# Patchwork copulas

Fabrizio Durante

School of Economics and Management
Free University of Bozen-Bolzano (Italy)
fabrizio.durante@unibz.it
http://sites.google.com/site/fbdurante

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# Outline

1. Introduction
2. Complete dependence and shuffles of Min
3. A measure-theoretic notion of shuffle of Min
4. Multivariate shuffles of copulas
5. The patchwork construction
6. Conclusions

# The main goal

Given a copula $C$, a patchwork copula derived from $C$ is any copula whose mass probability distribution coincides with the mass distribution of $C$ up to some $d$–dimensional boxes $B_i \subseteq \mathbb{I}^d$ (here $\mathbb{I} := [0, 1]$), in which the probability mass is distributed in a different way.

Applications:

- Modification of tail dependence behaviour
- Approximation of copulas

Patchwork copulas include ordinal sum constructions, orthogonal grid constructions, gluing copulas, upper comonotonicity, piecing–together, etc.

Aim: provide a general framework for patchwork copulas.

# The main tool

Given a copula $C$, a probability measure $\mu_C$ is defined on all boxes $B$ of $\mathbb{I}^d$ via

$$\mu_C(B) = V_C(B) = \mathbb{P}(\mathbf{U} \in B),$$

where $\mathbf{U} \sim C$, and extended by classical arguments to all Borel sets. Moreover, such a measure is *d*–fold stochastic, i.e.

$$\mu_C(p_i^{-1}(A)) = \lambda(A)$$

for any Borel set $A \subseteq \mathbb{I}$ ($\lambda$ = Lebesgue measure) and for any canonical projection $p_i$.

Conversely, given a $d$–fold stochastic measure $\mu$, a copula is defined via

$$C_\mu(\mathbf{u}) = \mu([\mathbf{0}, \mathbf{u}]) \quad \text{for all } \mathbf{u} \in \mathbb{I}^d.$$

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# Complete dependence

## Definition

A r.v. $Y$ is defined to be completely dependent on a r.v. $X$ if there exists a measurable function $f$ such that

$$\mathbb{P}(Y = f(X)) = 1.$$

The r.v.'s $X$ and $Y$ are mutually completely dependent (in short, MCD) if there exists a bijective measurable function $f$ such that

$$\mathbb{P}(Y = f(X)) = 1.$$

(Lancaster, 1963)

In other words, two r.v.'s are MCD if one variable is perfectly predictable from the other one, and conversely.

# Shuffle of Min

Shuffles of Min are bivariate copulas constructed by means of a rearrangement of the probability mass of the Fréchet upper bound $M_2(u_1, u_2) = \min\{u_1, u_2\}$.

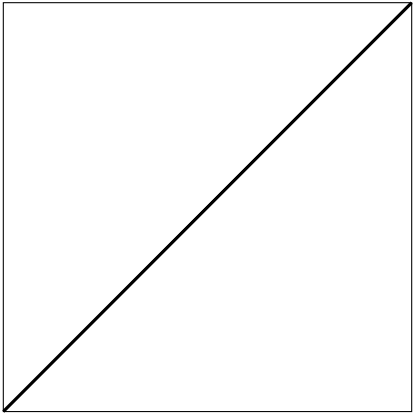
## Proposition

*Let $(X, Y)$ be a random pair distributed according to the copula $C$. Then $C$ is a shuffle of Min iff $\mathbb{P}(Y = f(X)) = 1$ for some bijective piece-wise continuous function $f$.*

*(Mikusinski, Sherwood and Taylor, 1992)*

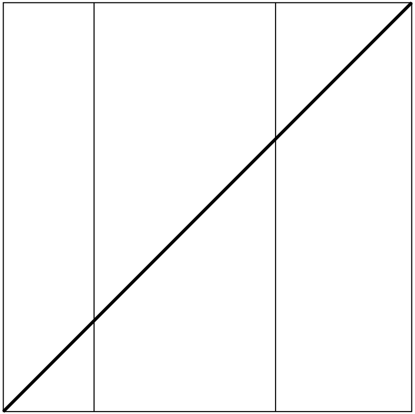
If two r.v.'s are coupled by means of a shuffle of Min, then they are MCD.

# Geometric interpretation of shuffles of Min



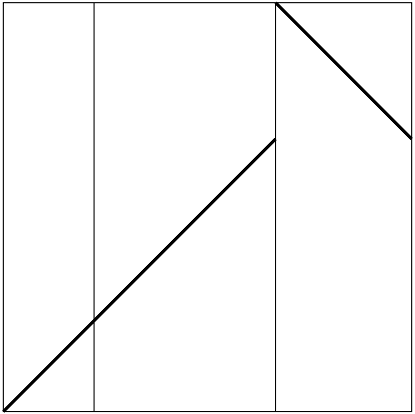
Placing the mass of the copula $M_2$ on $\mathbb{I}^2$.

# Geometric interpretation of shuffles of Min



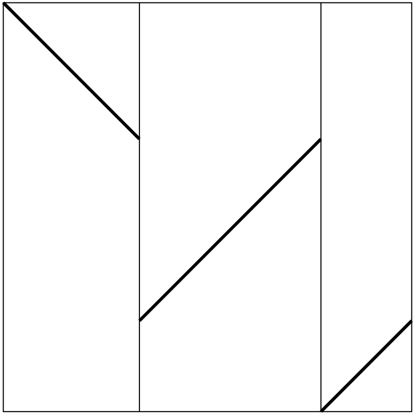
Cut $\mathbb{I}^2$ into a finite number of vertical strips.

# Geometric interpretation of shuffles of Min



Eventually, flip some strips around their vertical axis of symmetry.

# Geometric interpretation of shuffles of Min



"Shuffle" the strips and reassemble them to reform $\mathbb{I}^2$. The resulting picture represents the probability mass distribution of a "shuffle of Min".

# Kimeldorf & Sampson's shuffle of Min

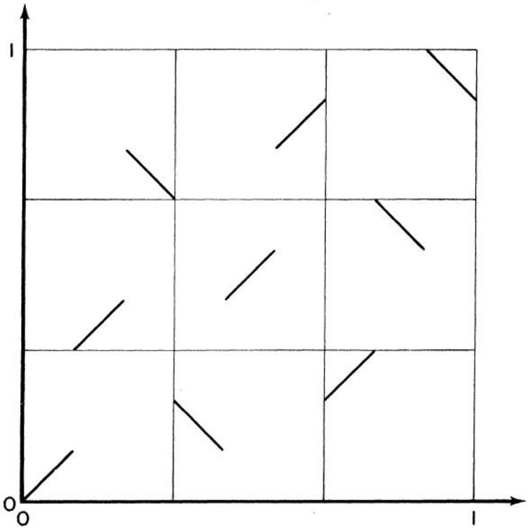



FIG. 1. Support of the distribution of $(U_3, V_3)$.

# Kimeldorf & Sampson's shuffle of Min

**Theorem**

*There are sequences* $(U_n)_n$ *and* $(V_n)_n$ *of r.v.'s all having uniform distribution on* $(0,1)$ *such that:*

- *for each n,* $U_n$ *and* $V_n$ *are mutually completely dependent,*
- *the pairs* $(U_n, V_n)$ *converge in law to a pair* $(U,V)$ *of independent r.v.'s each having a uniform distribution on* $(0,1)$.

*(Kimeldorf and Sampson, 1978; Vitale, 1991)*

In other words, independent r.v.'s can be approximated by means of a sequence of MCD r.v.'s.

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# The shuffling transformation

Let $T \in \mathcal{T}_\mathrm{p}$, the class of measure-preserving bijections of $\mathbb{I}$, i.e. for all Borel sets $A \subseteq \mathbb{I}$,

$$\lambda(T^{-1}(A)) = \lambda(A)$$

We define the shuffling transformation $S_T\colon \mathbb{I}^2 \to \mathbb{I}^2$ via

$$S_T(u_1, u_2) = (T(u_1), u_2)$$

for every $(u_1, u_2) \in \mathbb{I}^2$.

## Proposition

*A copula $C$ is a shuffle of Min iff there exists a piece-wise continuous $T \in \mathcal{T}_\mathrm{p}$ such that $\mu_C = S_T * \mu_M$, i.e. for all Borel $A \subseteq \mathbb{I}^2$*

$$\mu_C(A) = \mu_M(S_T^{-1}(A))$$

*(Durante, Sarkoci and Sempi, 2009)*

# Shuffles of copulas

## Definition

Let $C$ be any copula.
A copula $D$ is a shuffle of $C$ if there exists $T \in \mathcal{T}_{\mathrm{p}}$ such that

$$\mu_D = S_T * \mu_C .$$

(Durante, Sarkoci and Sempi, 2009)

In other words, any copula can be modified by cutting in a countable number of stripes its probability mass and by shuffling the resulting stripes.

# Representation of shuffles of copulas

We recall that any copula $C$ can be represented in the form

$$C(\mathbf{u}) = C_{f_1,\dots,f_d} = \lambda(f_1^{-1}[0,u_1] \cap \cdots \cap f_d^{-1}[0,u_d])$$

for suitable mpt's $f_1, \dots, f_d$.

(Vitale, 1996; Kolesárová, Mesiar, Sempi, 2008)

## Proposition

*Let $D = D_{f,g}$ be a copula represented in terms of mpt's in the following way:*

$$D_{f,g}(u_1,u_2) = \lambda\left(f^{-1}[0,u_1] \cap g^{-1}[0,u_2]\right)$$

*Then any shuffle of D via $T \in \mathcal{T}_{\mathrm{p}}$ (write: $D_T$) can be represented in the form*

$$D_T(u_1,u_2) = \lambda\left((T \circ f)^{-1}[0,u_1] \cap g^{-1}[0,u_2]\right)$$

# Remark

The mapping

$$\varphi \colon \mathcal{T}_\mathrm{p} \times \mathcal{C}_2 \to \mathcal{C}_2, \quad \varphi(T, C) = C_T$$

defines an action of the group $\mathcal{T}_\mathrm{p}$ on the set of all copulas.

The orbit of a copula $C$ with respect to this action is the set

$$\mathcal{T}_\mathrm{p}(C) = \{C_T \mid T \in \mathcal{T}_\mathrm{p}\}$$

formed by all shuffles of $C$.

## Properties of shuffles of copulas

- $\mathcal{T}_\mathrm{p}(C) = \{C\}$ iff $C = \Pi_2$.
- If $C$ is abs continuous, then every copula belonging to $\mathcal{T}_\mathrm{p}(C)$ is abs continuous.
- If $C \neq \Pi_2$, then $\mathcal{T}_\mathrm{p}(C)$ contains non-symmetric copulas.

# Approximation by means of shuffles

## Proposition

*For every copula $C$, the independence copula $\Pi$ can be approximated uniformly by elements of $\mathcal{T}_{\mathrm{p}}(C)$.*

*(Durante, Sarkoci and Sempi, 2009)*

The proof is based on ergodic theory and uses the following facts:

- the existence of suitable weakly mixing transformations in $\mathcal{T}_{\mathrm{p}}$;
- the following Lemma by Walters (1982).

## Lemma

*Let $(\Omega, \mathcal{F}, \nu)$ be a measure space. Let $T \colon \Omega \to \Omega$ be a weakly mixing transformation. Then there exists a subset $D$ of $\mathbb{Z}_+$ of density zero such that*

$$\lim_{\substack{n\to\infty \\ n\notin D}} \int_\Omega (f \circ T^n)(x) g(x)\, d\nu = \int_\Omega f(x)\, d\nu \int_\Omega g(x)\, d\nu$$

*for all real functions $f$ and $g$ in $L^2(\nu)$.*

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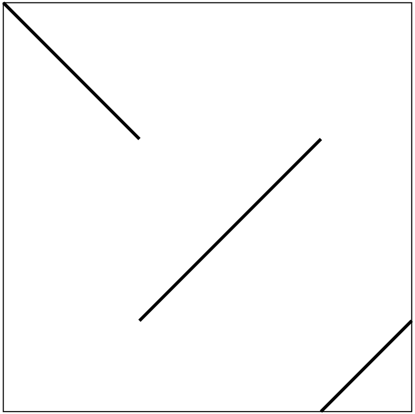
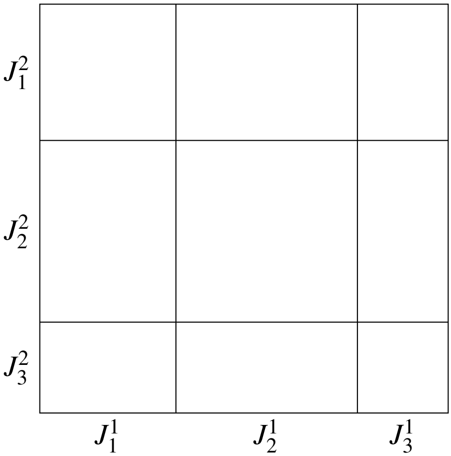
# Another look at shuffles of Min

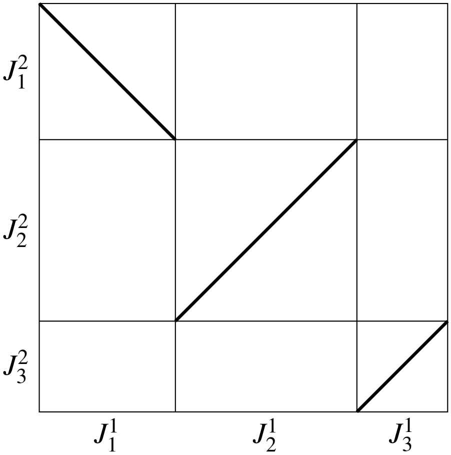


# Another look at shuffles of Min



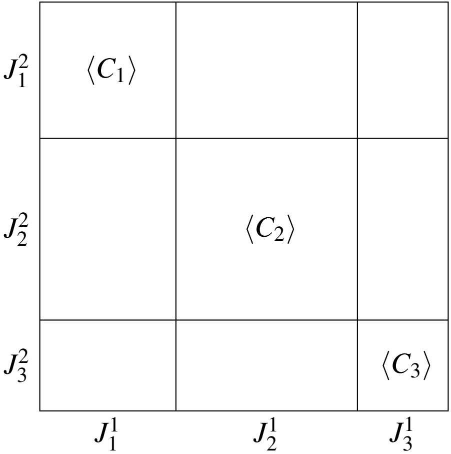

Consider two suitable partitions $\mathcal{J}^1 = (J_i^1)$ and $\mathcal{J}^2 = (J_i^2)$ such that $J_i^1 \times J_i^2$ is a square.

# Another look at shuffles of Min




Plug an affine transformation of the probability mass of $M_2$ or $W_2$ in the square $J_i^1 \times J_i^2$.

# Visual definition of shuffle of copulas




For $i = 1, 2, 3$, take $C_i \in \mathcal{C}_2$ and plug an affine transformation of the measure induced by $C_i$ in the square $J_i^1 \times J_i^2$. The resulting measure is doubly stochastic.

# Definition of shuffle of copulas

Let $\mathcal{J}^1, \dots, \mathcal{J}^d$ be systems of closed and non-empty intervals of $\mathbb{I}$,

$$\mathcal{J}^i = \left(J_n^i = \left[a_n^i, b_n^i\right]\right)_{n\in N}$$

such that:

(S1) $N$ represents a finite or countable index set, i.e. $N = \{0, 1, \dots, \widetilde{n}\}$ or $N = \mathbb{Z}_+$;

(S2) for every $i \in \{1, 2, \dots, d\}$ and $n, m \in N$, $n \neq m$, $J_n^i$ and $J_m^i$ have at most one endpoint in common;

(S3) for every $i \in \{1, 2, \dots, d\}$, $\sum_{n\in N} \lambda(J_n^i) = 1$;

(S4) for every $n \in N$, $\lambda(J_n^1) = \lambda(J_n^2) = \cdots = \lambda(J_n^d)$.

Let $(C_n)_{n\in N}$ be a system of $d$–copulas.

# Definition of a shuffle of copulas

For all $\mathbf{u} \in \mathbb{I}^d$,

$$C(\mathbf{u}) = \sum_{n \in N} \lambda(J_n^1)\, C_n\left(\frac{u_1 - a_n^1}{\lambda(J_n^1)}, \dots, \frac{u_d - a_n^d}{\lambda(J_n^1)}\right)$$

is a copula, called shuffling copula related to the partitions $\left(\mathfrak{J}^i\right)_{i=1}^d$ and the system $(C_n)_{n \in N}$.

(Durante, Fernández–Sánchez, 2010)

The multivariate shuffling copula is a patchwork copula since it can be also viewed as a modification of the probability mass of $M_d$ in some suitable boxes.

# Probabilistic interpretation of a shuffle of copulas

Assume that:

- for every $n \in N$, $\mathbf{U}^n = \left(U_1^n, \dots, U_d^n\right) \sim C_n$
- $Z$ is a discrete random variable assuming values in $N$ such that, for every $n \in N$, $P(Z = n) = \lambda(J_n^1)$.

For every $n \in N$, consider the random vector

$$\mathbf{V}^n = (V_1^n, \dots, V_d^n) = \left(\lambda(J_n^1)U_1^n + a_n^1, \dots, \lambda(J_n^1)U_d^n + a_n^d\right).$$

Finally, let us consider the random vector $\mathbf{W}$ given by

$$\mathbf{W} = \sum_{n \in N} \sigma_n(Z)\mathbf{V}^n,$$

where, for every $n \in N$, $\sigma_n(x) = 1$ if $x = n$, $\sigma_n(x) = 0$ otherwise.
Then $\mathbf{W}$ is distributed according to a shuffle of $(C_n)_n$.

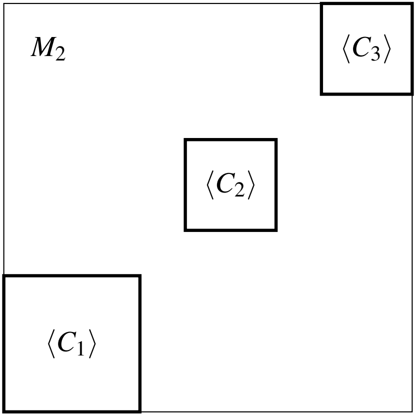
# Ordinal sums of bivariate copulas

Let $(]a_i, b_i[)_{i\in\mathcal{J}}$ be a family of non-empty, pairwise disjoint, open subintervals of $\mathbb{I}$ and let $(T_i)_{i\in\mathcal{J}}$ be a family of copulas. Then the function

$$C(u_1,u_2) = \begin{cases} a_i + (b_i - a_i)\, C_i\left(\dfrac{u_1 - a_i}{b_i - a_i}, \dfrac{u_2 - a_i}{b_i - a_i}\right) & \text{if } (u_1,u_2) \in \,]a_i, b_i[^2\,, \\ \min(u_1,u_2) & \text{otherwise,} \end{cases}$$

is a copula, called ordinal sum of the summands $(\langle a_i, b_i, C_i\rangle)_{i\in\mathcal{J}}$.

# Geometric interpretation of ordinal sums



# Multivariate ordinal sum of copulas

An ordinal sum of multivariate copulas can be introduced in the following way:

Let $L$ be a finite or countable set, let $([a_k, b_k])_{k\in L}$ be a system of sub–intervals of $\mathbb{I}$, and let $(C_k)_{k\in L}$ be a system in $\mathcal{C}_d$.

Then the ordinal sum $C$ of $(C_k)_{k\in L}$ wrt the family of intervals $([a_k, b_k])_{k\in L}$ is the $d$–copula defined, for all $\mathbf{u} \in \mathbb{I}^d$ by

$$C(\mathbf{u}) = \begin{cases} a_k + (b_k - a_k)\, C_k\left(\frac{\min\{u_1,b_k\}-a_k}{b_k-a_k}, \dots, \frac{\min\{u_d,b_k\}-a_k}{b_k-a_k}\right), \\ \qquad\qquad \text{if } \min\{u_1, u_2, \dots, u_d\} \in \,]a_k, b_k[\, \text{ for some } k \in L, \\ \min\{u_1, u_2, \dots, u_d\}, \qquad \text{elsewhere.} \end{cases}$$

(Mesiar and Sempi, 2010; Jaworski and Rychlik, 2008)

# Approximation by means of shuffles

## Proposition

*Fix a d–copula C. Then any copula can be approximated uniformly by means of shuffles of the system $(C_n)_{n\in N}$, where $C_n = B$ for every n.*

*(Durante, Fernández–Sánchez, 2010)*

## Corollary

*Any copula can be approximated uniformly by means of shuffles of copulas that are absolutely continuous.*

*(Durante, Fernández–Sánchez, 2010)*

# Remark

The approximation of copulas by means of shuffles strongly depends on the topology over $\mathcal{C}_d$.

## Example

Consider the topology induced on $\mathcal{C}_d$ by the Sobolev norm

$$\|C\| = \left( \int_{\mathbb{I}^d} |\nabla C(u)|^2 du \right)^{1/2}.$$

If $C$ is a shuffle of Min, then $\|C\| = 1$; but $\|\Pi_2\| = 2/3$.

In the Sobolev norm, shuffles of Min do not approximate $\Pi_2$.

(Siburg and Stoimenov, 2008)

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# Definition

Let $C$ and $C_B$ be $d$–dimensional copulas and let $B = [\mathbf{a}, \mathbf{b}]$ be a non-empty box contained in $\mathbb{I}^d$ such that $\mu_C(B) = \alpha > 0$. The function $C^* : \mathbb{I}^d \to \mathbb{I}$ given by

$$C^*(\mathbf{u}) = \mu_C\left([\mathbf{0}, \mathbf{u}] \cap B^c\right) + \alpha\, C_B\left(\widetilde{F}_B^1(u_1), \dots, \widetilde{F}_B^d(u_d)\right)$$

is a copula, where

$$\widetilde{F}_B^i(x_i) = \frac{1}{\alpha}\mu_C\left([a_1, b_1] \times \cdots \times [a_{i-1}, b_{i-1}] \times [a_i, x_i] \times [a_{i+1}, b_{i+1}] \times \cdots \times [a_d, b_d]\right),$$

for every $x_i \in [a_i, b_i]$.

The copula $C^*$ is called patchwork of $(B, C_B)$ into $C$ and it is denoted by the symbol $C^* = \langle B, C_B \rangle^C$.

(Durante, Fernández–Sánchez and Sempi, 2013)

# Illustration: tail modification

Consider the patchwork $C^* = \langle B, C_B \rangle^C$, where $B = [\mathbf{a}, \mathbf{1}]$ given by

$$C^*(\mathbf{u}) = \mu_C([\mathbf{0}, \mathbf{u}] \setminus [\mathbf{a}, \mathbf{1}]) + \alpha\, C_B\left(\widetilde{F}_B^1(u_1), \dots, \widetilde{F}_B^d(u_d)\right),$$

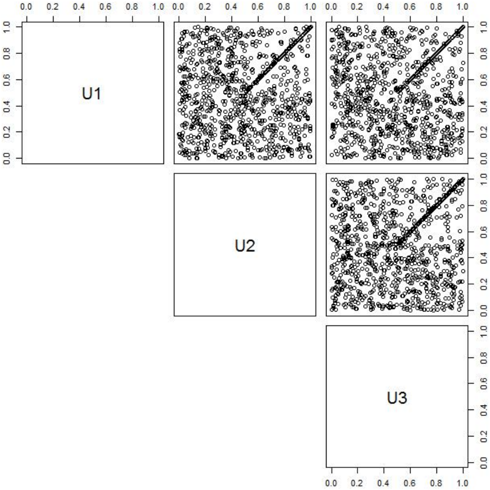
where $\alpha = V_C(B)$ and, for every $i \in \{1, \dots, d\}$, one has

$$\widetilde{F}_B^i(x_i) = \frac{1}{\alpha} V_C([a_1, 1] \times \dots [a_i, x_i] \times \dots \times [a_d, 1]) .$$

An algorithm for generating a random sample from $C^*$ goes as follows.

1. Generate $\mathbf{u}$ from the copula $C$.
2. Generate $\mathbf{v}$ from the copula $C_B$.
3. For $i = 1, 2, \dots, d$ set $w_i = (\widetilde{F}_B^i)^{-1}(v_i)$.
4. If $\mathbf{u} \in B$, then return $\mathbf{w}$.
   Otherwise, return $\mathbf{u}$.

# Illustration: tail modification




Random sample of 1000 realizations from the copula $\langle B, C_B \rangle^C$ where $B = [0.5, 1]^3$, $C$ is the independence copula and $C_B$ is the comonotone copula.

# Illustration: worst-case VaR scenario

Given the vector of losses $(L_1, L_2)$ having fixed marginals, the worst-possible *VaR* (at level $\alpha$) for the sum $L^+ = L_1 + L_2$ (write: $\overline{VaR}_\alpha(L^+)$) is given when $(L_1, L_2)$ is coupled by $\langle[\alpha, 1]^2, W_2\rangle^{M_2}$.

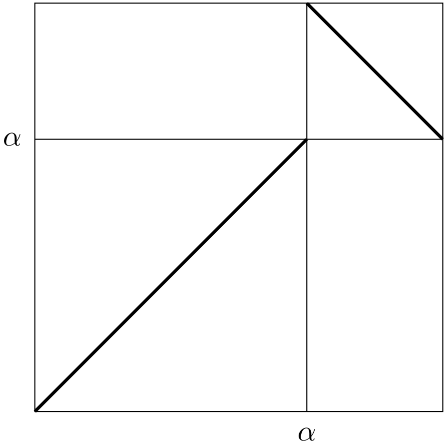
Interestingly, it is well known that

$$VaR_\alpha(L_1) + VaR_\alpha(L_2) \leq \overline{VaR}_\alpha(L^+),$$

where the left hand side corresponds to the comonotone case.

Now, for any copula $C$, the patchwork $C^* = \langle[\alpha, 1]^2, C\rangle^{M_2}$ can be used in order to interpolate between the comonotonic scenario and the worst-case scenario for $VaR_\alpha(L^+)$.

# Illustration: worst-case VaR scenario



# Illustration: worst-case VaR scenario

| | $\tau = 1$ | $\tau = 0.50$ | $\tau = 0.00$ | $\tau = -0.50$ | $\tau = -1$ |
|---|---|---|---|---|---|
| $VaR_\alpha(L_1^{C^*}, L_2^{C^*})$ | 2.5631 | 2.5663 | 2.5749 | 3.0340 | 3.2897 |

Numerical approximation of $VaR_{0.90}(L_1^{C^*}, L_2^{C^*})$ where $L_1, L_2, \sim N(0,1)$, $C^* = \langle [0.90, 1]^2, C \rangle^{M_2}$ for a Clayton copula $C$ with Kendall's $\tau$ equal to the indicated value. Results based on $10^6$ simulation from the given copula.

# Illustration: upper comonotonicity

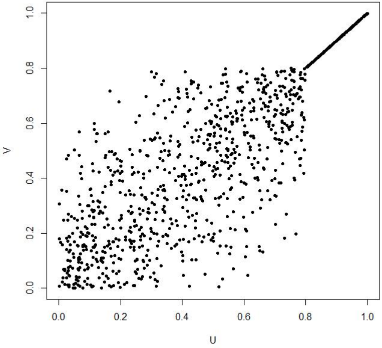
Let $C_B$ be an arbitrary $d$–copula and let $M_d$ be the comonotone copula. Consider the patchwork of copulas of type $\langle B, C_B \rangle^{M_d}$, where $B = [\mathbf{0}, \mathbf{a}]$. Then

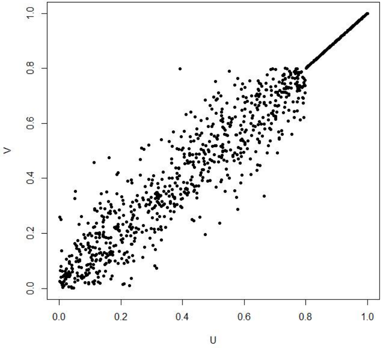
$$
\begin{aligned}
C^*(\mathbf{u}) =& \mu_C\left([\mathbf{0}, \mathbf{u}] \cap B^c\right) \\
& + \alpha\, C_B\left(\frac{\min\{a_1, \dots, u_1, \dots, a_d\}}{\alpha}, \dots, \frac{\min\{a_1, \dots, u_d, \dots, a_d\}}{\alpha}\right).
\end{aligned}
$$

Notice that in this case, $\alpha = V_{M_d}(B) = \min\{a_1, \dots, a_d\}$.

When all the components of $\mathbf{a}$ are equal to $a$, constructions of copulas of this type describe upper comonotonic random vectors (Cheung, 2009).

# Illustration: upper comonotonicity




Random sample of 1000 realizations from the copula $\langle B, C_B \rangle^{M_2}$ where $B = [0, 0.8]^2$, $C_B$ is a Frank with Kendall's tau equal to: 0.5 (left) and 0.75 (right).

# Definition: the general case

Let $C$ and $C_{B_s}$ $(s \in S)$ be $d$–dimensional copulas and let $B_s$ $(s \in S)$ be a system (finite or countable) of non–empty boxes contained in $\mathbb{I}^d$ such that $\lambda_d(B_s \cap B_{s'}) = 0$ if $s \neq s'$. Let $B = \cup_{s\in S} B_s$. Then the function $C^* : \mathbb{I}^d \to \mathbb{I}$ given by

$$C^*(\mathbf{u}) = \mu_C\left([\mathbf{0}, \mathbf{u}] \cap B^c\right) + \sum_{s\in S} \alpha_s\, C_s\left(\widetilde{F}^1_{B_s}(u_1), \ldots, \widetilde{F}^d_{B_s}(u_d)\right),$$

is a copula.

The copula $C^*$ is called patchwork of $(B_s, C_{B_s})_{s\in S}$ into $C$ and it is denoted by the symbol $C^* = \langle B_s, C_{B_s}\rangle^C_{s\in S}$.

# The patchwork construction

Define $\mathcal{C}_d^S := \{(C_s)_{s\in S}\}$, where, for every $s \in S$, $C_s$ is a $d$–copula. Let $C \in \mathcal{C}_d$ and let $B_s$ $(s \in S)$ be a system of $d$–boxes.
Formally, the patchwork is defined as the mapping $T_C : \mathcal{C}_d^S \to \mathcal{C}_d$ given by

$$T_C\left((C_s)_{s\in S}\right) := \langle B_s, C_s \rangle_{s\in S}^{C} .$$

$T_C$ is uniformly continuous when $\mathcal{C}_d$ is endowed by the uniform distance $d_\infty$, and $\mathcal{C}_d^S$ by the distance

$$d_S\left((C_s)_{s\in S}, (\widetilde{C}_s)_{s\in S}\right) := \sup_{s\in S} d_\infty(C_s, \widetilde{C}_s) .$$

(Durante, Fernández–Sánchez and Sempi, 2013)

# The patchwork construction and approximation

Then patchwork constructions are the general setting when an approximation of a copula $C$ can be considered, by using, for instance, the following scheme:

- Divide the domain $\mathbb{I}^d$ in several boxes $\mathbb{I}^d = \cup_i B_i$ such that each $B_i$ is sufficiently small.
- For any $B_i$ approximate $\mu_{C|B_i}$ with another convenient measure $\mu_i$.
- Join all the measure $\mu_i$'s by obtaining a suitable $d$–fold stochastic measure $\mu = \sum_i \mu_i$.

Depending on the expression of $\mu_i$'s several approximations of copulas can be obtained (Bernstein copulas, checkerboard copulas, etc.).

# Outline


1. Introduction

2. Complete dependence and shuffles of Min

3. A measure-theoretic notion of shuffle of Min

4. Multivariate shuffles of copulas

5. The patchwork construction

6. Conclusions

# To summarize

We have revisited the notion of patchwork copulas by using measure-theoretic techniques.

The introduced construction principle:

- works in any dimension
- induces specific tail behaviour in the dependence structure
- can be used in the approximation of copulas

# Bibliography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# Questions? Comments?

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