hal.archives-ouvertes.fr/hal-03250663`, 2021.

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M. Allouche, S. Girard, and E. Gobet.
Generative model for fBm with deep ReLU neural networks.
`https://hal.archives-ouvertes.fr/hal-03237854`, 2021.

`bousebata:enjolras:2020`
M. Bousebata, G. Enjolras, and S. Girard.
The dependence structure between yields and prices: A copula-based model of French farm income.
In *Annual Meeting of the Agricultural and Applied Economics Association (AAEA)*, Kansas City, USA, august 2020.

`bousebata:enjolras:2021`
M. Bousebata, G. Enjolras, and S. Girard.
Extreme Partial Least-Squares regression.
`https://hal.inria.fr/hal-03165399`, 2021.

`daouia:girard:2021`
A. Daouia, S. Girard, and G. Stupfler.
ExpectHill estimation, extreme risk and heavy tails.
Journal of Econometrics, 221(1):97–117, 2021.

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L. Gardes and S. Girard.
On the estimation of the variability in the distribution tail.
Test, 2021.
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Estimation of extreme quantiles from heavy-tailed distributions in a location-dispersion regression model.
In *StressTest-2020: International Workshop on Stress Test and Risk Management*, Paris, november 2020.

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S. Girard and E. Gobet.
Estimation of the tail-index and extreme quantiles from a mixture of heavy-tailed distributions.
In *RESIM 2021: 13th International Workshop on Rare-Event Simulation*, Virtual, may 2021.

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S. Girard, G. Stupfler, and A. Usseglio-Carleve.
Extreme conditional expectile estimation in heavy-tailed heteroscedastic regression models.
The Annals of Statistics, 2021.
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S. Girard, G. Stupfler, and A. Usseglio-Carleve.
Extreme L^p-quantile kernel regression.
https://hal.archives-ouvertes.fr/hal-03182032, 2021.

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S. Girard, G. Stupfler, and A. Usseglio-Carleve.
Functional estimation of extreme conditional expectiles.
Econometrics and Statistics, 2021.
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Nonparametric extreme conditional expectile estimation.
Scandinavian Journal of Statistics, 2021.
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S. Girard, G. Stupfler, and A. Usseglio-Carleve.
On automatic bias reduction for extreme expectile estimation.
http://hal.archives-ouvertes.fr/hal-03086048, 2021.

usseglio:girard:2020 .....
A. Usseglio-Carlveve, S. Girard, and G. Stupfler.
On second-order automatic bias reduction for extreme expectile estimation.
In *13th International Conference of the ERCIM WG on computing and statistics*, London, UK, december 2020.