Conditional simulations of max-stable processes

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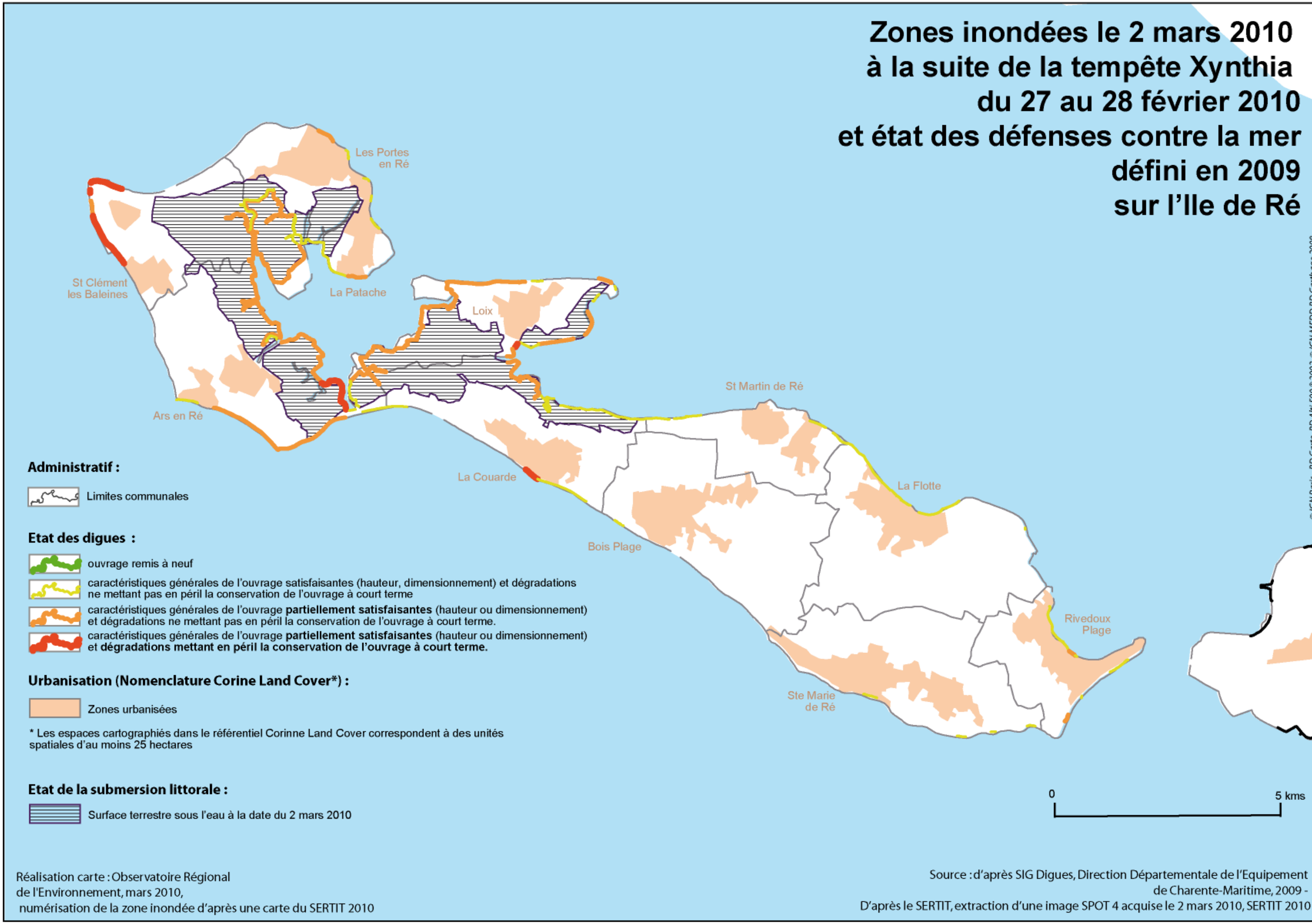
Zones inondées le 2 mars 2010
à la suite de la tempête Xynthia
du 27 au 28 février 2010
et état des défenses contre la mer
défini en 2009
sur l'Ile de Ré
Les Portes en Ré
St Clément les Baleines
La Patache
Loix
Ars en Ré
St Martin de Ré
La Couarde
La Flotte
Bois Plage
Rivedoux Plage
Ste Marie de Ré
Administratif :
Limites communales
Etat des digues :
ouvrage remis à neuf
caractéristiques générales de l'ouvrage satisfaisantes (hauteur, dimensionnement) et dégradations ne mettant pas en péril la conservation de l'ouvrage à court terme
caractéristiques générales de l'ouvrage partiellement satisfaisantes (hauteur ou dimensionnement) et dégradations ne mettant pas en péril la conservation de l'ouvrage à court terme.
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Urbanisation (Nomenclature Corine Land Cover*) :
Zones urbanisées
* Les espaces cartographiés dans le référentiel Corinne Land Cover correspondent à des unités spatiales d'au moins 25 hectares
Etat de la submersion littorale :
Surface terrestre sous l'eau à la date du 2 mars 2010
0
5 kms
© IGN Paris - BD Carto, BD Alti 500 ® 2002, © IGN MEDD BD Carthage 2008
Réalisation carte : Observatoire Régional de l'Environnement, mars 2010, numérisation de la zone inondée d'après une carte du SERTIT 2010
Source : d'après SIG Digues, Direction Départementale de l'Equipement de Charente-Maritime, 2009 - D'après le SERTIT, extraction d'une image SPOT 4 acquise le 2 mars 2010, SERTIT 2010

Motivation

- Geostatistics of extremes: many developments since 2002.
- But conditional simulations of max-stable processes were not available until 2011.
- Wang and Stoev (2011) give a first answer for max-linear processes, i.e.,

$$Z(x) = \max_{j=1,\dots,p} a_{x,j} X_j, \qquad X_j \overset{\text{iid}}{\sim} \text{unit Fréchet}.$$

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- But discrete spectral measure might be too restrictive for concrete applications.

☞ Can we get a procedure for max-stable processes with continuous spectral measure?

Setup

- Given a study region $\mathcal{X} \subset \mathbb{R}^d$, we want to sample from

$$Z(\cdot) \mid \{Z(x_1) = z_1, \ldots, Z(x_k) = z_k\},$$

for some $z_1, \ldots, z_k > 0$ and k conditioning locations $x_1, \ldots, x_k \in \mathcal{X}$.

- Recall that any max-stable process with unit Fréchet margins has the following spectral characterization

$$Z(\cdot) = \max_{i \geq 1} \zeta_i Y_i(\cdot),$$

where

  - $Y_i(\cdot)$ are independent copies of a non negative stochastic process such that $\mathbb{E}[Y(x)] = 1$ for all $x \in \mathcal{X}$;
  - $\{\zeta_i\}_{i \geq 1}$ are the points of a Poisson process on $(0, \infty)$ with intensity $\mathrm{d}\Lambda(\zeta) = \zeta^{-2}\mathrm{d}\zeta$.

Outline

1. Conditional distributions of max-stable processes

$$Z(x) = \max_{i\geq 1} \zeta_i Y_i(x) = \max_{\varphi \in \Phi} \varphi(x), \qquad x \in \mathcal{X},$$

where Φ is a point process whose atoms are $\varphi_i(\cdot) = \zeta_i Y_i(\cdot)$.

- Consider the two following Poisson point processes

$$\Phi^- = \left\{\varphi \in \Phi \colon \varphi(x_i) < z_i, \text{ for all } i \in \{1,\ldots,k\}\right\}, \text{(sub-extremal functions)}$$
$$\Phi^+ = \left\{\varphi \in \Phi \colon \varphi(x_i) = z_i, \text{ for some } i \in \{1,\ldots,k\}\right\}. \text{(extremal functions)}$$

- Clearly $\Phi = \Phi^- \cup \Phi^+$.

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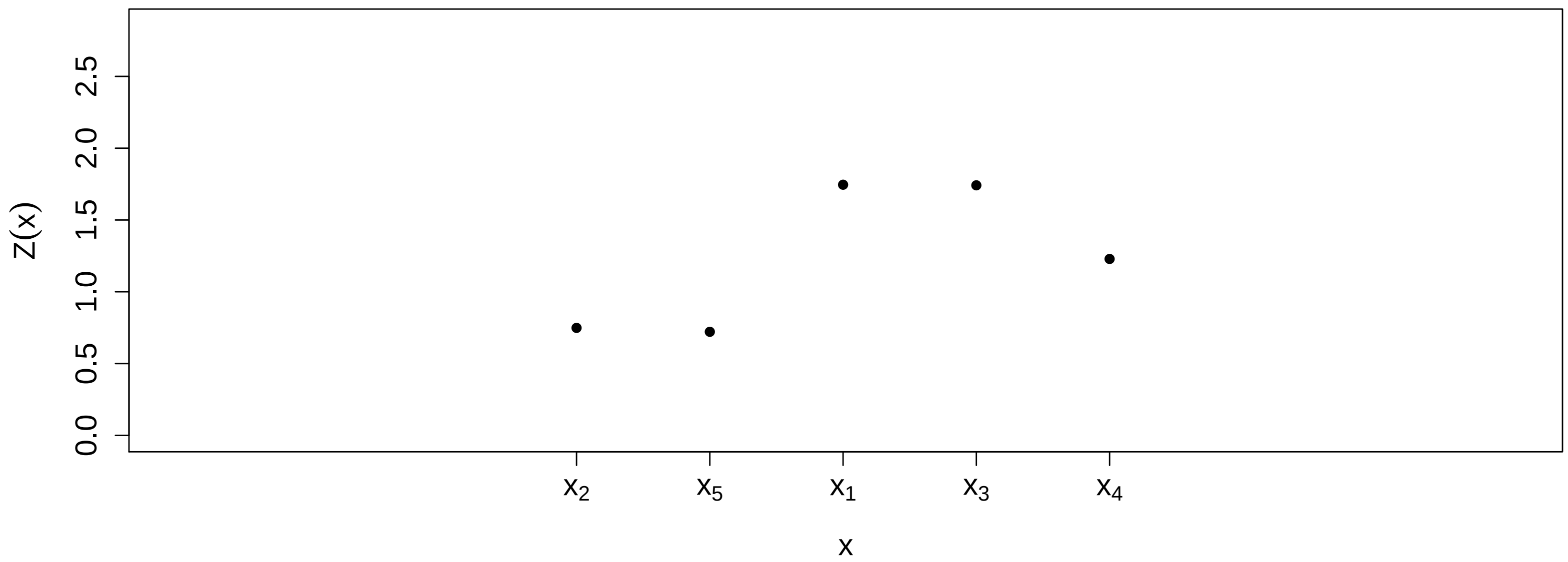
□ Clearly $\Phi = \Phi^- \cup \Phi^+$.

☞ Key point #1: Conditionally on $Z(\mathbf{x}) = \mathbf{z}$, Φ^- and Φ^+ are independent.

Why should we bother about Φ^-?



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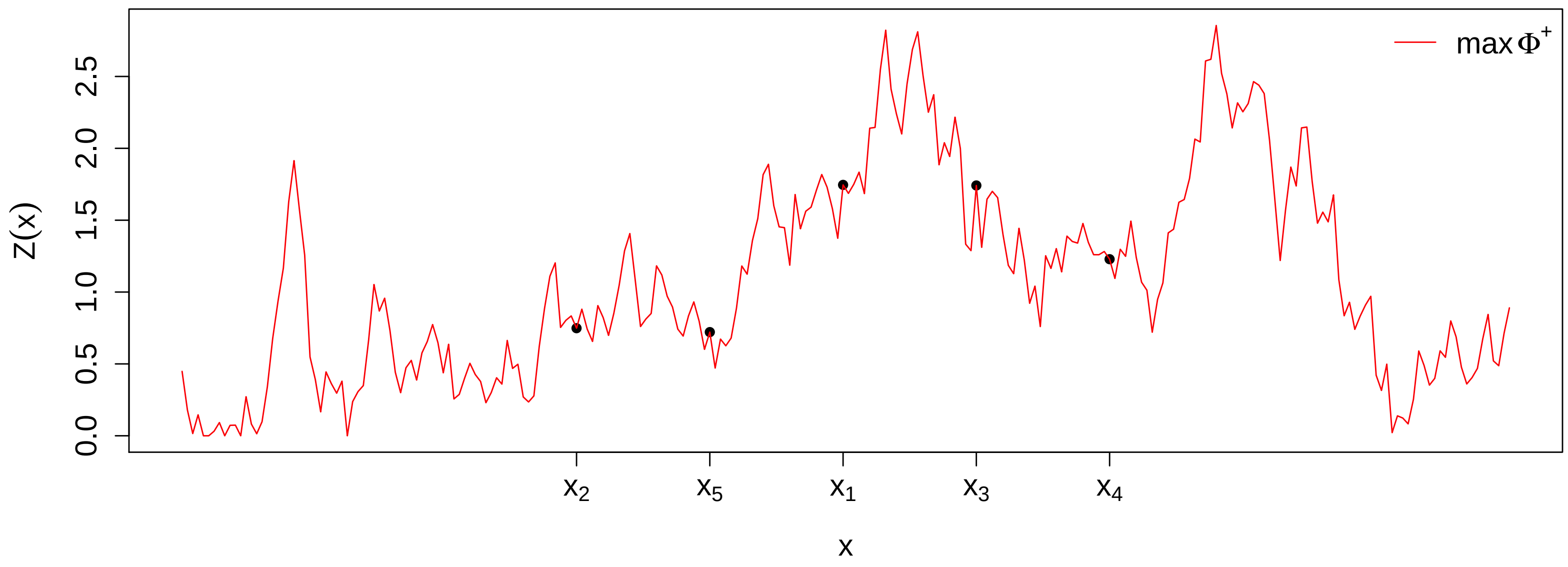

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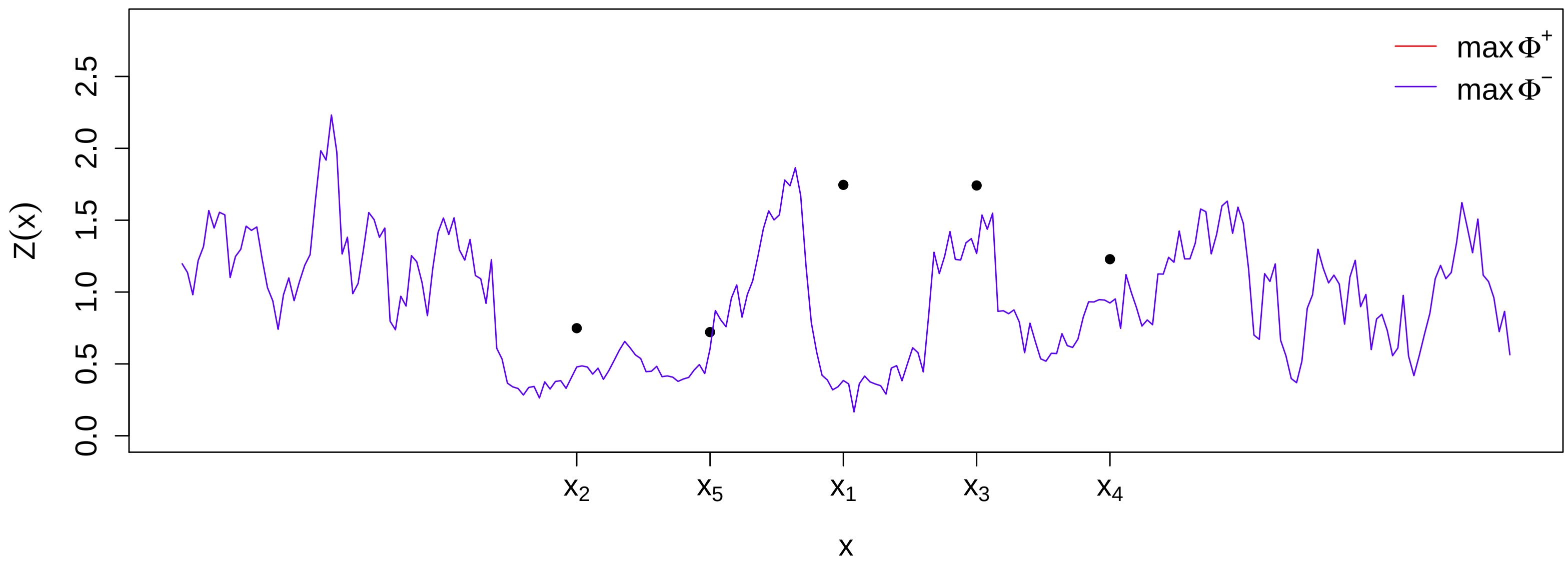

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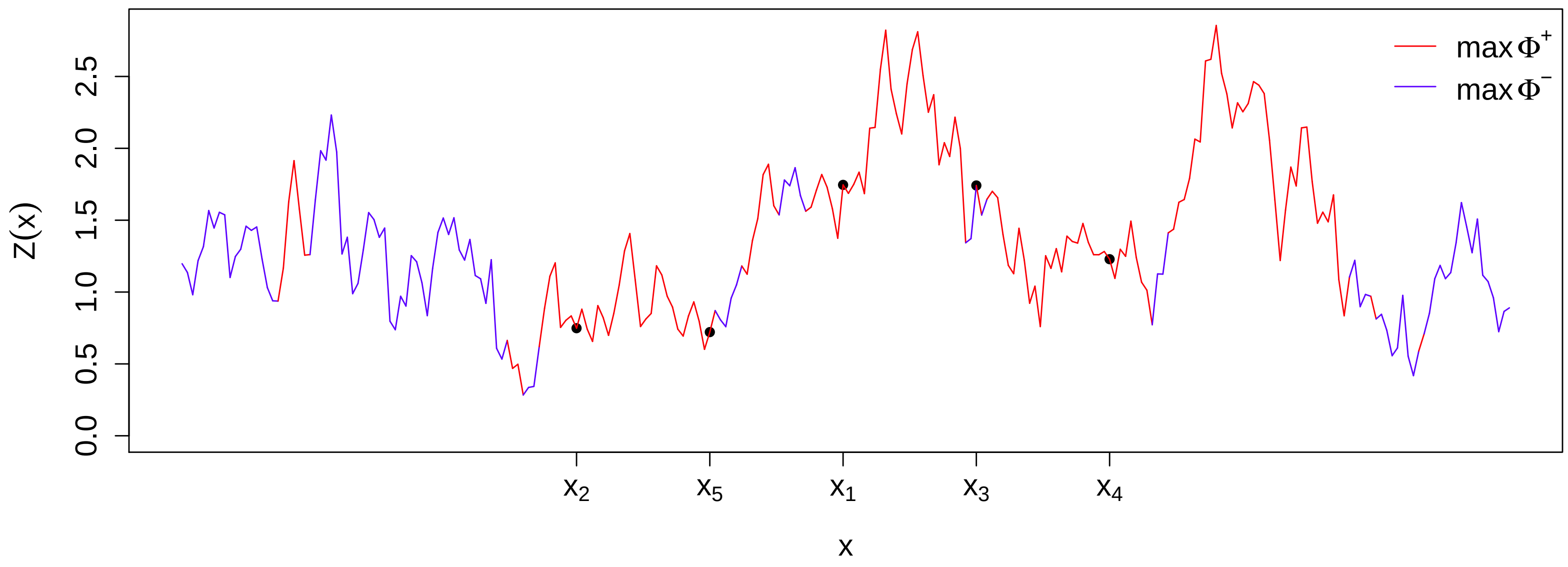

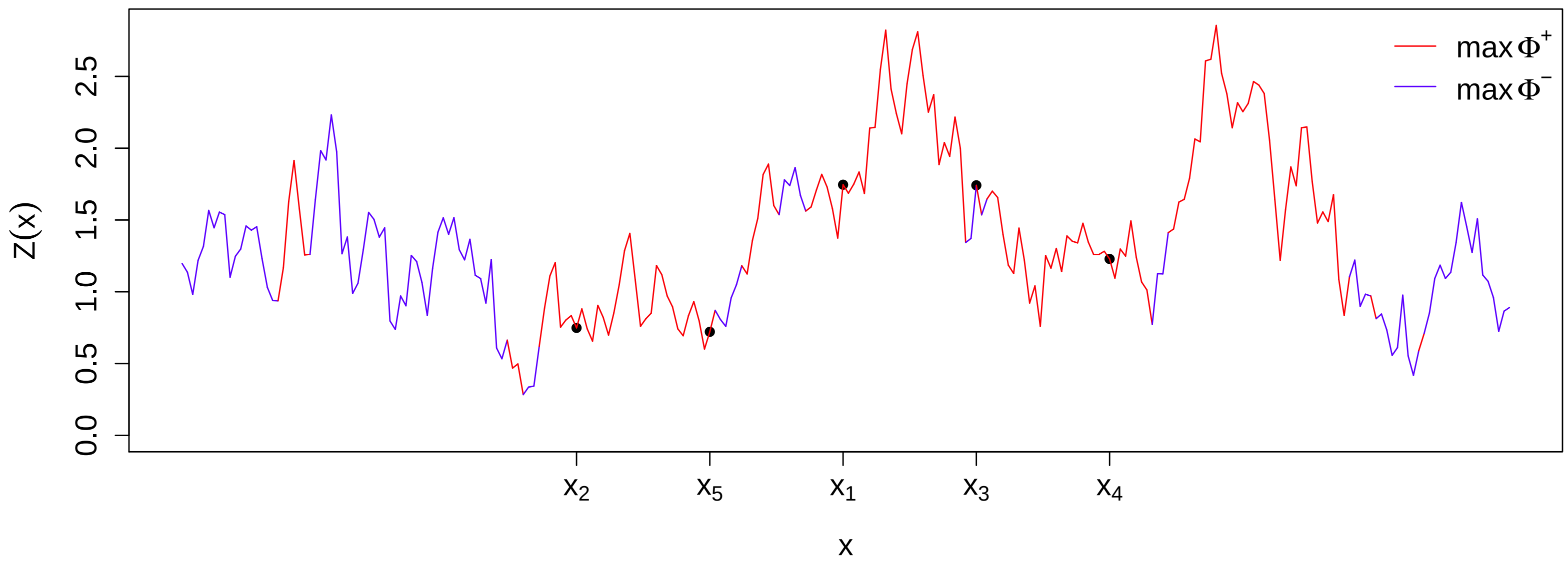

- The atoms of Φ^+ are only of interest if we restrict our attention to the conditioning points $\mathbf{x}$;
- But most often one would like to get realizations at $\mathbf{s} \neq \mathbf{x}$.

☞ The atoms of Φ^- are needed since it is likely that

$$\max \Phi^-(\mathbf{s}) > \max \Phi^+(\mathbf{s})!$$

Conditional intensity function

$$Z(\mathbf{x}) = \max_{i\geq 1} \zeta_i Y_i(\mathbf{x}) = \max_{i \geq 1} \varphi_i(\mathbf{x})$$

- The Poisson point process $\{\varphi_i(\mathbf{x})\}_{i\geq 1}$ has intensity measure

$$\Lambda_{\mathbf{x}}(A) = \int_0^\infty \Pr\{\zeta Y(\mathbf{x}) \in A\} \zeta^{-2} \mathrm{d}\zeta, \qquad \text{Borel set } A \subset \mathbb{R}^k.$$

- We assume that Φ is regular, i.e., $\Lambda_{\mathbf{x}}(\mathrm{d}\mathbf{z}) = \lambda_{\mathbf{x}}(\mathbf{z})\mathrm{d}\mathbf{z}$, for all $\mathbf{x} \in \mathcal{X}^k$.

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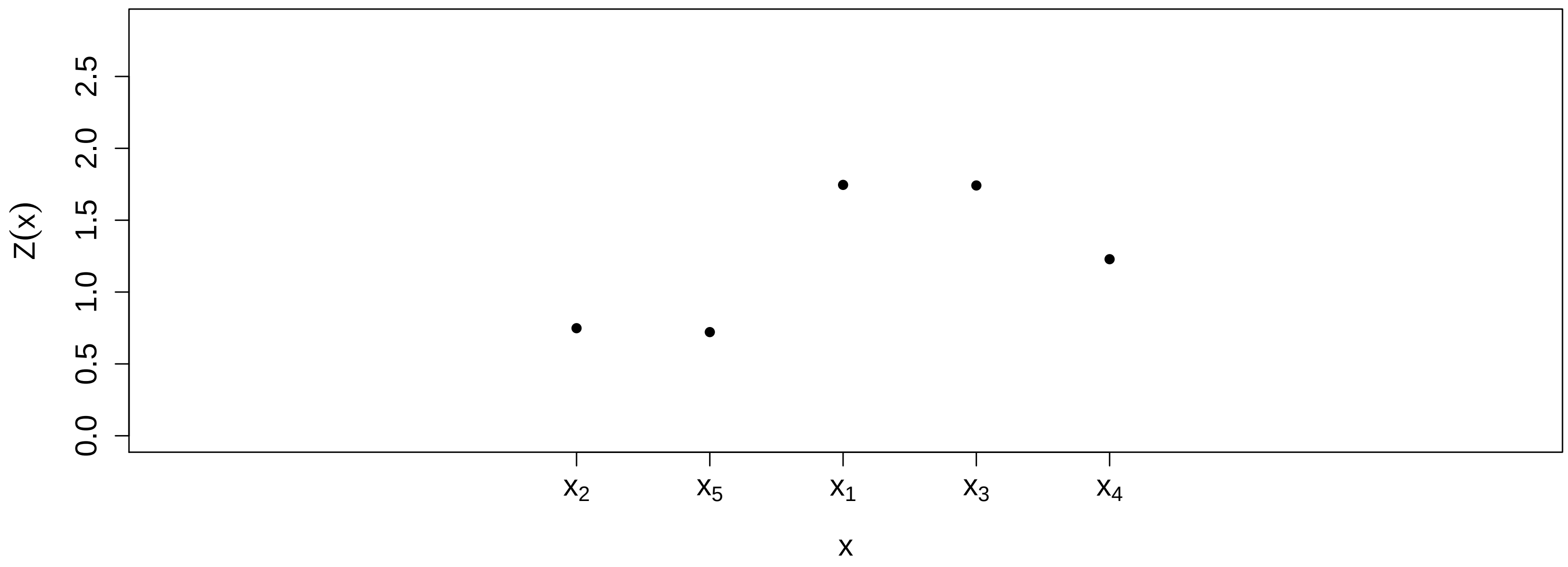
☞ Key point #2: The conditional intensity function

$$\lambda_{\mathbf{x}_1|\mathbf{x}_2,\mathbf{z}_2}(\mathbf{u}) = \frac{\lambda_{(\mathbf{x}_1,\mathbf{x}_2)}(\mathbf{u},\mathbf{z}_2)}{\lambda_{\mathbf{x}_2}(\mathbf{z}_2)}, \qquad \mathbf{x} = (\mathbf{x}_1,\mathbf{x}_2),\ \mathbf{z} = (\mathbf{z}_1,\mathbf{z}_2),$$

is the distribution of $Z(\mathbf{x})$—if we integrate w.r.t. all possible partitions of $\mathbf{x}$. But not that of $Z(\cdot)$!!!

Random partitions?



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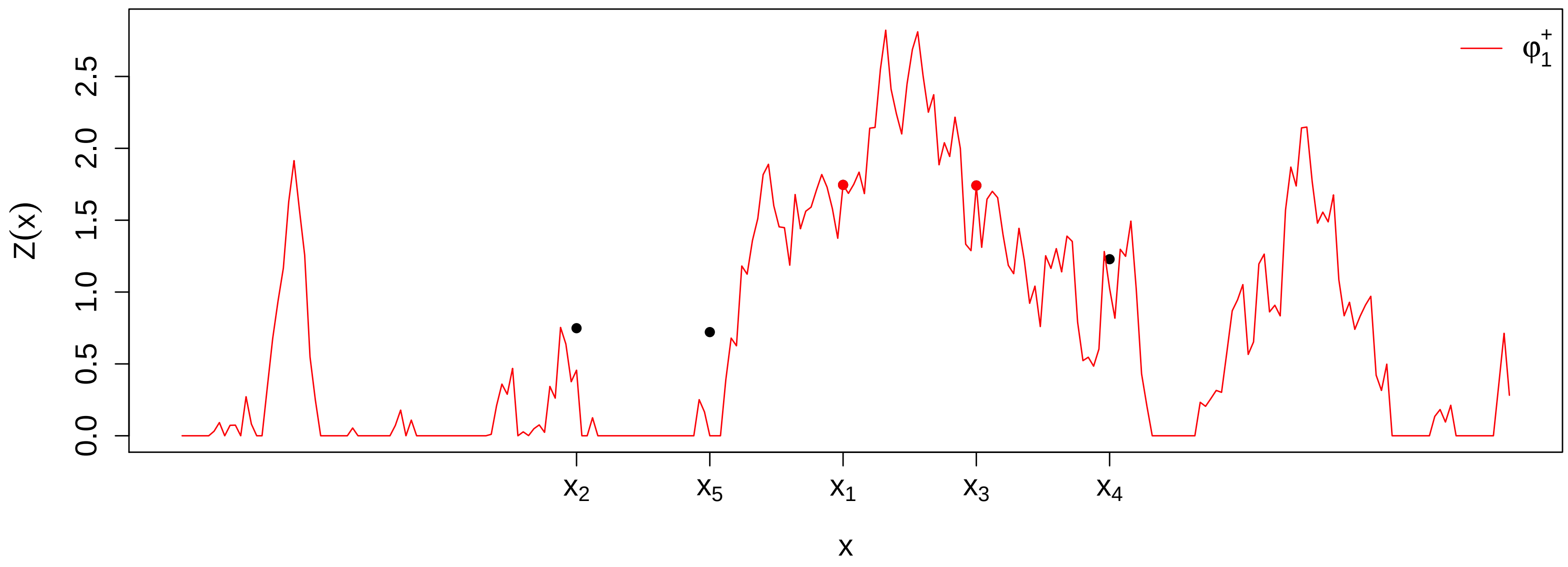
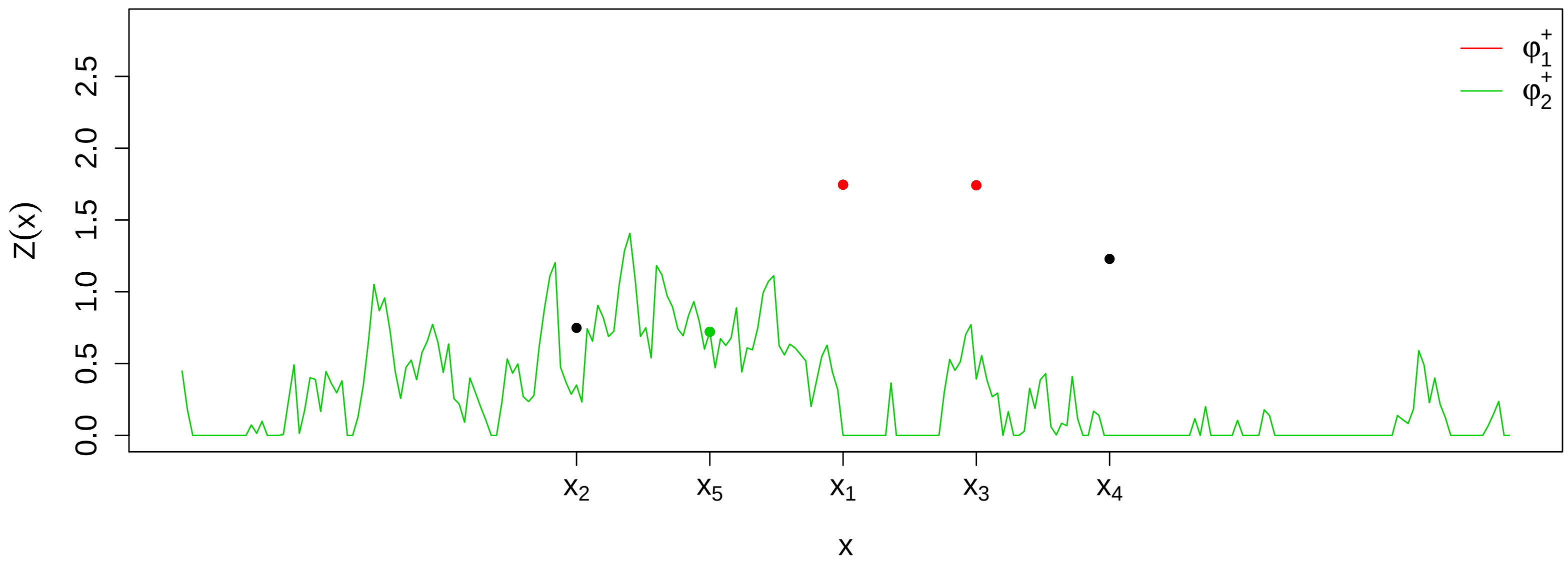
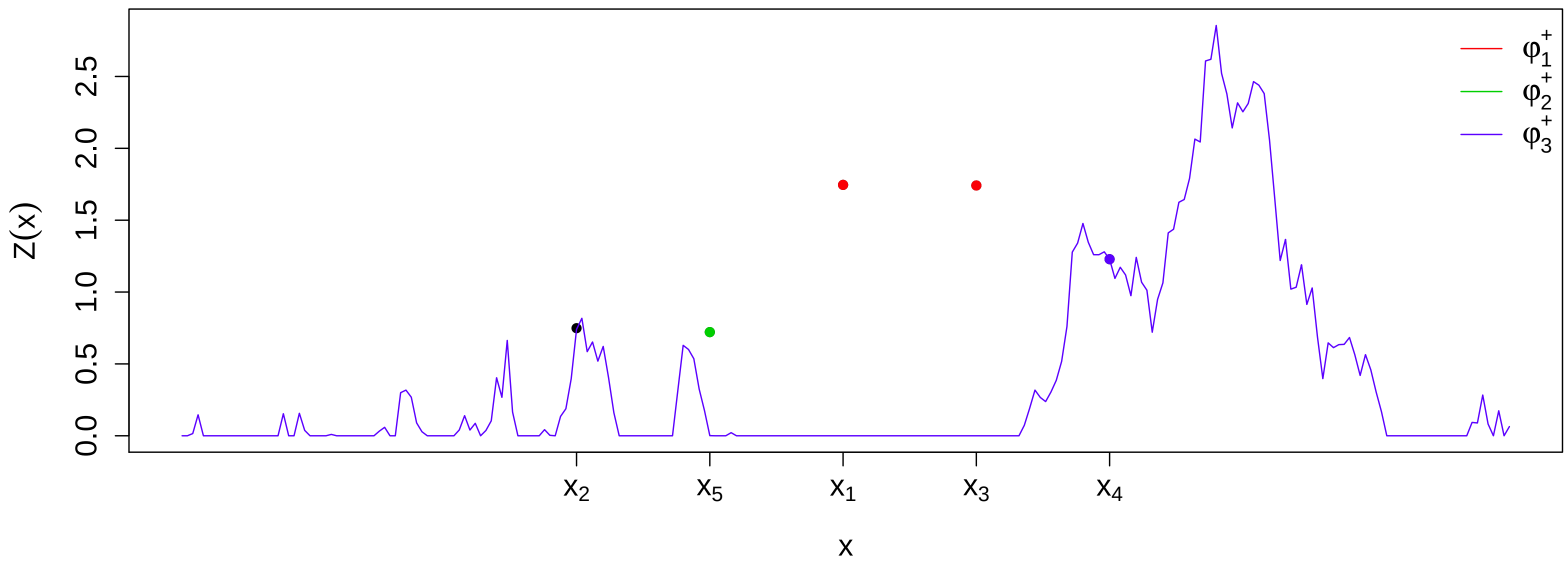
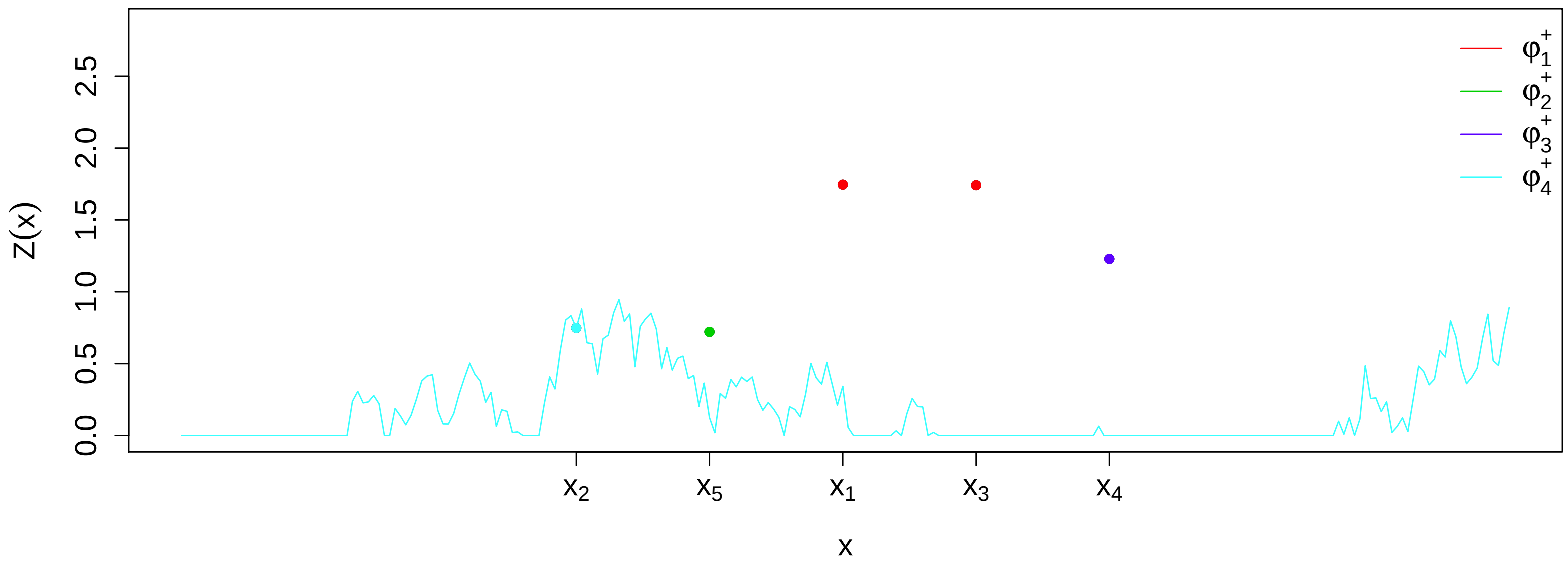


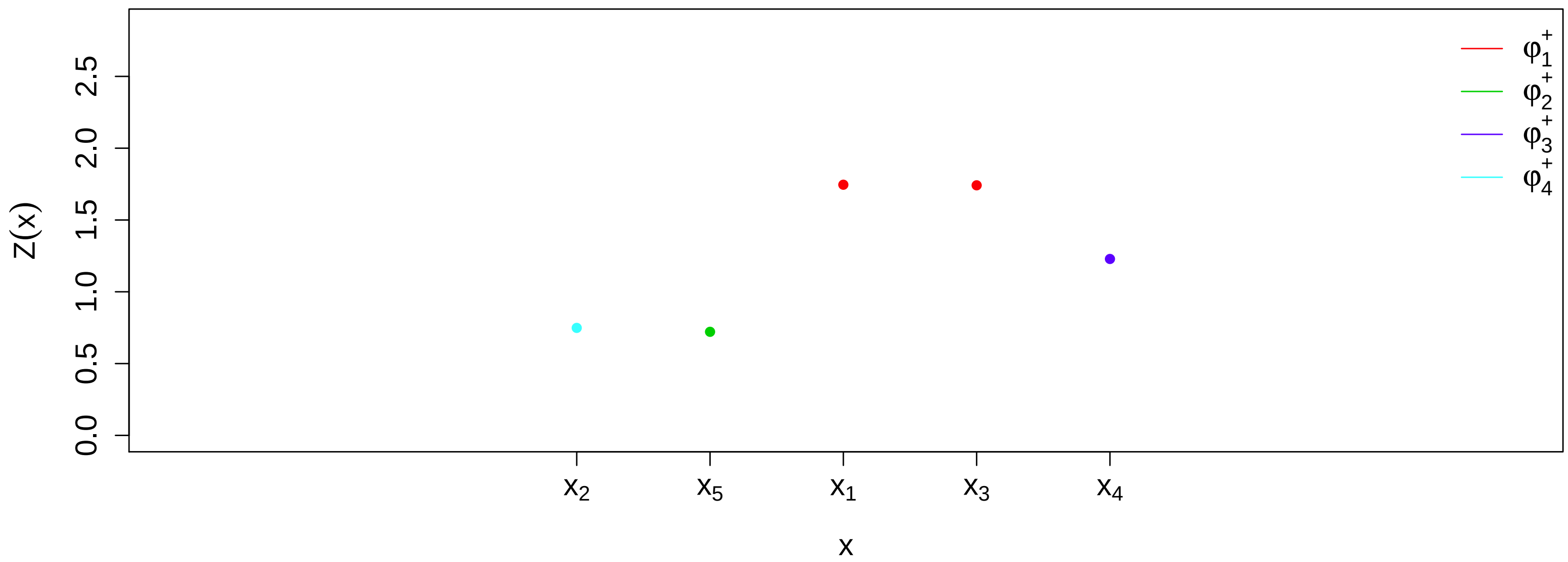
Random partitions?



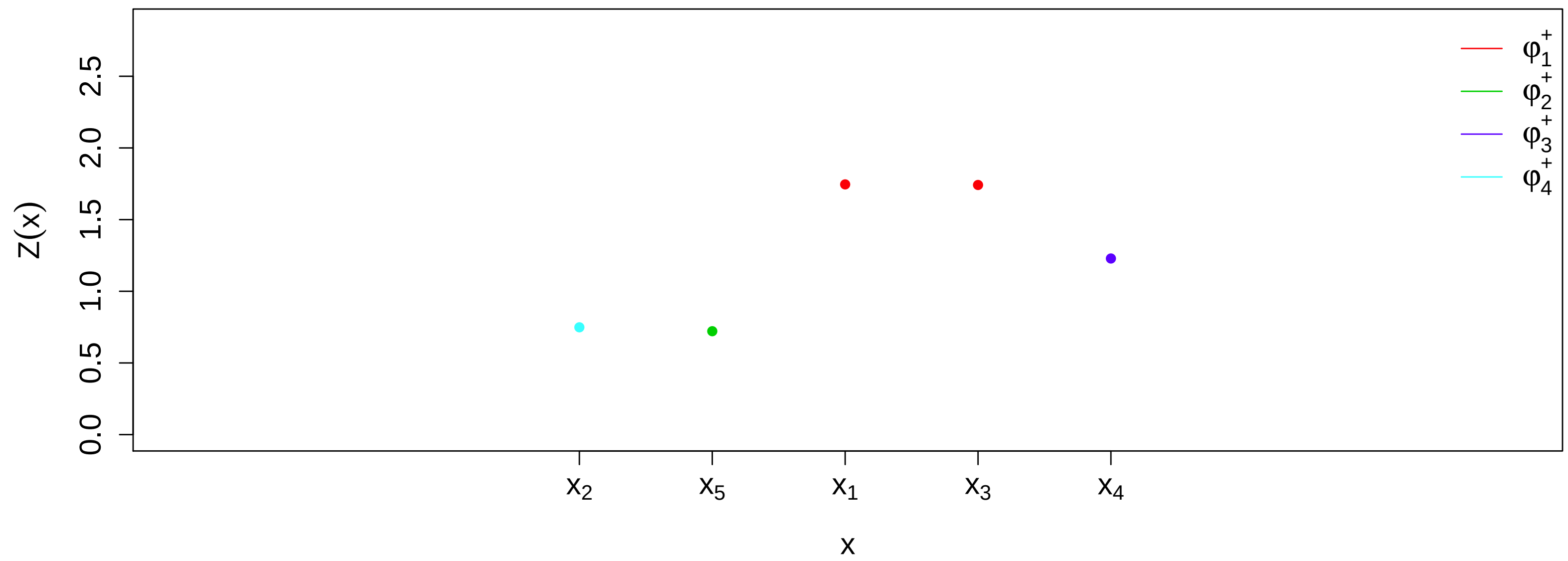

Z(x)
0.0
0.5
1.0
1.5
2.0
2.5
x_2
x_5
x_1
x_3
x_4
x
φ_1^+
φ_2^+
φ_3^+

Random partitions?





Here the set $\{x_1, \ldots, x_5\}$ is partitioned into $(\{x_1, x_3\}, \{x_2\}, \{x_4\}, \{x_5\})$



Here the set $\{x_1, \ldots, x_5\}$ is partitioned into $(\{x_1, x_3\}, \{x_2\}, \{x_4\}, \{x_5\})$

- The hitting bounds $\{z_i\}_{i=1,\ldots,k}$ might be reached by several extremal functions, i.e., $\Phi^+ = \{\varphi_1^+, \ldots, \varphi_k^+\} = \{\varphi_1^+, \ldots, \varphi_\ell^+\}$ a.s., $1 \leq \ell \leq k$.
- So we need to take into account all possible ways these hitting bounds are reached: the hitting scenarios

- The Poisson point process Φ^+, whose atoms are the extremal functions φ_i^+, is defined through a random partition of the set $\{x_1, \ldots, x_k\}$.
- The extremal functions have a distribution fully characterized by the conditional intensity;
- Given $Z(\mathbf{x}) = \mathbf{z}$, Φ^- and Φ^+ are independent.
- This suggests a three step sampling scheme:

- The Poisson point process Φ^+, whose atoms are the extremal functions φ_i^+, is defined through a random partition of the set $\{x_1, \ldots, x_k\}$.
- The extremal functions have a distribution fully characterized by the conditional intensity;
- Given $Z(\mathbf{x}) = \mathbf{z}$, Φ^- and Φ^+ are independent.
- This suggests a three step sampling scheme:

Step 1 Draw a random partition τ, i.e., a hitting scenario;

Step 2 Given τ of size ℓ, draw the extremal functions $\varphi_1^+, \ldots, \varphi_\ell^+$ independently;

Step 3 Independently from Steps 1 & 2, draw the sub-extremal functions φ_i^-, $i \geq 1$.

Step 1: The random partitions

- Let $\mathscr{P}_k$ the set of all possible partitions of the set $\{x_1, \ldots, x_k\}$.
- Draw a random partition $\tau \in \mathscr{P}_k$ with distribution

$$\pi_{\mathbf{x}}(\mathbf{z}, \tau) = \frac{1}{C(\mathbf{x}, \mathbf{z})} \prod_{j=1}^{|\tau|} \underbrace{\lambda_{\mathbf{x}_{\tau_j}}(\mathbf{z}_{\tau_j})}_{\text{density that some bounds are reached, i.e., the } \mathbf{z}_{\tau_j}} \underbrace{\int_{\{\mathbf{u} < \mathbf{z}_{\tau_j^c}\}} \lambda_{\mathbf{x}_{\tau_j^c} \mid \mathbf{x}_{\tau_j}, \mathbf{z}_{\tau_j}}(\mathbf{u}) \mathrm{d}\mathbf{u}}_{\text{probability to lie below the remaining bounds, i.e., below the } \mathbf{z}_{\tau_j^c}},$$

where the normalization constant $C(\mathbf{x}, \mathbf{z})$ is given by

$$C(\mathbf{x}, \mathbf{z}) = \sum_{\theta \in \mathscr{P}_k} \prod_{j=1}^{|\theta|} \lambda_{\mathbf{x}_{\theta_j}}(\mathbf{z}_{\theta_j}) \int_{\{\mathbf{u} < \mathbf{z}_{\theta_j^c}\}} \lambda_{\mathbf{x}_{\theta_j^c} \mid \mathbf{x}_{\theta_j}, \mathbf{z}_{\theta_j}}(\mathbf{u}) \mathrm{d}\mathbf{u},$$

and $|\tau|$ is the “size” of the partition τ.

- Given $\tau = (\tau_1, \ldots, \tau_\ell)$, draw ℓ independent random vectors $\varphi_1^+(\mathbf{s}), \ldots, \varphi_\ell^+(\mathbf{s})$ from the distribution

$$\Pr\left[\varphi_j^+(\mathbf{s}) \in \mathrm{d}\mathbf{v}_j\right] = \frac{1}{C_j}\left\{\int 1_{\{\mathbf{u}<\mathbf{z}_{\tau_j^c}\}} \underbrace{\lambda_{(\mathbf{s},\mathbf{x}_{\tau_j^c})|\mathbf{x}_{\tau_j},\mathbf{z}_{\tau_j}}(\mathbf{v}_j, \mathbf{u})}_{\text{density of an atom } \varphi \in \Phi \text{ given that } \varphi(\mathbf{x}_{\tau_j}) = \mathbf{z}_{\tau_j}} \mathrm{d}\mathbf{u}\right\}\mathrm{d}\mathbf{v}_j,$$

where $1_{\{\cdot\}}$ is the indicator function and

$$C_j = \int 1_{\{\mathbf{u}<\mathbf{z}_{\tau_j^c}\}} \lambda_{(\mathbf{s},\mathbf{x}_{\tau_j^c})|\mathbf{x}_{\tau_j},\mathbf{z}_{\tau_j}}(\mathbf{v}_j, \mathbf{u}) \mathrm{d}\mathbf{u}\mathrm{d}\mathbf{v}_j.$$

- Define the random vector

$$Z^+(\mathbf{s}) = \max_{j=1,\ldots,\ell} \varphi_j^+(\mathbf{s}), \qquad \mathbf{s} \in \mathscr{X}^m.$$

- Independently draw $\{\zeta_i\}_{i\geq 1}$ a Poisson point process on $(0,\infty)$ with intensity $\zeta^{-2}\mathrm{d}\zeta$ and $\{Y_i(\cdot)\}_{i\geq 1}$ independent copies of $Y(\cdot)$
- Define the random vector

$$Z^{-}(\mathbf{s}) = \max_{i\geq 1} \zeta_i Y_i(\mathbf{s}) 1_{\{\zeta_i Y_i(\mathbf{x})<\mathbf{z}\}}, \quad \mathbf{s} \in \mathscr{X}^m.$$

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☞ Then provided Φ is regular, the random vector

$$\tilde{Z}(\mathbf{s}) = \max\left\{Z^+(\mathbf{s}), Z^-(\mathbf{s})\right\}$$

follows the conditional distribution of $Z(\mathbf{s})$ given $Z(\mathbf{x}) = \mathbf{z}$.

- The conditional cumulative distribution function is

$$\Pr\{Z(\mathbf{s}) \leq \mathbf{a} \mid Z(\mathbf{x}) = \mathbf{z}\} = \underbrace{\left\{ \sum_{\tau \in \mathscr{P}_k} \pi_{\mathbf{x}}(\mathbf{z}, \tau) \prod_{j=1}^{|\tau|} F_{\tau,j}(\mathbf{a}) \right\}}_{\text{Steps 1 \& 2}} \underbrace{\frac{\Pr[Z(\mathbf{s}) \leq \mathbf{a}, Z(\mathbf{x}) \leq \mathbf{z}]}{\Pr[Z(\mathbf{x}) \leq \mathbf{z}]}}_{\text{Step 3}},$$

where

$$F_{\tau,j}(\mathbf{a}) = \frac{\int_{\{\mathbf{y} < \mathbf{z}_{\tau_j^c}, \mathbf{u} < \mathbf{a}\}} \lambda_{(\mathbf{s}, \mathbf{x}_{\tau_j^c}) | \mathbf{x}_{\tau_j}, \mathbf{z}_{\tau_j}}(\mathbf{u}, \mathbf{y}) \mathrm{d}\mathbf{y} \mathrm{d}\mathbf{u}}{\int_{\{\mathbf{y} < \mathbf{z}_{\tau_j^c}\}} \lambda_{\mathbf{t}_{\tau_j^c} | \mathbf{x}_{\tau_j}, \mathbf{z}_{\tau_j}}(\mathbf{y}) \mathrm{d}\mathbf{y}}.$$

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Remark. It is “clear” that $Z(\cdot) \mid \{Z(\mathbf{x}) = \mathbf{z}\}$ is not max-stable.

Proposition 1. *If Z is a Brown–Resnick process, i.e.,*

$$Z(x) = \max_{i \geq 1} \zeta_i \exp\{\varepsilon_i(x) - \gamma(x)\}, \qquad x \in \mathscr{X},$$

then the intensity function is

$$\lambda_{\mathbf{x}}(\mathbf{z}) = C_{\mathbf{x}} \exp\left(-\frac{1}{2} \log \mathbf{z}^T Q_{\mathbf{x}} \log \mathbf{z} + L_{\mathbf{x}} \log \mathbf{z}\right) \prod_{i=1}^{k} \mathbf{z}_i^{-1}, \qquad \mathbf{z} \in (0,\infty)^k,$$

and the conditional intensity function is

$$\lambda_{\mathbf{s}|\mathbf{x},\mathbf{z}}(\mathbf{u}) = (2\pi)^{-m/2} |\Sigma_{\mathbf{s}|\mathbf{x}}|^{-1/2} \exp\left\{-\frac{1}{2}(\log \mathbf{u} - \mu_{\mathbf{s}|\mathbf{x},\mathbf{z}})^T \Sigma_{\mathbf{s}|\mathbf{x}}^{-1} (\log \mathbf{u} - \mu_{\mathbf{s}|\mathbf{x},\mathbf{z}})\right\} \prod_{i=1}^{m} \mathbf{u}_i^{-1},$$

i.e., the extremal functions are log-Normal processes.

Proposition 2. *If Z is a Schlather process, i.e.,*

$$Z(x) = \sqrt{2\pi} \max_{i \geq 1} \zeta_i \max\{0, \varepsilon_i(x)\}, \qquad x \in \mathcal{X},$$

then the intensity function is

$$\lambda_{\mathbf{x}}(\mathbf{z}) = \pi^{-(k-1)/2} |\Sigma_{\mathbf{x}}|^{-1/2} a_{\mathbf{x}}(\mathbf{z})^{-(k+1)/2} \Gamma\left(\frac{k+1}{2}\right), \qquad \mathbf{z} \in \mathbb{R}^k,$$

where $a_{\mathbf{x}}(\mathbf{z}) = \mathbf{z}^T \Sigma_{\mathbf{x}}^{-1} \mathbf{z}$, and the conditional intensity function is

$$\lambda_{\mathbf{s}|\mathbf{x},\mathbf{z}}(\mathbf{u}) = \pi^{-m/2} (k+1)^{-m/2} |\tilde{\Sigma}|^{-1/2} \left\{1 + \frac{(\mathbf{u}-\mu)^T \tilde{\Sigma}^{-1} (\mathbf{u}-\mu)}{k+1}\right\}^{-(m+k+1)/2} \frac{\Gamma\left(\frac{m+k+1}{2}\right)}{\Gamma\left(\frac{k+1}{2}\right)},$$

i.e., the extremal functions are Student processes.

2. Markov chain Monte–Carlo sampler (for Step 1)

Do you recognize these numbers?

1	1	2	5	15
52	203	877	4140	21147
115975	678570	4213597	27644437	190899322
1382958545	10480142147	82864869804	682076806159	5832742205057
...				

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...				

☞ These are the first 20 Bell numbers.

Remark. Recall that Bell(k) is the number of partitions of a set with k elements. Hence with our notations we have

$$\# \text{ hitting scenarios} = \text{Card}\,(\mathscr{P}_k) = \text{Bell}(k).$$

Computational burden

- In Step 1, we need to sample from a discrete distribution whose state space is $\mathscr{P}_k$, i.e., all possible hitting scenarios.

Combinatorial explosion

Hence we cannot compute $C(\mathbf{x}, \mathbf{z})$ in

$$\pi_{\mathbf{x}}(\mathbf{z}, \tau) = \frac{1}{C(\mathbf{x}, \mathbf{z})} \prod_{j=1}^{|\tau|} \lambda_{\mathbf{x}_{\tau_j}}(\mathbf{z}_{\tau_j}) \int_{\{\mathbf{u} < \mathbf{z}_{\tau_j^c}\}} \lambda_{\mathbf{x}_{\tau_j^c} | \mathbf{x}_{\tau_j}, \mathbf{z}_{\tau_j}}(\mathbf{u}) \mathrm{d}\mathbf{u}.$$

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💣💣💣 Combinatorial explosion 💣💣💣

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☞ Use of MCMC samplers to sample from the target $\pi_{\mathbf{x}}(\mathbf{z},\cdot)$.

Remark. We will use a Gibbs sampler since the full conditional distributions are especially convenient.

Full conditional distributions

- For $\tau \in \mathscr{P}_k$ of size ℓ, let τ_{-j} be the restriction of τ to the set $\{x_1, \ldots, x_k\} \setminus \{x_j\}$, e.g., $\tau = (\{x_1, x_2\}, \{x_3\})$, $\tau_{-2} = (\{x_1\}, \{x_3\})$.
- We aim at sampling from $\Pr[\theta \in \cdot \mid \theta_{-j} = \tau_{-j}]$, $\theta \sim \pi_{\mathbf{x}}(\mathbf{z}, \cdot)$.
- The number of possible states for θ is

$$b^+ = \begin{cases} \ell & \text{if } \{x_j\} \text{ is a partitioning set of } \tau, \\ \ell + 1 & \text{otherwise,} \end{cases}$$

since $\{x_j\}$ is reallocated to any partitioning set or to a new one—if possible.

Example 1. For $\tau = (\{x_1, x_2\}, \{x_3\})$ we have $\ell = 2$ and

	Restriction	
	$\tau^*_{-2} = \tau_{-2}$	$\tau^*_{-3} = \tau_{-3}$
Possible states	$(\{x_1, x_2\}, \{x_3\})$	$(\{x_1, x_2\}, \{x_3\})$
	$(\{x_1\}, \{x_2, x_3\})$	$(\{x_1, x_2, x_3\})$
	$(\{x_1\}, \{x_2\}, \{x_3\})$	——

- For all $\tau^* \in \mathscr{P}_k$ such that $\tau^*_{-j} = \tau_{-j}$,

$$\Pr[\theta = \tau^* \mid \theta_{-j} = \tau_{-j}] = \frac{\pi_{\mathbf{x}}(\mathbf{z}, \tau^*)}{\sum_{\tilde\tau \in \mathscr{P}_k} \pi_{\mathbf{x}}(\mathbf{z}, \tilde\tau) 1_{\{\tilde\tau_{-j} = \tau_{-j}\}}} \propto \frac{\prod_{j=1}^{|\tau^*|} w_{\tau^*,j}}{\prod_{j=1}^{|\tau|} w_{\tau,j}},$$

where $w_{\tau,j} = \lambda_{\mathbf{x}_{\tau_j}}(\mathbf{z}_{\tau_j}) \int_{\{\mathbf{u} < \mathbf{z}_{\tau_j^c}\}} \lambda_{\mathbf{x}_{\tau_j^c} \mid \mathbf{x}_{\tau_j}, \mathbf{z}_{\tau_j}}(\mathbf{u}) \mathrm{d}\mathbf{u}$.

☞ In particular at most 4 weights $w_{\cdot,\cdot}$ need to be evaluated and the Gibbs sampler is especially convenient!

Remark. The most CPU demanding part is the computation of

$$\int_{\{\mathbf{u} < \mathbf{z}_{\tau_j^c}\}} \lambda_{\mathbf{x}_{\tau_j^c} \mid \mathbf{x}_{\tau_j}, \mathbf{z}_{\tau_j}}(\mathbf{u}) \mathrm{d}\mathbf{u}.$$

This is done following the lines of Genz (1992), i.e.,

Quasi M.–C. + Sep. of Var. + Var. ordering + Antithetic

…the state space $\mathscr{P}_k$ isn't! (really?)

But how do I implement a Gibbs sampler whose states are partitions of a set???

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Lemma 1. *There is a one-one mapping between $\mathscr{P}_k$ and*

$$\mathscr{P}_k^* = \left\{(a_1, \ldots, a_k), \forall i \in \{2, \ldots, k\} : a_1 \leq a_i \leq \max_{1 \leq j < i} a_j + 1, a_i \in \mathbb{Z}\right\},$$

where $a_1 = 1$ by convention.

Example 2. $(\{x_1, x_2\}, \{x_3\})$ is identified to $(1, 1, 2)$ while $(\{x_1, x_3\}, \{x_2\})$ is identified to $(1, 2, 1)$.

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Remark. When updating $\tau \in \mathscr{P}_k^*$, the updated partition doesn't necessarily lie in $\mathscr{P}_k^*$; but corresponds to a unique element of $\mathscr{P}_k$. For instance, $(1,1,2) \mapsto (1,3,2) \leftrightarrow (1,2,3)$.

3. Simulation Study

What we expect

- Less variability in regions close to some conditioning points;
- The coverage is OK, i.e., pointwise confidence intervals have the nominal coverage;
- "Unconditional like behavior" in regions far away from any conditioning point.

Table 1: *Spatial dependence structures of Brown–Resnick processes with (semi) variogram* $\gamma(h) = (h/\lambda)^\kappa$. *The variogram parameters are set to ensure that the extremal coefficient function satisfies* $\theta(115) = 1\cdot 7$.

	Sample path properties		
	γ_1: Very wiggly	γ_2: Wiggly	γ_3: Smooth
λ	25	54	69
κ	0·5	1·0	1·5

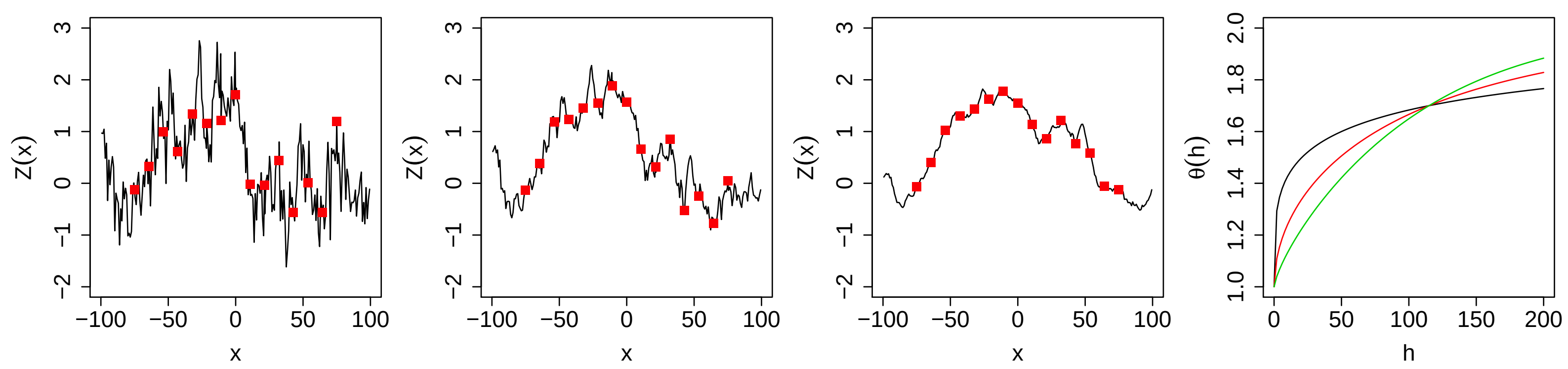

Figure 1: *Three realizations of a Brown–Resnick process with standard Gumbel margins and (semi) variograms* γ_1, γ_2 *and* γ_3. *The squares correspond to the 15 conditioning values that will be used in the simulation study. The right panel shows the associated extremal coefficient functions.*

Table 2: *Spatial dependence structures of Schlather processes with correlation function* $\rho(h) = \exp\{-(h/\lambda)^\kappa\}$. *The correlation function parameters are set to ensure that the extremal coefficient function satisfies* $\theta(100) = 1{\cdot}5$.

	Sample path properties		
	ρ_1: Very wiggly	ρ_2: Wiggly	ρ_3: Smooth
λ	208	144	128
κ	0·5	1·0	1·5

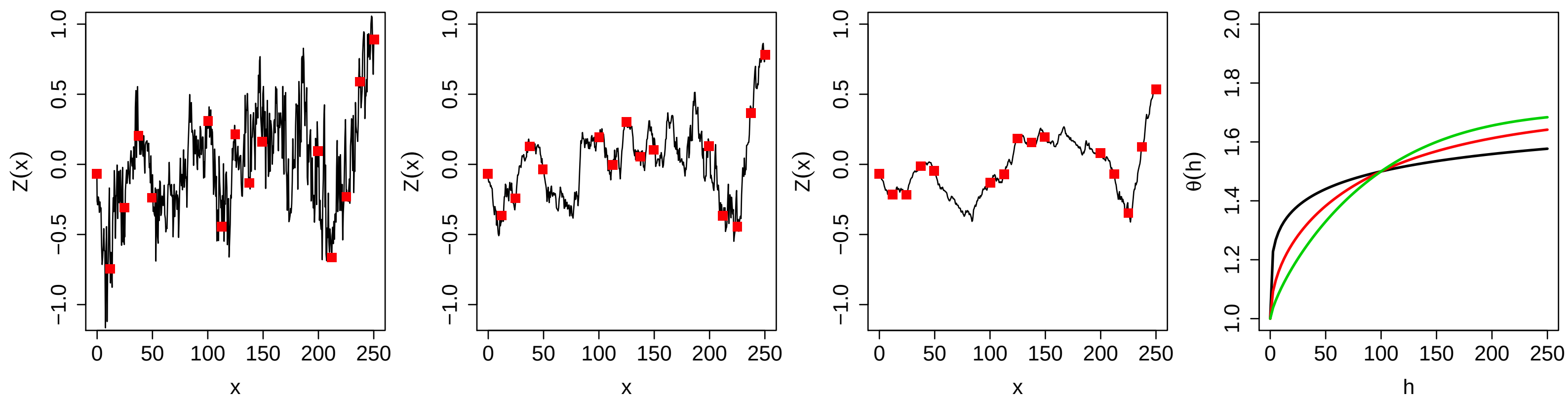

Figure 2: *Three realizations of a Schlather process with standard Gumbel margins and correlation functions* ρ_1, ρ_2 *and* ρ_3. *The squares correspond to the 15 conditioning values that will be used in the simulation study. The right panel shows the associated extremal coefficient functions.*

What we get: Brown–Resnick

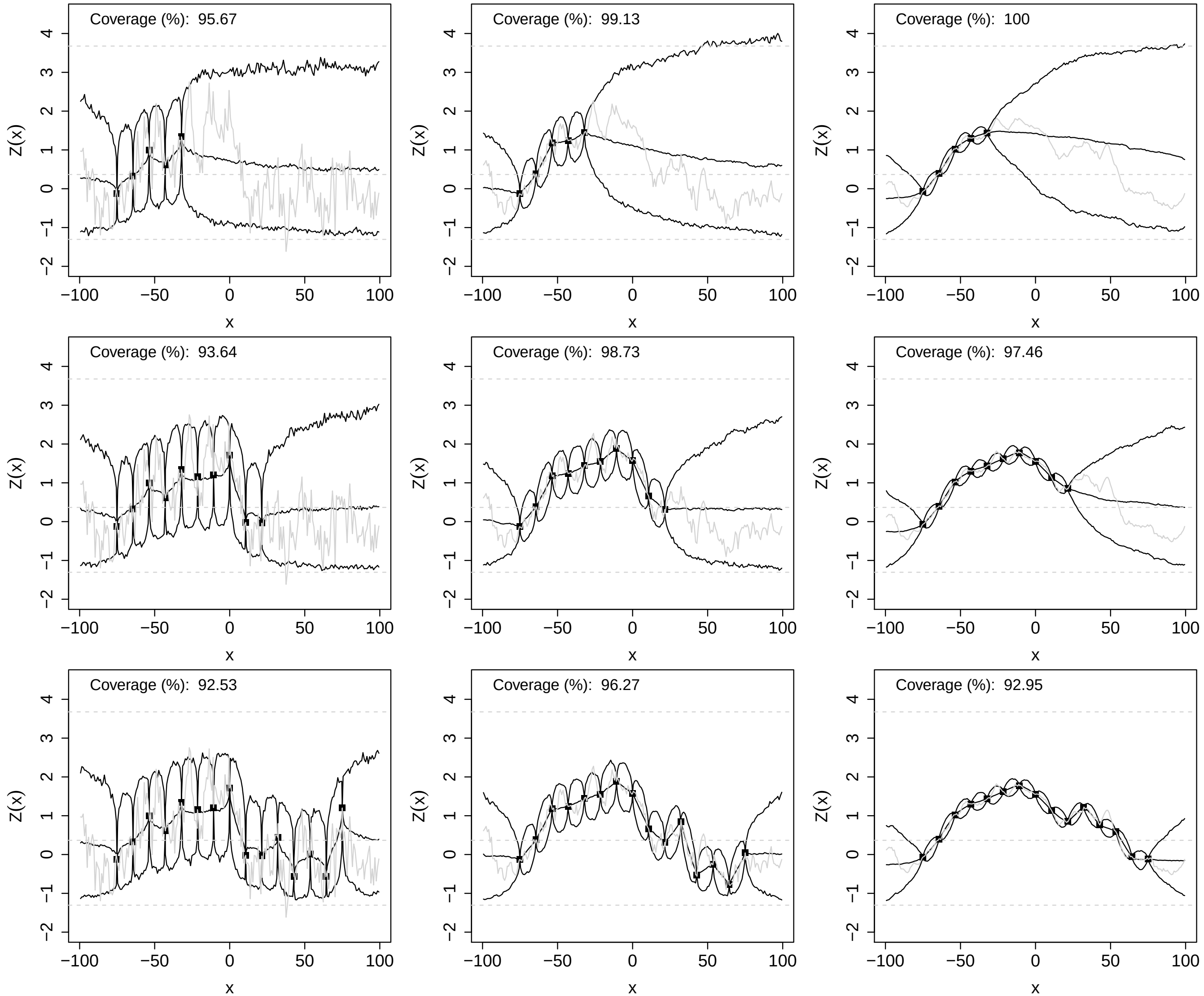

Figure 3: *Pointwise sample quantiles (0.025, 0.5, 0.975) estimated from 1000 conditional simulations of Brown–Resnick processes.*

What we get: Schlather

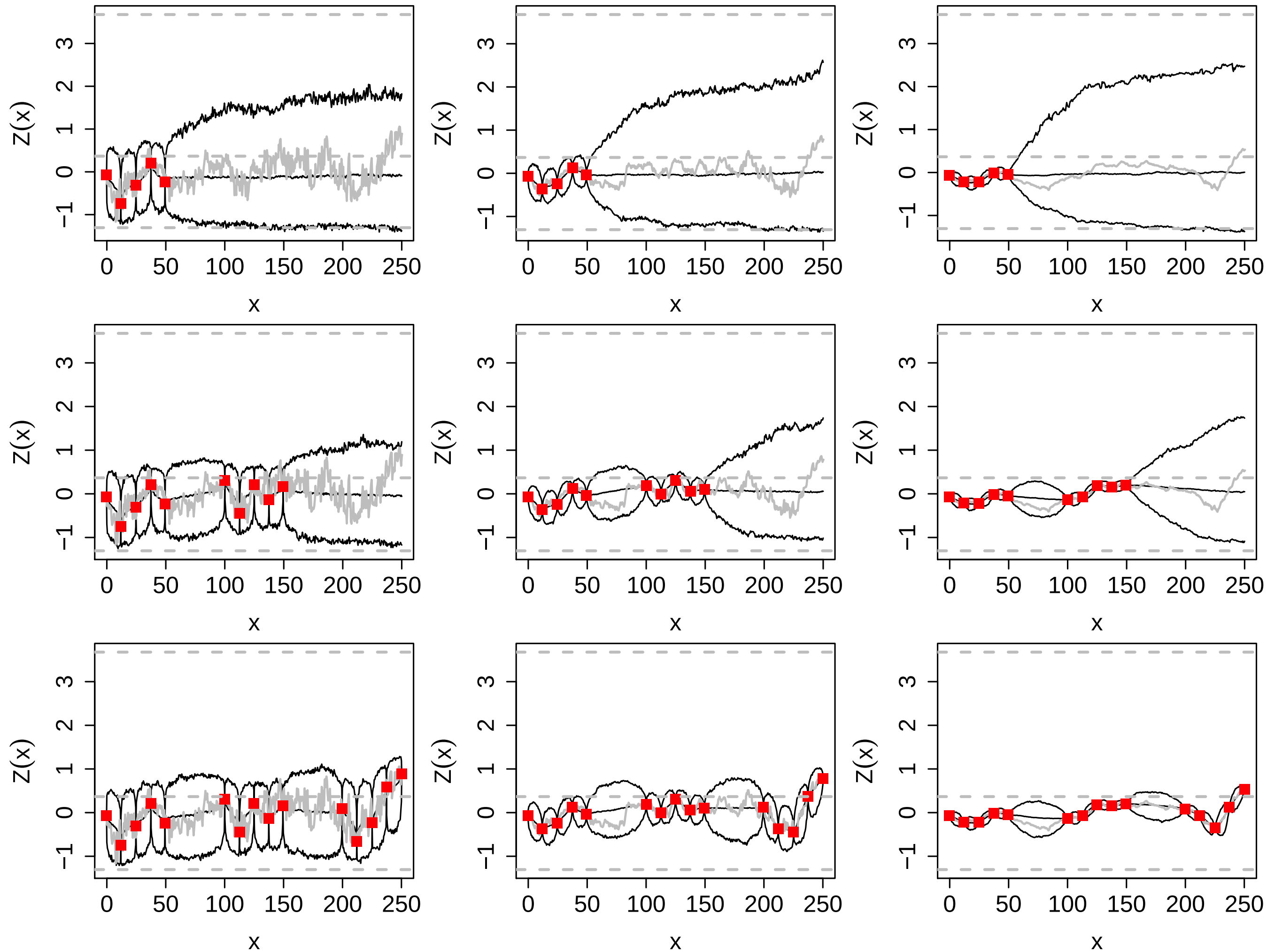

Figure 4: *Pointwise sample quantiles (0.025, 0.5, 0.975) estimated from 1000 conditional simulations of Schlather processes.*

- Is the spatial dependence correct?
- Want to compare the theoretical extremal coefficient function $\theta(\cdot)$ to the pairwise extremal coefficient estimates.

☞ But recall, $Z(\cdot) \mid \{Z(\mathbf{x}) = \mathbf{z}\}$ is not max-stable and the extremal coefficient function does not exist!!!

One last point ;-)

- Is the spatial dependence correct?
- Want to compare the theoretical extremal coefficient function $\theta(\cdot)$ to the pairwise extremal coefficient estimates.
- ☞ But recall, $Z(\cdot) \mid \{Z(\mathbf{x}) = \mathbf{z}\}$ is not max-stable and the extremal coefficient function does not exist!!!

Since

$$f(x) = \int f(x \mid y) f(y) \mathrm{d}y,$$

and to recover the max-stability property, we

1. generate 1000 independent conditional events;
2. and for each such conditional event, one conditional realization—hence having 1000 independent conditional realizations at the end.

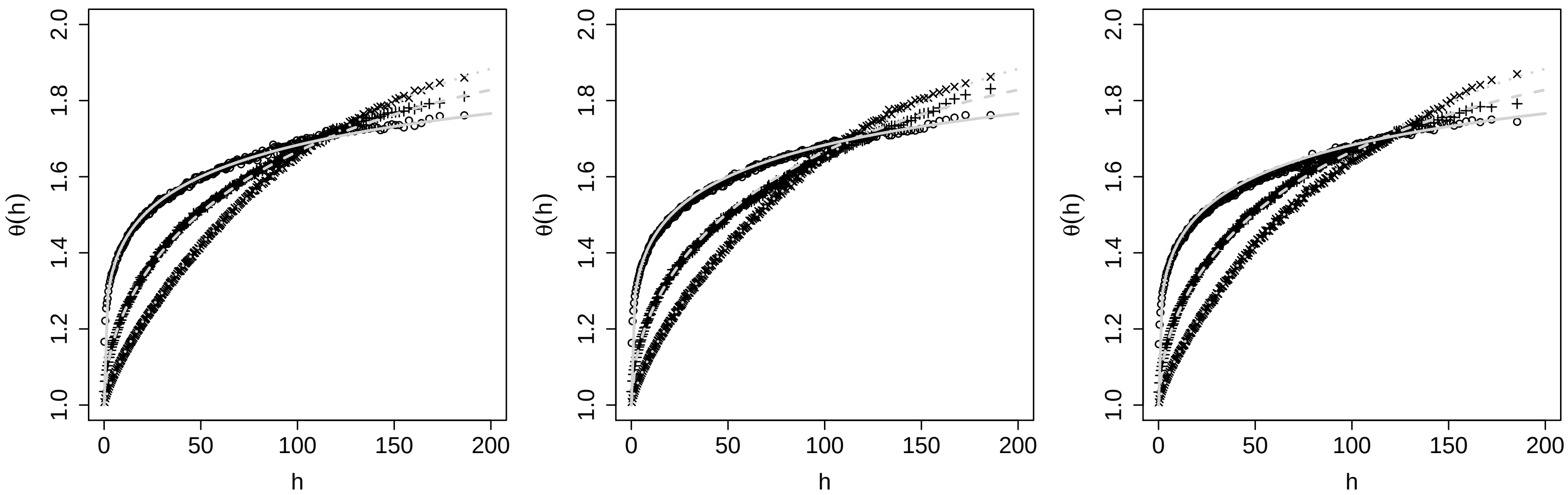

Figure 5: *Comparison of the extremal coefficient estimates (using a binned F-madogram with 250 bins) and the theoretical extremal coefficient function for a varying number of conditioning locations and different (semi) variograms. From left to right,* $k = 5, 10, 15$. *The* 'o', '+' *and* 'x' *symbols correspond respectively to* γ_1, γ_2 *and* γ_3. *The solid, dashed and dotted grey lines correspond to the theoretical extremal coefficient functions for* γ_1, γ_2 *and* γ_3.

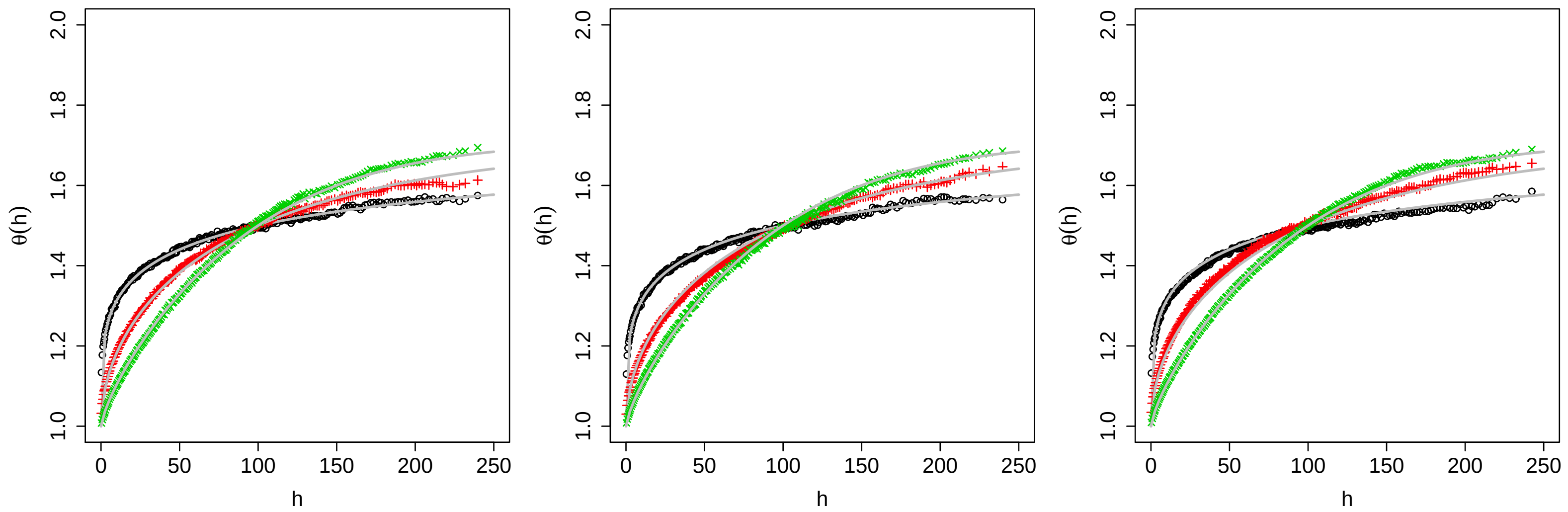

Figure 6: *Comparison of the extremal coefficient estimates (using a binned F-madogram with 250 bins) and the theoretical extremal coefficient function for a varying number of conditioning locations and different correlation functions. From left to right, $k = 5, 10, 15$. The 'o', '+' and 'x' symbols correspond respectively to ρ_1, ρ_2 and ρ_3.*

CPU times

Table 3: *Timings*† *for conditional simulations of max-stable processes on a* 50×50 *grid defined on the square* $[0, 100 \times 2^{1/2}]^2$ *for a varying number* k *of conditioning locations uniformly distributed over the region. The times, in seconds, are mean values over 100 conditional simulations; standard deviations are reported in brackets.*

	Brown–Resnick: $\gamma(h) = (h/25)^{0\cdot5}$				Schlather: $\rho(h) = \exp\{-(h/208)^{0\cdot50}\}$			
	Step 1	Step 2	Step 3	Overall	Step 1	Step 2	Step 3	Overall
$k = 5$	0·21 (0·01)	49 (11)	1·4 (0·1)	50 (11)	1·4 (0·02)	1·9 (0·7)	0·9 (0·3)	4·2 (0·8)
$k = 10$	8 (2)	76 (18)	1·4 (0·1)	85 (19)	12 (4)	2·4 (0·8)	1·0 (0·3)	15 (4)
$k = 25$	95 (38)	117 (30)	1·4 (0·1)	214 (61)	86 (42)	4 (1)	1·0 (0·3)	90 (43)
$k = 50$	583 (313)	348 (391)	1·5 (0·1)	931 (535)	367 (222)	62 (113)	1·0 (0·3)	430 (262)

†Conditional simulations with $k = 5$ do not use a Gibbs sampler.

4. Applications

□ We re-analyze the data of Davison et al. (2012), i.e., summer precipitation around Zurich.

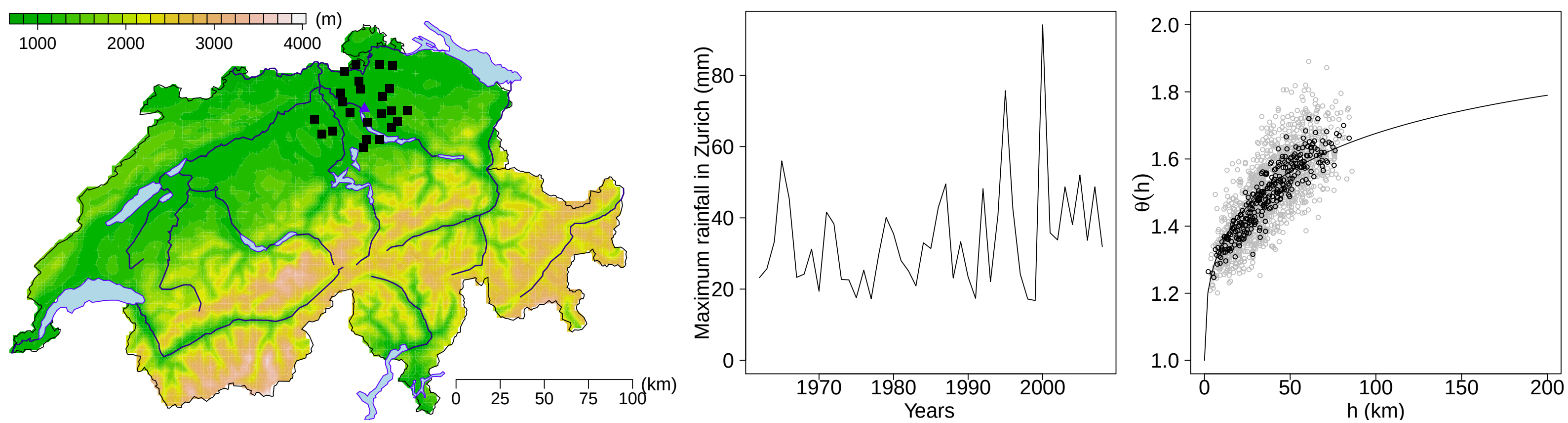

Figure 7: *Left: Map of Switzerland showing the stations of the 24 rainfall gauges used for the analysis, with an insert showing the altitude. The station marked with a blue square corresponds to Zurich. Middle: Summer maximum daily rainfall values for 1962–2008 at Zurich. Right: Comparison between the pairwise extremal coefficient estimates for the 51 original weather stations and the extremal coefficient function derived from a fitted Brown–Resnick processes having (semi) variogram* $\gamma(h) = (h/\lambda)^{\kappa}$. *The grey points are pairwise estimates; the black ones are binned estimates and the red curve is the fitted extremal coefficient function.*

- We fit a Brown–Resnick process by maximizing the pairwise likelihood with the following trend surfaces

$$\eta(x) = \beta_{0,\eta} + \beta_{1,\eta}\text{lon}(x) + \beta_{2,\eta}\text{lat}(x),$$
$$\sigma(x) = \beta_{0,\sigma} + \beta_{1,\sigma}\text{lon}(x) + \beta_{2,\sigma}\text{lat}(x),$$
$$\xi(x) = \beta_{0,\xi},$$

where $\eta(x), \sigma(x), \xi(x)$ are the location, scale and shape parameters of the generalized extreme value distribution and $\text{lon}(x), \text{lat}(x)$ the longitude and latitude of the stations at which the data are observed.

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  where $\eta(x), \sigma(x), \xi(x)$ are the location, scale and shape parameters of the generalized extreme value distribution and $\text{lon}(x), \text{lat}(x)$ the longitude and latitude of the stations at which the data are observed.
- Take as conditional event the values observed during year 2000.
- Simulate a Markov chain of length 15000 from $\pi_{\mathbf{x}}(\mathbf{z}, \cdot)$ to estimate the distribution of the partition size.
- And perform a bunch of conditional simulations from our fitted model to get a nice map!

Table 4: *Empirical distribution of the partition size for the rainfall data estimated from a simulated Markov chain of length 15000.*

Partition size	1	2	3	4	5	6	7–24
Empirical probabilities (%)	66·2	28·0	4·8	0·5	0·2	0·2	$<$0·05

- Around 65% of the time, the maxima at the 24 locations are a consequence of a single extremal function, i.e., only one storm, and around 30% of the time of two extremal functions.
- Focusing only on partitions of size 2, around 65% of the time at least one of the four up-north locations are impacted by a first extremal function while the remaining 20 stations are always influenced by a second extremal function.

Conditional map

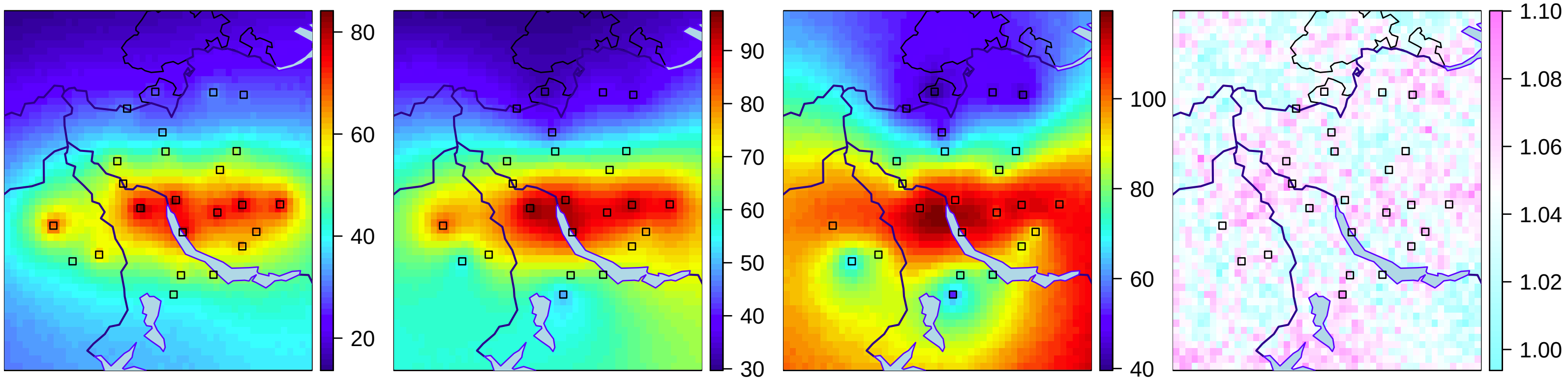

Figure 8: *From left to right, maps on a* 50 × 50 *grid of the pointwise 0·025, 0·5 and 0·975 sample quantiles for rainfall (mm) obtained from* 10000 *conditional simulations of Brown–Resnick processes having semi variogram* $\gamma(h) = (h/38)^{0\cdot 69}$. *The rightmost panel plots the ratio of the width of the pointwise confidence intervals with and without taking estimation uncertainties into account. The squares show the conditional locations.*

- We re-analyze the data of Davison and Gholamrezaee (2012), i.e., annual maxima temperature in Switzerland.

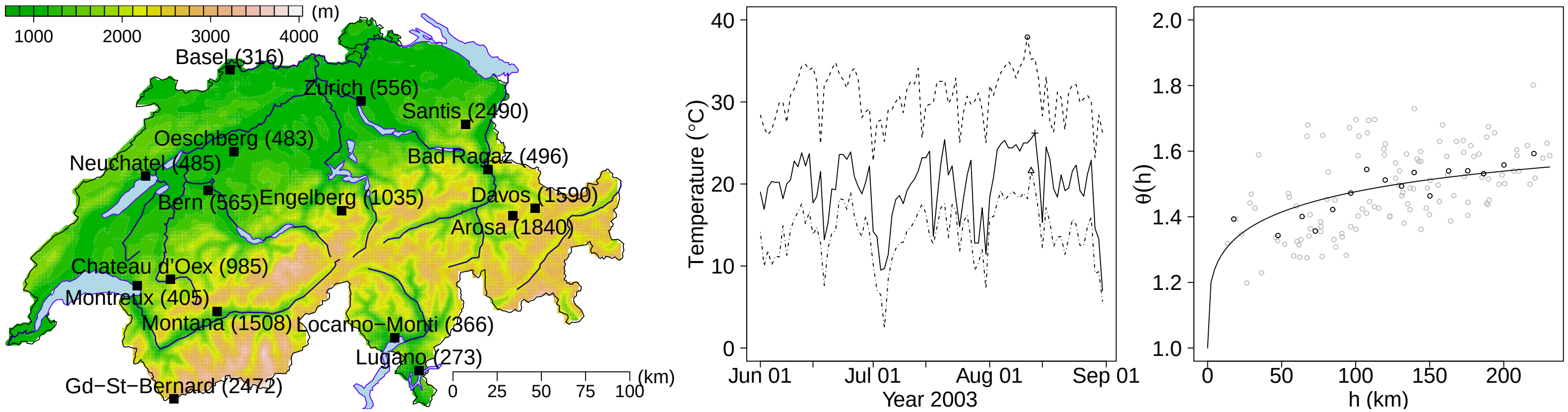

Figure 9: *Left: Topographical map of Switzerland showing the sites and altitudes in metres above sea level of 16 weather stations for which annual maxima temperature data are available. Middle: Times series of the daily maxima temperatures at the 16 weather stations for year 2003. The 'o', '+' and 'x' symbols indicate the annual maxima that occurred the 11th, 12th and 13th of August respectively. Right: Comparison between the fitted extremal coefficient function from a Schlather process (solid red line) and the pairwise extremal coefficient estimates (gray circles). The black circles denote binned estimates with 16 bins.*

- We fit a Schlather process by maximizing the pairwise likelihood with the following trend surfaces

$$\eta(x) = \beta_{0,\eta} + \beta_{1,\eta}\text{alt}(x),$$
$$\sigma(x) = \beta_{0,\sigma},$$
$$\xi(x) = \beta_{0,\xi} + \beta_{1,\xi}\text{alt}(x),$$

where $\eta(x), \sigma(x), \xi(x)$ are the location, scale and shape parameters of the generalized extreme value distribution and $\text{alt}(x)$ the altitude of the stations at which the data are observed.

- We fit a Schlather process by maximizing the pairwise likelihood with the following trend surfaces

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$$\xi(x) = \beta_{0,\xi} + \beta_{1,\xi}\text{alt}(x),$$

  where $\eta(x), \sigma(x), \xi(x)$ are the location, scale and shape parameters of the generalized extreme value distribution and $\text{alt}(x)$ the altitude of the stations at which the data are observed.
- Take as conditional event the values observed during the 2003 European heatwave.
- Simulate a Markov chain of length 10000 from $\pi_{\mathbf{x}}(\mathbf{z}, \cdot)$ to estimate the distribution of the partition size.
- And perform a bunch of conditional simulations from our fitted model to get a nice map!

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Table 5: *Empirical distribution of the partition size for the temperature data estimated from a simulated Markov chain of length 10000.*

Partition size	1	2	3	4	5–16
Empirical probabilities (%)	2·47	21·55	64·63	10·74	0·61

- Around 90% of the time, the conditional simulations are a consequence of at most 3 extremal functions;
- Inspecting the data, we found that the annual maxima in 2003 occured between the 11th and 13rd of August

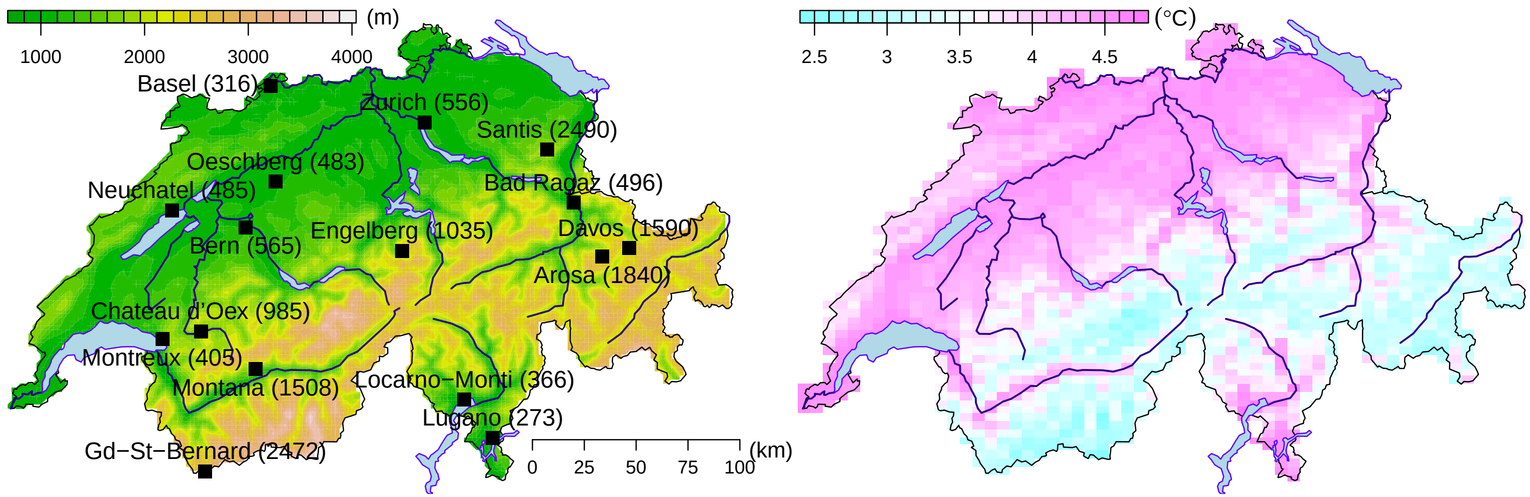
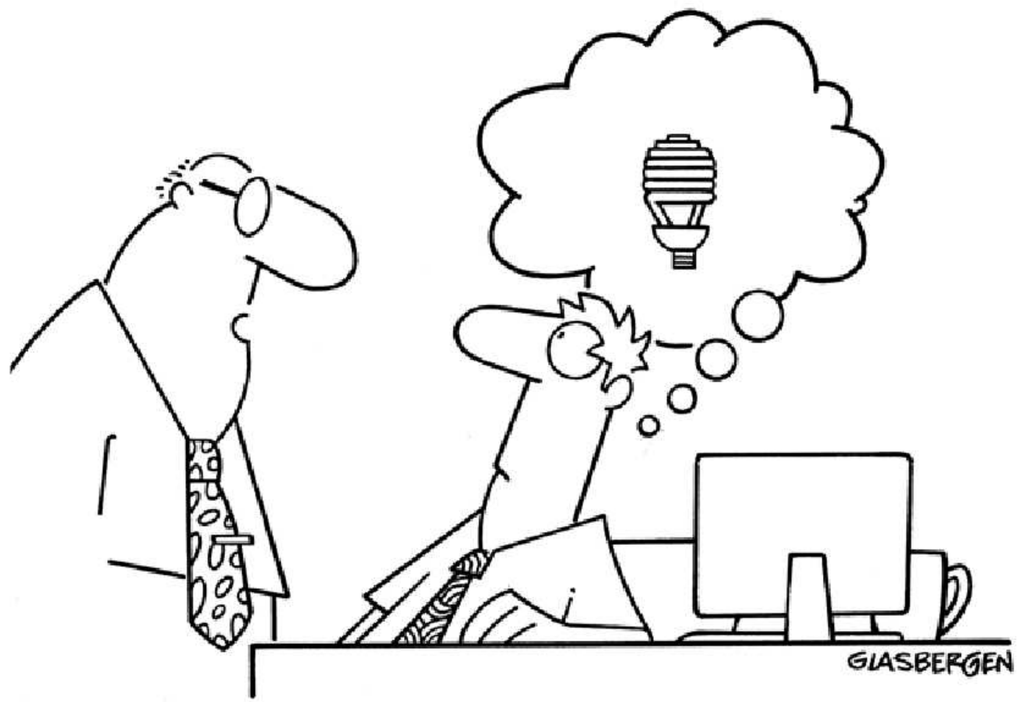


Figure 10: *Left: Topographical map of Switzerland showing the sites and altitudes in metres above sea level of 16 weather stations for which annual maxima temperature data are available. Right: Map of temperature anomalies* (°C)*, i.e., the difference between the pointwise medians obtained from 10000 conditional simulations and unconditional medians estimated from the fitted Schlather process.*

- As expected the largest deviations occur in the plateau region of Switzerland
- The differences range between 2·5°C and 4·75°C

Global warming mitigation: the academic way…



"Your bright ideas don't seem as bright as they used to."

THANK YOU !

Dombry, C. Éyi-Minko, F. and Ribatet, M. *Conditional simulation of max-stable processes.* To appear in Biometrika.

(http://arxiv.org/abs/1208.5376)