Some nonparametric tests for copulas

Ivan Kojadinovic

Université de Pau et des Pays de l'Adour, France

Based on joint work with Axel Bücher, Mark Holmes, Johan Segers and Jun Yan.

Grenoble, November 2013

Outline

Outline

Introduction I

- The copula-based modeling of multivariate distributions is finding extensive applications in many fields such as **hydrology** (Salvadori, Michele, Kottegoda, and Rosso, 2007), **finance and insurance** (McNeil, Frey, and Embrechts, 2005) or **actuarial sciences** (Frees and Valdez, 1998).
- Let $\mathbf{X}$ be a d-dimensional random vector with continuous marginal cumulative distribution functions (c.d.f.s) $F_1, \ldots, F_d$. From the work of Sklar (1959), the c.d.f. F of $\mathbf{X}$ can be written in a unique way as

$$F(\mathbf{x}) = C\{F_1(x_1), \ldots, F_d(x_d)\}, \qquad \mathbf{x} \in \mathbb{R}^d,$$

where the function $C : [0,1]^d \to [0,1]$ is a **copula**.

Copula

A d-dimensional copula is a c.d.f. on $[0,1]^d$ with standard uniform marginal c.d.f.s.

Introduction II

- Assume that C and $F_1, \dots, F_d$ are unknown and let $\mathbf{X}_1, \dots, \mathbf{X}_n$ be drawn from a strictly stationary sequence of continuous d-dimensional random vectors with c.d.f. F.
- For any $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, d\}$, denote by $R_{ij}^{1:n}$ the rank of X_{ij} among $X_{1j}, \dots, X_{nj}$ and let $\hat{U}_{ij}^{1:n} = R_{ij}^{1:n}/n$.
- The random vectors $\hat{\mathbf{U}}_i^{1:n} = (\hat{U}_{i1}^{1:n}, \dots, \hat{U}_{id}^{1:n})$, $i \in \{1, \dots, n\}$, are often referred to as **pseudo-observations** from the copula C, and a natural nonparametric estimator of C is the **empirical copula** of $\mathbf{X}_1, \dots, \mathbf{X}_n$, i.e.,

$$C_{1:n}(\mathbf{u}) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(\hat{\mathbf{U}}_i^{1:n} \leq \mathbf{u}), \qquad \mathbf{u} \in [0,1]^d.$$

- The empirical copula plays a key role in most nonparametric inference procedures on C. The asymptotics of these procedures typically follow from the asymptotics of the **empirical copula process**.

Introduction III

- One of the key issues is: given data $\mathbf{X}_1, \dots, \mathbf{X}_n$, **which parametric copula family should be used?**
- There are many families of copulas: **Archimedean**, **elliptical**, **extreme-value**, etc (see e.g. Joe, 1997; Nelsen, 2006).
- Any family, if exchangeable, can be made **asymmetrical** using Khoudraji's device (Khoudraji, 1995; Genest, Ghoudi, and Rivest, 1998; Liebscher, 2008).
- To guide the choice of a parametric copula family, nonparametric tests based on the **empirical copula** can be used.
- Most of the tests to be mentioned are implemented in the `copula` R package but the computation of their approximate p-value is valid only when $\mathbf{X}_1, \dots, \mathbf{X}_n$ are i.i.d. random vectors.
- The extension to **strongly mixing observations** is possible as we shall see.

Outline

Some nonparametric tests I

- A first natural step would be to test for **independence**: Is the unknown copula C significantly different from the **independence copula** $\Pi(\mathbf{u}) = \Pi_{j=1}^d u_j$, $\mathbf{u} \in [0,1]^d$?
- Such a test can be based on the empirical process

$$\sqrt{n}\{C_{1:n}(\mathbf{u}) - \Pi_{j=1}^d u_j\}$$

and a natural test statistic is

$$n \int_{[0,1]^d} \{C_{1:n}(\mathbf{u}) - \Pi_{j=1}^d u_j\}^2 \mathrm{d}\mathbf{u}.$$

- More details can be found for instance in Genest and Rémillard (2004); K and Holmes (2009); Quessy (2010).
- A second step would for instance be to test for **exchangeability**.

Some nonparametric tests II

- In the bivariate case, such a test can be based on the empirical process

$$\sqrt{n}\{C_{1:n}(u_1, u_2) - C_{1:n}(u_2, u_1)\}.$$

- More details can be found in Genest, Nešlehová, and Quessy (2012).
- One could also test for **extreme-value dependence** (Ben Ghorbal, Genest, and Nešlehová, 2009; K and Yan, 2010; K, Segers, and Yan, 2011a).
- One avenue consists of testing if C is **max-stable**, i.e., if

$$C(\mathbf{u}) = \{C(u_1^{1/r}, \dots, u_d^{1/r})\}^r, \qquad \forall\, \mathbf{u} \in [0,1]^d, \qquad \forall\, r > 0.$$

- Several empirical processes could be used. One of these is

$$\mathbb{E}_{r,n}(\mathbf{u}) = \sqrt{n}\left[\{C_{1:n}(\mathbf{u}^{1/r})\}^r - C_{1:n}(\mathbf{u})\right], \qquad \mathbf{u} \in [0,1]^d.$$

- More details can be found in K, Segers, and Yan (2011a).

Some nonparametric tests III

- More powerful tests can be obtained by focusing on the underlying Pickands dependence function.
- Fix $d = 2$. Bivariate extreme-value copulas are characterized by a convex function $A : [0,1] \to [1/2, 1]$ satisfying $\max(t, 1-t) \leq A(t) \leq 1$ for all $t \in [0,1]$, and can be represented as

$$C(u_1, u_2) = \exp\left[\log(u_1 u_2) A\left\{\frac{\log(u_2)}{\log(u_1 u_2)}\right\}\right], \quad (u_1, u_2) \in (0,1]^2 \setminus \{(1,1)\}.$$

The function A is commonly referred to as the **Pickands dependence function** (Pickands, 1981).

Some nonparametric tests IV

- Given an estimator A_n of A (see Gudendorf and Segers, 2012, for nice multivariate estimators), tests for extreme-value dependence can then naturally be based on the process

$$\sqrt{n}\left(C_{1:n}(u_1, u_2) - \exp\left[\log(u_1 u_2) A_n\left\{\frac{\log(u_2)}{\log(u_1 u_2)}\right\}\right]\right).$$

- More details can be found in K and Yan (2010) and in the PhD thesis of Gordon Gudendorf who extended the approach to arbitrary dimensions.
- Nonparametric tests of Archimedeanity for bivariate copulas can be found in Bücher et al. (2012).
- ...

Outline

A class of goodness-of-fit test I

- Let $\mathcal{C} = \{C_\theta : \theta \in \mathcal{O}\}$ be a chosen parametric copula family, where $\mathcal{O}$ is an open subset of $\mathbb{R}^p$ for some integer $p > 0$.
- We wish to test

$$H_0 : C \in \mathcal{C} \qquad \text{against} \qquad H_1 : C \notin \mathcal{C}.$$

- A relatively large number of testing procedures have been proposed in the literature. See Berg (2009); Genest, Rémillard, and Beaudoin (2009) for reviews and Monte Carlo studies.
- These authors advocate the use of **"blanket tests"** (no strategic choice of smoothing parameter, weight function, kernel, window, etc).
- One approach that appears to perform well consists of comparing $C_{1:n}$ with an estimation C_{θ_n} of C obtained assuming that $H_0 : C \in \mathcal{C}_0$ holds.

A class of goodness-of-fit test II

- The quantity θ_n is an estimator of θ computed from the pseudo-observations $\hat{\mathbf{U}}_1^{1:n}, \dots, \hat{\mathbf{U}}_n^{1:n}$.
- This amounts to using the empirical process

$$\sqrt{n}\{C_{1:n}(\mathbf{u}) - C_{\theta_n}(\mathbf{u})\}, \qquad u, v \in [0,1].$$

- See for instance Genest, Rémillard, and Beaudoin (2009) or K, Yan, and Holmes (2011b) for more details.

Outline

Ok but what about change-point detection?

- A broad class of nonparametric tests for change-point detection particularly sensitive to changes in the copula can be derived from the process

$$\mathbb{D}_n(s,\mathbf{u}) = \sqrt{n}\,\lambda_n(0,s)\lambda_n(s,1)\{C_{1:\lfloor ns \rfloor}(\mathbf{u}) - C_{\lfloor ns \rfloor+1:n}(\mathbf{u})\}, \quad (s,\mathbf{u}) \in [0,1]^{d+1},$$

  where $\lambda_n(s,t) = (\lfloor nt \rfloor - \lfloor ns \rfloor)/n$ and with the convention that $C_{k:k-1}(\mathbf{u}) = 0$ for all $\mathbf{u} \in [0,1]^d$ and all $k \in \{1,\ldots,n\}$.
- The above definition is a mere transposition to the copula context of the "classical construction" adopted for instance in Csörgő and Horváth (1997, Section 2.6).

Outline

The sequential empirical copula process I

- Under the corresponding null hypotheses, all of the previously mentioned empirical processes can be rewritten in terms of the **two-sided sequential empirical copula process**. It is defined, for any $(s,t) \in \Delta = \{(s,t) \in [0,1]^2 : s \leq t\}$ and $\mathbf{u} \in [0,1]^d$, by

$$\mathbb{C}_n(s,t,\mathbf{u}) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \left\{ \mathbf{1}(\hat{\mathbf{U}}_i^{\lfloor ns \rfloor+1:\lfloor nt \rfloor} \leq u) - C(\mathbf{u}) \right\}.$$

- The latter process can be rewritten in terms of the empirical copula $C_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}$ of the sample $\mathbf{X}_{\lfloor ns \rfloor+1}, \dots, \mathbf{X}_{\lfloor nt \rfloor}$ as

$$\mathbb{C}_n(s,t,\mathbf{u}) = \sqrt{n}\lambda_n(s,t)\{C_{\lfloor ns \rfloor+1:\lfloor nt \rfloor}(\mathbf{u}) - C(\mathbf{u})\},$$

  where $\lambda_n(s,t) = (\lfloor nt \rfloor - \lfloor ns \rfloor)/n$ and with the convention that $C_{k:k-1}(\mathbf{u}) = 0$ for all $\mathbf{u} \in [0,1]^d$ and all $k \in \{1, \dots, n\}$.
- The quantity $\mathbb{C}_n(0,1,\cdot,\cdot)$ is the "standard" empirical copula process.

The sequential empirical copula process II

- For instance, under the null hypothesis of no change in the distribution, the process for detecting changes in the dependence structure can be simply rewritten as

$$\mathbb{D}_n(s,\mathbf{u}) = \lambda_n(s,1)\mathbb{C}_n(0,s,\mathbf{u}) + \lambda_n(0,s)\mathbb{C}_n(s,1,\mathbf{u}), \quad (s,\mathbf{u}) \in [0,1]^{d+1}.$$

- Similarly, all of the previously mentioned processes can be rewritten in terms of $\mathbb{C}_n$ under the corresponding null hypotheses.
- **It is therefore crucial to obtain the weak limit of $\mathbb{C}_n$.**
- Let $\mathbf{U}_1,\dots,\mathbf{U}_n$ be the unobservable sample obtained from $\mathbf{X}_1,\dots,\mathbf{X}_n$ by the probability integral transforms $U_{ij} = F_j(X_{ij})$. The corresponding sequential empirical process is then defined as

$$\tilde{\mathbb{B}}_n(s,\mathbf{u}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor ns \rfloor} \{\mathbf{1}(\mathbf{U}_i \leq \mathbf{u}) - C(\mathbf{u})\}, \qquad (s,\mathbf{u}) \in [0,1]^{d+1}.$$

The sequential empirical copula process III

Condition (1)

The unobservable sample $\mathbf{U}_1, \dots, \mathbf{U}_n$ *is drawn from a strictly stationary sequence* $(\mathbf{U}_i)_{i \in \mathbb{Z}}$ *such that* $\tilde{\mathbb{B}}_n$ *converges weakly in* $\ell^\infty([0,1]^{d+1})$ *to a tight centered Gaussian process* $\mathbb{B}_C$ *concentrated on*

$$\{\alpha^\star \in \mathcal{C}([0,1]^{d+1}) : \alpha^\star(s, \mathbf{u}) = 0 \text{ if one of the components of } (s, \mathbf{u}) \text{ is 0, ar}$$
$$\alpha^\star(s, 1, \dots, 1) = 0 \text{ for all } s \in (0,1]\}.$$

Condition (2)

For any $j \in \{1, \dots, d\}$, *the partial derivatives* $\dot{C}_j = \partial C / \partial u_j$ *exist and are continuous on* $V_j = \{\mathbf{u} \in [0,1]^d : u_j \in (0,1)\}$.

The sequential empirical copula process IV

Theorem (Bücher and K (2013))

Assume that the unobservable sample $\mathbf{U}_1, \dots, \mathbf{U}_n$ *satisfies Condition 1 and that* C *satisfies Condition 2. Then,*
$\sup_{(s,t,\mathbf{u}) \in \Delta \times [0,1]^d} \left| \mathbb{C}_n(s,t,\mathbf{u}) - \tilde{\mathbb{C}}_n(s,t,\mathbf{u}) \right| \overset{\Pr}{\to} 0$, *where*

$$\tilde{\mathbb{C}}_n(s,t,\mathbf{u}) = \{\tilde{\mathbb{B}}_n(t,\mathbf{u}) - \tilde{\mathbb{B}}_n(s,\mathbf{u})\} - \sum_{j=1}^d \dot{C}_j(\mathbf{u}) \{\tilde{\mathbb{B}}_n(t,\mathbf{u}^{(j)}) - \tilde{\mathbb{B}}_n(s,\mathbf{u}^{(j)})\}.$$

Consequently, $\mathbb{C}_n \rightsquigarrow \mathbb{C}_C$ *in* $\ell^\infty(\Delta \times [0,1]^d)$*, where, for* $(s,t,\mathbf{u}) \in \Delta \times [0,1]^d$,

$$\mathbb{C}_C(s,t,\mathbf{u}) = \{\mathbb{B}_C(t,\mathbf{u}) - \mathbb{B}_C(s,\mathbf{u})\} - \sum_{j=1}^d \dot{C}_j(\mathbf{u}) \{\mathbb{B}_C(t,\mathbf{u}^{(j)}) - \mathbb{B}_C(s,\mathbf{u}^{(j)})\},$$

where $\mathbb{B}_C$ *is the weak limit of* $\tilde{\mathbb{B}}_n$*.*

The sequential empirical copula process V

- The following corollary is an immediate consequence of the strong approximation result of Dhompongsa (1984), which states that, if $\mathbf{U}_1, \dots, \mathbf{U}_n$ is drawn from a strictly stationary sequence $(\mathbf{U}_i)_{i \in \mathbb{Z}}$ whose strong mixing coefficients satisfy $\alpha_r = O(r^{-a})$, $a > 2 + d$, then $\tilde{\mathbb{B}}_n \rightsquigarrow \mathbb{B}_C$ in $\ell^\infty([0,1]^{d+1})$, that is, $\mathbf{U}_1, \dots, \mathbf{U}_n$ satisfies Condition 1.

Corollary

Assume that $\mathbf{X}_1, \dots, \mathbf{X}_n$ is drawn from a strictly stationary sequence $(\mathbf{X}_i)_{i \in \mathbb{Z}}$ whose strong mixing coefficients satisfy $\alpha_r = O(r^{-a})$, $a > 2 + d$. Then, provided C satisfies Condition 2,

$$\sup_{(s,t,\mathbf{u}) \in \Delta \times [0,1]^d} \left| \mathbb{C}_n(s,t,\mathbf{u}) - \tilde{\mathbb{C}}_n(s,t,\mathbf{u}) \right| \overset{\Pr}{\to} 0.$$

Outline

A dependent multiplier bootstrap for $\mathbb{C}_n$ I

- The weak limit of $\mathbb{C}_n$ is unwiedly. We need a resampling scheme to carry out the previously mentioned tests.
- Starting from the seminal work of Bühlmann (1993, Section 3.3), Bücher and K (2013) have studied a dependent multiplier bootstrap for $\mathbb{C}_n$ which extends the multiplier bootstrap of Rémillard and Scaillet (2009) to the sequential and strongly mixing setting.
- The key idea in Bühlmann (1993) is to replace i.i.d. multipliers by suitably serially dependent multipliers that will capture the serial dependence in the data.
- We say that a sequence of random variables $(\xi_{i,n})_{i\in\mathbb{Z}}$ is a **dependent multiplier sequence** if:

(M1) The sequence $(\xi_{i,n})_{i\in\mathbb{Z}}$ is strictly stationary with $\mathrm{E}(\xi_{0,n}) = 0$, $\mathrm{E}(\xi_{0,n}^2) = 1$ and $\mathrm{E}(|\xi_{0,n}|^\nu) < \infty$ for all $\nu \geq 1$, and is independent of the available sample $\mathbf{X}_1, \ldots, \mathbf{X}_n$.

A dependent multiplier bootstrap for $\mathbb{C}_n$ II

(M2) There exists a sequence $\ell_n \to \infty$ of strictly positive constants such that $\ell_n = o(n)$ and the sequence $(\xi_{i,n})_{i \in \mathbb{Z}}$ is ℓ_n-dependent, i.e., $\xi_{i,n}$ is independent of $\xi_{i+h,n}$ for all $h > \ell_n$ and $i \in \mathbb{N}$.

(M3) There exists a function $\varphi : \mathbb{R} \to [0,1]$, symmetric around 0, continuous at 0, satisfying $\varphi(0) = 1$ and $\varphi(x) = 0$ for all $|x| > 1$ such that $\mathrm{E}(\xi_{0,n}\xi_{h,n}) = \varphi(h/\ell_n)$ for all $h \in \mathbb{Z}$.

- Let M be a large integer and let $(\xi_{i,n}^{(1)})_{i \in \mathbb{Z}}, \dots, (\xi_{i,n}^{(M)})_{i \in \mathbb{Z}}$ be M independent copies of the same **dependent multiplier sequence**.
- For any $m \in \{1, \dots, M\}$ and any $(s, \mathbf{u}) \in [0,1]^{d+1}$, let

$$\hat{\mathbb{B}}_n^{(m)}(s, \mathbf{u}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor ns \rfloor} \xi_{i,n}^{(m)} \{\mathbf{1}(\hat{\mathbf{U}}_i^{1:n} \leq \mathbf{u}) - C_{1:n}(\mathbf{u})\}.$$

- To define "almost" independent copies of $\mathbb{C}_n$ for large n in the spirit of Rémillard and Scaillet (2009), we additionally need to estimate the partial derivatives $\dot{C}_j$.

A dependent multiplier bootstrap for $\mathbb{C}_n$ III

- As we continue, we consider estimators $\dot{C}_{j,n}$ of $\dot{C}_j$ satisfying the following condition put forward in Segers (2012):

Condition (3)

For any $j \in \{1,\dots,d\}$, there exists a constant $K > 0$ such that $|\dot{C}_{j,n}(\mathbf{u})| \leq K$ for all $n \geq 1$ and $\mathbf{u} \in [0,1]^d$, and, for any $\delta \in (0,1/2)$,

$$\sup_{\substack{\mathbf{u}\in[0,1]^d \\ u_j\in[\delta,1-\delta]}} |\dot{C}_{j,n}(\mathbf{u}) - \dot{C}_j(\mathbf{u})| \xrightarrow{\Pr} 0.$$

A dependent multiplier bootstrap for $\mathbb{C}_n$ IV

- We can now define empirical processes that can be fully computed and that, under appropriate conditions, can be regarded as "almost" independent copies of $\mathbb{C}_n$ for large n. For any $m \in \{1, \dots, M\}$ and $(s, t, \mathbf{u}) \in \Delta \times [0,1]^d$, let

$$\hat{\mathbb{C}}_n^{(m)}(s,t,\mathbf{u}) = \{\hat{\mathbb{B}}_n^{(m)}(t,\mathbf{u}) - \hat{\mathbb{B}}_n^{(m)}(s,\mathbf{u})\} - \sum_{j=1}^{d} \dot{C}_{j,n}(\mathbf{u})\{\hat{\mathbb{B}}_n^{(m)}(t,\mathbf{u}^{(j)}) - \hat{\mathbb{B}}_n^{(m)}(s,\mathbf{u}^{(j)})\}.$$

A dependent multiplier bootstrap for $\mathbb{C}_n$ V

Proposition (Unconditional dependent multiplier bootstrap for $\mathbb{C}_n$)

Assume that $\ell_n = O(n^{1/2-\varepsilon})$ for some $0 < \varepsilon < 1/2$ and that $\mathbf{X}_1, \dots, \mathbf{X}_n$ is drawn from a strictly stationary sequence $(\mathbf{X}_i)_{i \in \mathbb{Z}}$ whose strong mixing coefficients satisfy $\alpha_r = O(r^{-a})$, $a > 3 + 3d/2$. Then, under Conditions 2 and 3,

$$\left(\mathbb{C}_n, \hat{\mathbb{C}}_n^{(1)}, \dots, \hat{\mathbb{C}}_n^{(M)}\right) \rightsquigarrow \left(\mathbb{C}_C, \mathbb{C}_C^{(1)}, \dots, \mathbb{C}_C^{(M)}\right)$$

in $\{\ell^\infty(\Delta \times [0,1]^d)\}^{M+1}$, where $\mathbb{C}_C$ is the weak limit of the two-sided sequential empirical copula process $\mathbb{C}_n$ and $\mathbb{C}_C^{(1)}, \dots, \mathbb{C}_C^{(M)}$ are independent copies of $\mathbb{C}_C$.

- A simple possible choice is to estimate the partial derivatives $\dot{C}_j$ by finite-differences as proposed by Rémillard and Scaillet (2009).

A dependent multiplier bootstrap for $\mathbb{C}_n$ VI

- **Back to change-point detection :** To be able to compute approximate p-values for statistics derived from $\mathbb{D}_n$, it is then natural to define the processes

$$\hat{\mathbb{D}}_n^{(m)}(s, \mathbf{u}) = \lambda_n(s, 1)\hat{\mathbb{C}}_n^{(m)}(0, s, \mathbf{u}) + \lambda_n(0, s)\hat{\mathbb{C}}_n^{(m)}(s, 1, \mathbf{u}),$$

  $m \in \{1, \ldots, M\}$, which could be thought of as "almost" independent copies of $\mathbb{D}_n$ under the null hypothesis of no change in the distribution.

- Under the null and the conditions of the previous Proposition, we immediately obtain from the continuous mapping theorem that $\mathbb{D}_n, \hat{\mathbb{D}}_n^{(1)}, \ldots, \hat{\mathbb{D}}_n^{(M)}$ weakly converge jointly to independent copies of the same limit.
- The latter result is the key step for establishing that "multiplier" tests based on $\mathbb{D}_n$ hold their level asymptotically.

A dependent multiplier bootstrap for $\mathbb{C}_n$ VII

- Note that the bandwidth parameter ℓ_n defined in Assumption (M2) plays a role similar to that of the **block length in the block bootstrap** of Künsch (1989).
- Its value has therefore a crucial influence on the finite-sample performance of the dependent multiplier bootstrap.
- To make testing procedures automatic, we have extended the approach of Politis and White (2004) to the empirical process setting and suggest an estimator of ℓ_n.

Bibliography I

M. Ben Ghorbal, C. Genest, and J. Nešlehová. On the test of Ghoudi, Khoudraji, and Rivest for extreme-value dependence. *The Canadian Journal of Statistics*, 37(4):534–552, 2009.

D. Berg. Copula goodness-of-fit testing: An overview and power comparison. *The European Journal of Finance*, 15:675–701, 2009.

A. Bücher and I. K. A dependent multiplier bootstrap for the sequential empirical copula process under strong mixing. *arXiv:1306.3930*, 2013.

A. Bücher, H. Dette, and S.Volgushev. A test for Archimedeanity in bivariate copula models. *Journal of Multivariate Analysis*, 110:121–132, 2012.

P. Bühlmann. *The blockwise bootstrap in time series and empirical processes*. PhD thesis, ETH Zürich, 1993. Diss. ETH No. 10354.

M. Csörgő and L. Horváth. *Limit theorems in change-point analysis*. Wiley Series in Probability and Statistics. John Wiley & Sons, Chichester, UK, 1997.

Bibliography II

S. Dhompongsa. A note of the almost sure approximation of the empirical process of weakly dependent random vectors. *Yokohama Mathematical Journal*, 32:113–121, 1984.

E.W. Frees and E.A. Valdez. Understanding relationships using copulas. *North American Actuarial Journal*, 2:1–25, 1998.

C. Genest and B. Rémillard. Tests of independence and randomness based on the empirical copula process. *Test*, 13(2):335–369, 2004.

C. Genest, K. Ghoudi, and L.-P. Rivest. Discussion of "Understanding relationships using copulas", by E. Frees and E. Valdez. *North American Actuarial Journal*, 3:143–149, 1998.

C. Genest, B. Rémillard, and D. Beaudoin. Goodness-of-fit tests for copulas: A review and a power study. *Insurance: Mathematics and Economics*, 44:199–213, 2009.

C. Genest, J. Nešlehová, and J.-F. Quessy. Tests of symmetry for bivariate copulas. *Annals of the Institute of Statistical Mathematics*, 64:811–834, 2012.

Bibliography III

G. Gudendorf and J. Segers. Nonparametric estimation of multivariate extreme-value copulas. *Journal of Statistical Planning and Inference*, 143:3073–3085, 2012.

H. Joe. *Multivariate models and dependence concepts*. Chapman and Hall, London, 1997.

I. K and M. Holmes. Tests of independence among continuous random vectors based on Cramér-von Mises functionals of the empirical copula process. *Journal of Multivariate Analysis*, 100(6):1137–1154, 2009.

I. K and J. Yan. Modeling multivariate distributions with continuous margins using the `copula` R package. *Journal of Statistical Software*, 34 (9):1–20, 2010.

I. K, J. Segers, and J. Yan. Large-sample tests of extreme-value dependence for multivariate copulas. *The Canadian Journal of Statistics*, 39(4):703–720, 2011a.

I. K, J. Yan, and M. Holmes. Fast large-sample goodness-of-fit for copulas. *Statistica Sinica*, 21(2):841–871, 2011b.

Bibliography IV

A. Khoudraji. *Contributions à l'étude des copules et à la modélisation des valeurs extrêmes bivariées*. PhD thesis, Université Laval, Québec, Canada, 1995.

H.R. Künsch. The jacknife and the bootstrap for general stationary observations. *Annals of Statistics*, 17(3):1217–1241, 1989.

E. Liebscher. Construction of asymmetric multivariate copulas. *Journal of Multivariate Analysis*, 99:2234–2250, 2008.

A.J. McNeil, R. Frey, and P. Embrechts. *Quantitative risk management*. Princeton University Press, New Jersey, 2005.

R.B. Nelsen. *An introduction to copulas*. Springer, New-York, 2006. Second edition.

J. Pickands. Multivariate extreme value distributions. With a discussion. Proceedings of the 43rd session of the Internatinal Statistical Institute. *Bull. Inst. Internat. Statist.*, 49:859–878, 894–902, 1981.

D.N. Politis and H. White. Automatic block-length selection for the dependent bootstrap. *Econometric Reviews*, 23(1):53–70, 2004.

Bibliography V

J.-F. Quessy. Applications and asymptotic power of marginal-free tests of stochastic vectorial independence. *Journal of Statistical Planning and Inference*, 140:3058–3075, 2010.

B. Rémillard and O. Scaillet. Testing for equality between two copulas. *Journal of Multivariate Analysis*, 100(3):377–386, 2009.

G. Salvadori, C. De Michele, N.T. Kottegoda, and R. Rosso. *Extremes in Nature: An Approach Using Copulas*. Water Science and Technology Library, Vol. 56. Springer, 2007.

J. Segers. Asymptotics of empirical copula processes under nonrestrictive smoothness assumptions. *Bernoulli*, 18:764–782, 2012.

A. Sklar. Fonctions de répartition à *n* dimensions et leurs marges. *Publications de l'Institut de Statistique de l'Université de Paris*, 8: 229–231, 1959.