Modèles de Markov cachés en théorie des valeurs extrêmes

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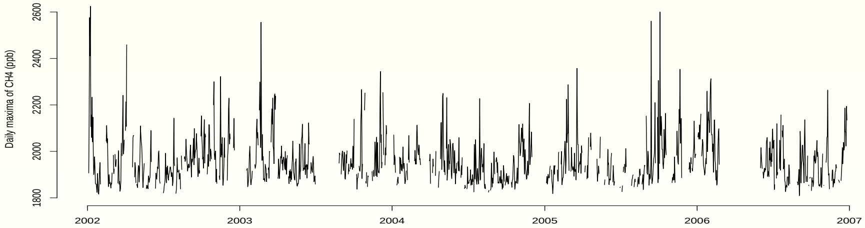
Outline

Outline

Series of daily maxima of CH4 in Gif-sur-Yvette



The GEV distribution

$$X_1, ..., X_n \sim F \in D(\gamma)$$

$$\lim_{n\to\infty} \mathbb{P}\{a_n^{-1}(\max_{1\leq i\leq n} X_i - b_n) \leq x\} = \lim_{n\to\infty} F^n(a_n x + b_n) = H_\gamma(x)$$

where

$$H_\gamma(x) = \begin{cases} \exp\Big(-(1+\gamma x)^{-\frac{1}{\gamma}}\Big) & \text{with } x \text{ such that } 1+\gamma x > 0, \text{ if } \gamma \neq 0 \\ \exp\Big(-\exp(-x)\Big) & \text{for all } x \in \mathbb{R}, \text{ if } \gamma = 0 \end{cases}$$

Domain of attraction

$$\begin{cases} \gamma > 0 & \text{Fréchet, heavy-tailed, } 1 - F(x) = x^{-\frac{1}{\gamma}} \ell_F(x), \\ \gamma = 0 & \text{Gumbel, light-tailed,} \\ \gamma < 0 & \text{Weibull, finite endpoint } \tau_F,\ 1 - F(x) = (\tau_F - x)^{-\frac{1}{\gamma}} \ell_F((\tau_F - x)^{-1}). \end{cases}$$

A function $\ell : (0, \infty) \to (0, \infty)$ is called slowly varying if for all $t > 0$

$$\lim_{x \to \infty} \frac{\ell(tx)}{\ell(x)} = 1.$$

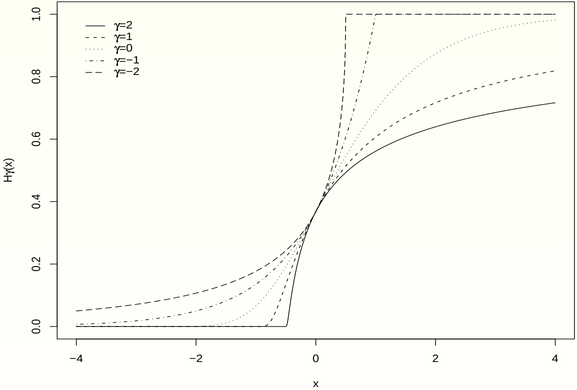


FIG. 1: *Generalized Extreme Value Distributions*

Distribution	$1-F(x)$	γ
Burr(β,τ,λ), $\beta>0$, $\tau>0$, $\lambda>0$	$\left(\frac{\beta}{\beta+x^\tau}\right)^\lambda$	$\frac{1}{\lambda\tau}$
Fréchet$\left(\frac{1}{\alpha}\right)$, $\alpha>0$	$1-\exp(-x^{-\alpha})$	$\frac{1}{\alpha}$
Pareto(α), $\alpha>0$	$x^{-\alpha}$	$\frac{1}{\alpha}$
Gumbel(μ,σ), $\mu\in\mathbb{R}$, $\sigma>0$	$1-\exp\left(-\exp\left(-\frac{x-\mu}{\sigma}\right)\right)$	0
Logistique	$\frac{2}{1+\exp(x)}$	0
Exp(λ), $\lambda>0$	$\exp(-\lambda x)$	0
ReverseBurr$(\beta,\tau,\lambda,\tau_F)$, $\beta>0$, $\tau>0$, $\lambda>0$	$\left(\frac{\beta}{\beta+(\tau_F-x)^{-\tau}}\right)^\lambda$	$-\frac{1}{\lambda\tau}$
Uniforme(a,b)	$1-\frac{x-a}{b-a}$	-1

TAB. 1: *Some distributions with their associated index*

Max-stability

• The max-stable distributions coincide with the extreme value distributions.

Definitions

i) Let $Y_1, Y_2...$ be iid copies of Y with distribution function G.
If for every positive integer k there exist $c_k > 0$ and d_k such that $\max(Y_1, \ldots, Y_k) \stackrel{d}{=} c_k Y + d_k$ then G is max-stable.

$\iff$

ii) If for every positive integer k, we can find $c_k > 0$ and d_k such that $G^k(c_k x + d_k) = G(x)$ then G is called max-stable.

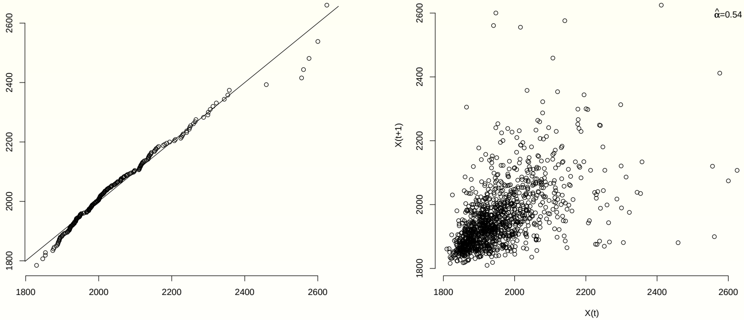


FIG. 2: Gumbel QQplot (on the left) and scatter plot of successive values, i.e. (X_t, X_{t+1}) (on the right) corresponding to the daily maxima of CH4.

State-space models

Our objective : Propose state-space models that preserve the nature of maxima distributions.

Classical state-space models :

$$\begin{aligned} Y_t &= F_t(X_t, \varepsilon_t) && \text{(observation equation)} \\ X_t &= G_t(X_{t-1}, \eta_t) && \text{(state equation)} \end{aligned}$$

where we suppose independence between and within noises ε_t and η_t.

Classical extra assumptions :

- Gaussian noises
- Linearity for the observation and state equations

If a record occurs in Y_t, then one expects Y_t to be a GEV. The problem is that a GEV distribution cannot be retrieved through a Gaussian additive state-space model.

A max-stable state-space model

Max-stable processes : Davis and Resnick (1989)

Naveau and Poncet (2007) introduced the following max-stable state-space model

$$Y_t = F_t X_t \vee \varepsilon_t$$
$$X_t = G_t X_{t-1} \vee \eta_t$$

where ε_t and η_t correspond to Frechet noises.

They proposed lower and upper bounds for X_t.

Morales' state-space model (2005)

$$Y_t = \mu + \sigma \frac{X_t^\gamma - 1}{\gamma} + \varepsilon_t$$
$$X_t = [\beta \eta_t] \vee [(1-\beta)\eta_{t-1}]$$

where $a \vee b = \max(a,b)$, η_t unit Fréchet iid r.v. and ε_t Gaussian noise.

Note that $\mu + \sigma \frac{X_t^\gamma - 1}{\gamma}$ is GEV(μ, σ, γ).

Use of α-stable variables

- A random variable S is said to be (α)-stable if and only if for all $k > 1$ there exist $c_k > 0$ and d_k such that $S_1 + \ldots + S_k \stackrel{d}{=} c_k S + d_k$ where S_1, S_2... are iid copies of S.

- Examples and special cases where one can write down explicit expressions for the density : Gaussian, Cauchy, Levy distributions.

A key linear relationship

$$\text{Gumbel}(\mu_1 + \mu_2, \sigma/\alpha) = \mu_2 + \sigma \log S + \text{Gumbel}(\mu_1, \sigma)$$

where Gumbel(μ_1, σ) denotes a Gumbel r.v. which is independent of S that is a positive α-stable r.v. ($\alpha \in (0, 1]$) with Laplace transform

$$\mathbb{E}(\exp(-uS)) = \exp(-u^{\alpha}), \text{ for all } u > 0.$$

Fougères et al. (2009) : If

$$Y_t = F_t \log\left(\sum_{a\in A} c_{t,a} S_a\right) + \varepsilon_t, \text{ with } t = 1, \dots, T,$$

where $c_{t,a} \geq 0$, where $\{S_a, a \in A\}$ are independent positive α-stable variables and where ε_t follows an iid Gumbel(μ_t, F_t), all these variables being mutually independent, then

$$\mathbb{P}(Y_1 \leq x_1, \dots, Y_T \leq x_T) = \prod_{a\in A} \exp\left(-\left(\sum_{t\in T} c_{t,a} e^{-\frac{x_t-\mu_t}{F_t}}\right)^{\alpha}\right).$$

Gumbel state-space model

•Naveau and Poncet (2007) proposed the following Gumbel state-space model :

$$Y_t = F_t \log U_t + \varepsilon_t$$
$$U_t = G_t U_{t-1} + S_t$$

where ε_t corresponds to an iid Gumbel noise and where S_t represents an iid positive α-stable noise.

•This implies

$$Y_t = F_t \log \left(\sum_{i=0}^{\infty} c_{t,i} S_{t-i} \right) + \varepsilon_t$$

where $c_{t,i} = 0$ for $i \geq t$, $c_{t,i} = 1$ for $i = 0$ and $c_{t,i} = \prod_{j=0}^{i-1} G_{t-j}$ otherwise.

•Y_t are Gumbel distributed and the state-space model is linear !

State-space models

Our objective : Propose linear state-space models that preserve the nature of maxima distributions.

Classical linear state-space models :

$$Y_t = F_t X_t + \varepsilon_t$$
$$X_t = G_t X_{t-1} + \eta_t$$

where we suppose independence between and within noises ε_t and η_t.

A key linear relationship

$$\text{Gumbel}(\mu_1 + \mu_2, \sigma/\alpha) = \mu_2 + \sigma \log S + \text{Gumbel}(\mu_1, \sigma)$$

where Gumbel(μ_1, σ) denotes a Gumbel r.v. which is independent of S that is a positive α-stable ($\alpha \in (0, 1]$) with Laplace transform

$$\mathbb{E}(\exp(-uS)) = \exp(-u^\alpha), \text{ for all } u > 0.$$

Outline

Gumbel linear model

Result (1)

Let $\{X_t, t \in \mathbb{Z}\}$ *be a stochastic process defined by the recurrence relation*

$$X_t = \alpha X_{t-1} + \alpha\sigma \log S_t \tag{1}$$

where $\sigma \in \mathbb{R}_*^+$*.*

Equation (1) has a unique strictly stationary solution,

$$X_t = \sigma \sum_{j=0}^{\infty} \alpha^{j+1} \log S_{t-j} \tag{2}$$

and X_t *follows a Gumbel* $(0, \sigma)$ *distribution,* $\forall\, t \in \mathbb{Z}$*.*

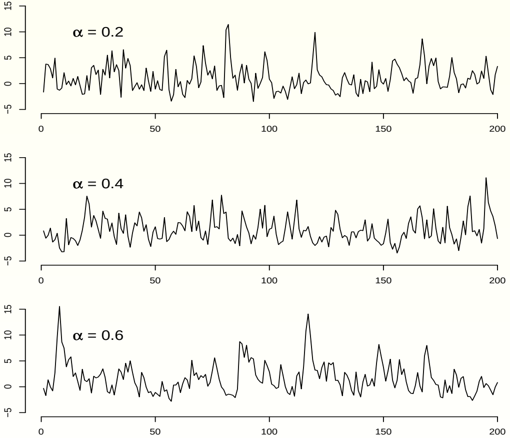


FIG. 3: Simulated time series X_t from proposed model, with $t = 1, \ldots, 200$. We set $\sigma = 2$.

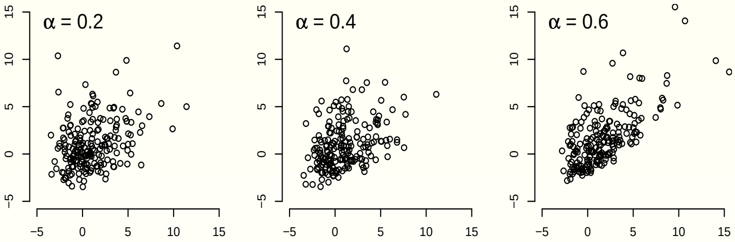


FIG. 4: Scatter plots of successive values, i.e. X_t versus X_{t+1} from the three simulated time series of the previous figure

Covariance $\mathbb{C}ov(X_t, X_{t-h}) = \mathbb{V}ar(X_0)\alpha^{|h|}$

Estimators of the three unknown parameters

Natural estimators of μ and σ based on the method of moments are

$$\widehat{\mu} = \overline{X} - \frac{\delta\sqrt{6}s}{\pi}$$

$$\widehat{\sigma} = \frac{\sqrt{6}s}{\pi}$$

where $\overline{X} = \frac{1}{T}\sum_{t=1}^{T} X_t$, $s^2 = \frac{1}{T}\sum_{t=1}^{T}(X_t - \overline{X})^2$ and where δ is the Euler's constant.

An estimator for α could be the following

$$\widehat{\alpha} = \frac{\frac{1}{T}\sum_{t=1}^{T-1}(X_t - \overline{X})(X_{t+1} - \overline{X})}{s^2}.$$

Asymptotic behavior of the three estimators

Result (2)

The estimators of the three parameters μ, σ and α are consistent (a.s.).

Result (3)

$$\sqrt{T}\begin{pmatrix}\widehat{\mu}-\mu\\ \widehat{\sigma}-\sigma\\ \widehat{\alpha}-\alpha\end{pmatrix}$$

is asymptotically normal with null expectation and covariance matrix defined as follows

$$\begin{pmatrix}
\frac{\pi^2\sigma^2}{6}\frac{1+\alpha}{1-\alpha}-\frac{12\delta\sigma^2\zeta(3)(1+\alpha+\alpha^2)}{\pi^2(1-\alpha^2)}+\frac{11\delta^2\sigma^2(1+\alpha^2)}{10(1-\alpha^2)} & \frac{6\sigma^2\zeta(3)(1+\alpha+\alpha^2)}{\pi^2(1-\alpha^2)}-\frac{11\delta\sigma^2(1+\alpha^2)}{10(1-\alpha^2)} & -\alpha\sigma\delta\\
\frac{6\sigma^2\zeta(3)(1+\alpha+\alpha^2)}{\pi^2(1-\alpha^2)}-\frac{11\delta\sigma^2(1+\alpha^2)}{10(1-\alpha^2)} & \frac{11\sigma^2(1+\alpha^2)}{10(1-\alpha^2)} & \alpha\sigma\\
-\alpha\sigma\delta & \alpha\sigma & 1-\alpha^2
\end{pmatrix}$$

Sketch of the proof (1/2)

$$Z_t = \sum_{j=0}^{\infty} \sigma \alpha^{j+1} \varepsilon_{t-j} = X_t - \delta\sigma \text{ with } \alpha \in (0,1)$$

$$\varepsilon_t = \log S_t - \frac{\delta}{\alpha}(1-\alpha) \overset{iid}{\sim} (0, \sigma_\varepsilon^2) \text{ with } \sigma_\varepsilon^2 = (\pi^2/6) \times (1/\alpha^2 - 1).$$

Lemma

Let $\mathbf{W_t} = \left(Z_t\,,\, Z_t^2 - \mathbb{E}Z_t^2\,,\, Z_t Z_{t+1} - \mathbb{E}Z_t Z_{t+1}\right)'$.

The random vector $T^{-1/2}\sum_{t=1}^{T} \mathbf{W_t}$ converges to a normal distribution with mean vector equal to $\mathbf{0}$ and covariance matrix $\mathbf{\Sigma_Z}$.

Sketch of the proof (2/2)

- We define the truncated sequence as follows

$\mathbf{W_{m,t}} = \left(Z_{m,t}\ ,\ Z_{m,t}^2 - \mathbb{E}Z_{m,t}^2\ ,\ Z_{m,t}Z_{m,t+1} - \mathbb{E}Z_{m,t}Z_{m,t+1}\right)'$ where $Z_{m,t} = \sum_{j=0}^{m} \sigma\alpha^{j+1}\varepsilon_{t-j}$.

- We prove first the asymptotic normality of $T^{-1/2}\sum_{t=1}^{T}\mathbf{W_{m,t}}$

$$T^{-1/2}\sum_{t=1}^{T}\mathbf{W_{m,t}} \xrightarrow{d} \Lambda_m \text{ with } \Lambda_m \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma_{Z_m}}).$$

- Then we let m tend to infinity. Since $\Lambda_m \xrightarrow{d} \Lambda$ as m tends to infinity with $\Lambda \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma_Z})$ and

$$\lim_{m\to\infty}\limsup_{T\to\infty}\mathbb{P}\left(\sqrt{T}\left|\frac{1}{T}\sum_{t=1}^{T}\mathbf{W_{m,t}} - \frac{1}{T}\sum_{t=1}^{T}\mathbf{W_t}\right| > \varepsilon\right) = 0.$$

- We conclude

$$T^{-1/2}\sum_{t=1}^{T}\mathbf{W_t} \xrightarrow{d} \Lambda \text{ with } \Lambda \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma_Z}).$$

Joint distribution of the vector $\mathbf{X_h} = (X_t, \ldots, X_{t-h})^t$, $h > 0$

Result (4)

The characteristic function of $\mathbf{X_h} = (X_t, \ldots, X_{t-h})^t$, $h > 0$, *noted* $\mathbb{E}\left(e^{i<u,\mathbf{X_h}>}\right)$ *is*

$$\Gamma\left(1 - i\sigma \sum_{j=0}^{h} u_j \alpha^{h-j}\right) \prod_{j=0}^{h-1} \frac{\Gamma\left(1 - i\sigma \sum_{k=0}^{j} u_k \alpha^{j-k}\right)}{\Gamma\left(1 - i\sigma \sum_{k=0}^{j} u_k \alpha^{j-k+1}\right)}.$$

Asymptotic dependence parameters

Result (5)

- *The upper tail dependence parameter* χ *defined as follows*

$$\chi = \lim_{x\to\infty} \frac{\mathbb{P}(X_{t-1} > x \quad \textit{and} \quad X_t > x)}{\mathbb{P}(X_{t-1} > x)}$$

is equal to zero (asymptotic independence).

- *The dependence parameter* $\overline{\chi}$ *defined as follows*

$$\overline{\chi} = \lim_{x\to\infty} \frac{2\log \mathbb{P}(X_{t-1} > x)}{\log \mathbb{P}(X_{t-1} > x, X_t > x)} - 1$$

provides a measure which increases with the dependence strength and which is equal to $\alpha/(2-\alpha) \in (0,1)$.

Sketch of the proof

- $\mathbb{P}(X_{t-1} > x, X_t > x)$

$$
\begin{aligned}
&= \mathbb{P}(X_{t-1} > x, \alpha X_{t-1} + \alpha\sigma \log S_t > x) \\
&= \int_x^\infty \mathbb{P}\left(\log S_t > \frac{1}{\alpha\sigma}(x - \alpha y) \mid X_{t-1} = y\right) dH(y) \\
&= \int_x^\infty \mathbb{P}\left(S_t > \exp\left(\frac{1}{\alpha\sigma}(x - \alpha y)\right)\right) dH(y) \\
&= \exp\left(-\frac{x}{\alpha\sigma}\right) \int_0^\infty \exp\left(-u \exp\left(-\frac{x}{\alpha\sigma}\right)\right) \mathbb{1}_{0 \le u \le e^{\frac{x}{\alpha\sigma}(1-\alpha)}} \mathbb{P}(S_t > u) du.
\end{aligned}
$$

- $\mathbb{P}(X_{t-1} > x) \overset{x\to\infty}{\sim} \exp\left(-\frac{x}{\sigma}\right)$.

- $\chi = \lim_{x\to\infty} \int_0^1 \exp\left(-\omega \exp\left(-\frac{x}{\sigma}\right)\right) \mathbb{P}\left(S_t > \omega e^{\frac{x}{\alpha\sigma}(1-\alpha)}\right) d\omega.$

Outline

Simulations

- 1000 samples
- Sample sizes $n = 50, 100, \ldots, 1000$
- $\alpha \in \{0.2, 0.5, 0.8\}$
- $\mu = 0$ and $\sigma = 2$
- For each parameter we compute the mean of the estimations and also the first and the third quartiles

Finite sample behavior of the estimators of μ and σ

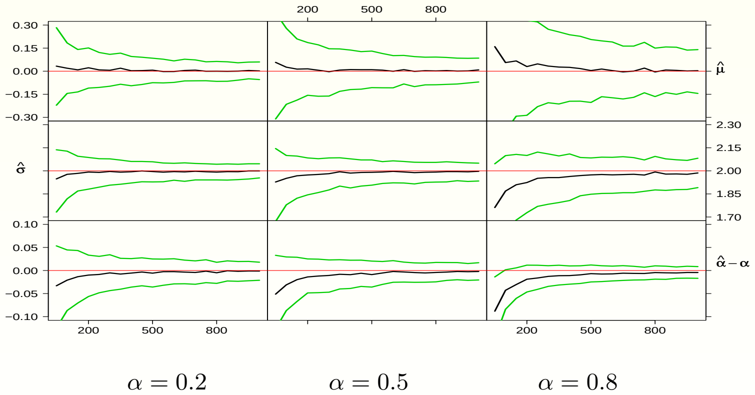


FIG. 5: Mean (black line), first and third quartiles (green lines) for different sample sizes in the abscissa $n \in \{50, 100, \ldots, 1000\}$.

Daily maxima of CH4

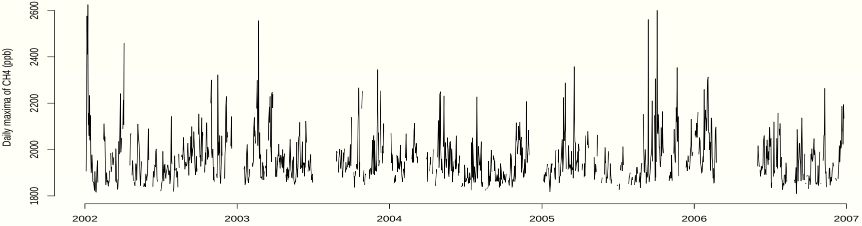


FIG. 6: Series of daily maxima of CH4 in Gif-sur-Yvette.

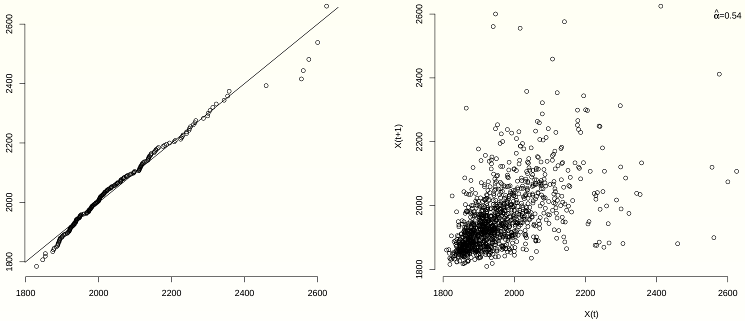


FIG. 7: Gumbel QQplot (on the left) and scatter plot of successive values, i.e. (X_t, X_{t+1}) (on the right) corresponding to the daily maxima of CH4.

One-step prediction (1/2)

We compute an estimation of $X_t | X_{t-1} = x_{t-1}$, $t = 1, \ldots, T-1$

- according to the proposed Gumbel model

$$X_t = \alpha X_{t-1} + \alpha\sigma \log S_t$$

- according to the classical Gaussian model

$$X_t = \alpha X_{t-1} + \varepsilon_t$$

$\hookrightarrow$ White histograms

We compare these white histograms with the shaded areas which are the histograms of our daily maxima of CH4.

Daily maxima of CH4

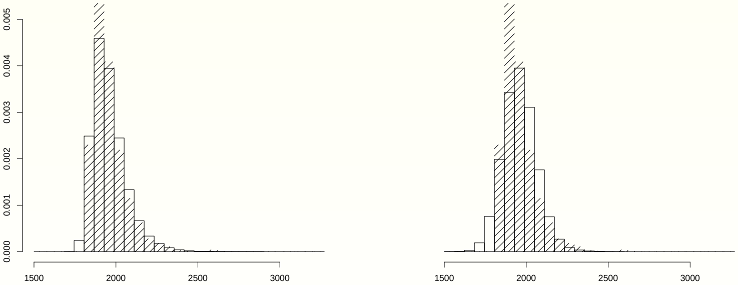


FIG. 8: Histograms of CH4 daily maxima previsions according to Gumbel model (on the left) and Gaussian model (on the right). The shaded histograms represent CH4 daily maxima.

One-step prediction (2/2)

- We compute the estimators of the three parameters on a first period, here from 2002 to the middle of 2006.
- For the second part of 2006, we compute 1000 $\widehat{X}_{t+1} = \widehat{\alpha} x_t + \widehat{\alpha}\widehat{\sigma} \log S_{t+1}$ with x_t the observed value at time t and S_{t+1} a positive $\widehat{\alpha}$-stable random variable.
- We deduce the empirical quartiles of the distribution of $\widehat{X}_{t+1} | X_t = x_t$.

Daily maxima of CH4

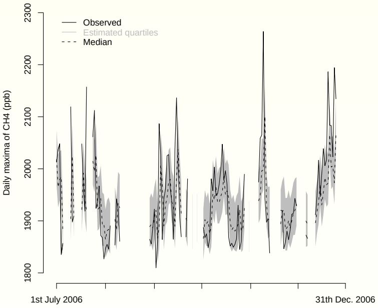


FIG. 9: Validation of the one-step previsions of methane daily maxima on the second part of the year 2006.

Outline

Extension to Gumbel ARMA models

Gumbel AR model

Let $S_{t,\alpha}$ be iid positive α-stable variables and let

$$X_t = \alpha X_{t-1} + \alpha\sigma \log S_{t,\alpha}.$$

X_t is Gumbel$(0, \sigma)$ distributed for any $t > 0$.

Gumbel ARMA model

Let S_{t,α_1} and S_{t,α_2} be independent sequences of iid positive α_i-stable variables with $i = 1, 2$ and let

$$X_t = \alpha_1\alpha_2 X_{t-1} + \alpha_1\alpha_2\sigma \log S_{t,\alpha_1} + \alpha_2\sigma \log S_{t-1,\alpha_2}.$$

X_t is Gumbel$(0, \sigma)$ distributed for any $t > 0$.

Generalization to the GEV distribution

Let E be a r.v. from a GEV(μ, σ, γ).

- If $\gamma < 0$ then $-\log(\mu - \sigma/\gamma - E) \sim$ Gumbel$(\log(-\gamma/\sigma), -\gamma)$.
- If $\gamma > 0$ then $\log(E - \mu + \sigma/\gamma) \sim$ Gumbel$(\log(\sigma/\gamma), \gamma)$.

Result (6)

The following equation

$$X_t = \mu - \frac{\sigma}{\gamma} + \left(X_{t-1} - \mu + \frac{\sigma}{\gamma}\right)^{\alpha} \times S_t^{\alpha\gamma} \times \left(\frac{\sigma}{\gamma}\right)^{1-\alpha}$$

where $(\mu, \sigma, \gamma) \in \mathbb{R} \times \mathbb{R}_*^+ \times \mathbb{R}_*$ *has a unique strictly stationary solution given by*

$$X_t = \mu - \frac{\sigma}{\gamma} + \frac{\sigma}{\gamma} \prod_{j=0}^{\infty} (S_{t-j})^{\gamma\alpha^{j+1}}$$

and X_t *follows a* $GEV(\mu, \sigma, \gamma)$ *distribution,* $\forall\, t \in \mathbb{Z}$.

Outline

Current work

Gumbel state-space model

$$\begin{aligned} Y_t &= \mu_t + H_t(X_t + \sigma \log S_{t,\alpha_2}) && \text{(observational equation)} \\ X_t &= \alpha_1 X_{t-1} + \alpha_1 \sigma \log S_{t,\alpha_1} && \text{(state equation)} \end{aligned}$$

with S_{t,α_1} and S_{t,α_2} two independent positive α_1 and α_2-stable noises.

Therefore, for all t, X_t follows a Gumbel$(0, \sigma)$ distribution and Y_t follows a Gumbel$(\mu_t, H_t \frac{\sigma}{\alpha_2})$ distribution.

- Additive model, X_t and Y_t are both Gumbel
- G_t has to be equal to $\alpha_1 \in (0, 1)$ and Y_t has a specific form.

Current work

This model can be rewritten as

$$\begin{aligned} Y_t &= \nu_t + H_t Z_t + \eta_{t,\alpha_2} && \text{(observational equation)} \\ Z_t &= \alpha_1 Z_{t-1} + \varepsilon_{t,\alpha_1} && \text{(state equation)} \end{aligned}$$

with

$$Z_t = X_t - (\delta\sigma)$$

$$\nu_t = \mu_t + H_t \left(\mu + \frac{\delta\sigma}{\alpha_2} \right)$$

$$\varepsilon_{t,\alpha_1} = \alpha_1 \sigma \log S_{t,\alpha_1} - \delta\sigma(1-\alpha_1) \overset{iid}{\sim} ExpS_{\alpha_1}\left(0, \frac{\sigma^2\pi^2}{6}(1-\alpha_1^2)\right)$$

$$\eta_{t,\alpha_2} = H_t \sigma \log S_{t,\alpha_2} - H_t \frac{\delta\sigma}{\alpha_2}(1-\alpha_2) \overset{iid}{\sim} ExpS_{\alpha_2}\left(0, \frac{H_t^2\sigma^2\pi^2}{6\alpha_2^2}(1-\alpha_2^2)\right)$$

- Distribution of $Z_t | Y_1, \ldots, Y_t$?

Thank you for your attention !

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