Bayesian modelling of financial extremes

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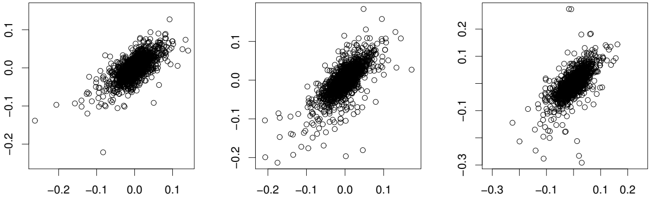
19/11/2013

Plan

Motivation

Some "stylized facts" about multivariate financial return data



- Scatterplots look elliptical.
- Distributions are heavy-tailed :

$$\mathrm{Pr}(X > x) \sim c_P / x^{\alpha_P}, \quad \mathrm{Pr}(X < -x) \sim c_L / x^{\alpha_L} \quad (x \to \infty).$$

- There is tail dependence in negative returns, i.e. in losses :

$$\lim_{u \downarrow 0} \mathrm{pr}(F_{X_1}(X_1) \le u \mid F_{X_2}(X_2) \le u) > 0.$$

("lower tail correlation")

Modelling financial data

Data are multivariate – often in relatively high dimension when we consider a portfolio (e.g. $D > 5$).

Due to the stylized facts, a t-copula seems an appropriate dependence model for financial data.

- Parameters are a correlation matrix and the degree of freedom.
- Parameters can be estimated with robust rank-based methods.

This means fitting a “global” model to the entire range of the data.

Question : Is extreme value behavior different from gobal behavior ?

Objectives in this talk

- characterize the tail dependence in **elliptical distributions** ;
- construction of the corresponding **limit distributions** :
  - **max-stable** for componentwise maxima ;
  - **Pareto** for threshold exceedances ;
- present an **efficient likelihood** with **partial censoring** ;
- **Bayesian inference** with a **nonparametric correlation structure** : application to loss data of 13 European stocks from the finance sector.

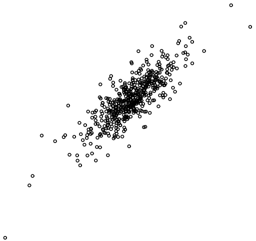
Elliptical distributions

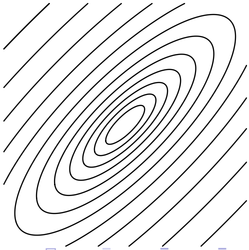
[Cambanis et al., 1981, Anderson and Fang, 1990], ...

Stochastic polar representation $\mathbf{X} \stackrel{d}{=} R\mathbf{A}\mathbf{U} + \mathbf{M}$

with

- a random radius $R \geq 0$;
- a dispersion matrix $\mathbf{\Sigma} = \mathbf{A}\mathbf{A}^T$, assumed to be invertible in the following;
- a random vector $\mathbf{U}$ uniform on $\{\mathbf{x} \mid \mathbf{x}^T\mathbf{x} = 1\}$, independent of R;
- a median vector $\mathbf{M}$.





Tail dependence in elliptical distributions

[Hult and Lindskog, 2002, Hashorva, 2006]

We have tail dependence in elliptical distributions if

$$\frac{\mathrm{Pr}(\mathrm{R} \geq \mathrm{tr})}{\mathrm{Pr}(\mathrm{R} \geq \mathrm{t})} \to r^{-\alpha}, \quad t \to \infty,$$

for all $r > 0$ with some fixed $\alpha > 0$.

Then for all components $j \in \{1, \ldots, D\}$, we observe

$$\frac{\mathrm{Pr}(\mathrm{X_j} \geq \mathrm{tx})}{\mathrm{Pr}(\mathrm{X_j} \geq \mathrm{t})} = \frac{\mathrm{Pr}(\mathrm{X_j} \leq -\mathrm{tx})}{\mathrm{Pr}(\mathrm{X_j} \leq -\mathrm{t})} \to x^{-\alpha}, \quad t \to \infty,$$

for all $x > 0$.

Multivariate elliptical t distributions are tail dependent with $\alpha = \mathrm{df}$.
The multivariate normal distribution is not tail dependent.

Limit distributions : the extremal elliptical model

Let $\mathbf{X}_i$ i.i.d. copies of a tail-dependent elliptical random vector $\mathbf{X}$.

- We get a **max-stable limit distribution** G for rescaled componentwise maxima :

$$\max_{i=1,\dots,n} a_n^{-1}\mathbf{X}_i \to \mathbf{Z} \sim G, \quad n \to \infty.$$

Max-stability : $\max_{i=1,\dots,n} \tilde{a}_n^{-1}\mathbf{Z}_i \overset{d}{=} \mathbf{Z}$ for i.i.d. copies $\mathbf{Z}_i$ of $\mathbf{Z}$.

- Equivalently, we get a **multivariate generalized Pareto limit distribution** H for rescaled threshold exceedances :

$$\mathbf{X}_+/u \mid \left(\max_{j=1,\dots,D} X_j \geq u\right) \to \mathbf{Y} \sim H, \quad u \to \infty.$$

Peaks-over-threshold stability : $[\mathbf{Y}/u \mid (\max_{j=1,\dots,D} Y_j \geq u)] \overset{d}{=} \mathbf{Y}$ for $u \geq 1$.

The limiting dependence structure

We can decouple the marginal behavior from the dependence structure.
Let $\mathbf{X}^*$ with $X_j^* = 1/\left[1 - F_{X_j}(X_j)\right]$ a standardized vector with standard Pareto margins.

We characterize the convergence of the dependence structure :

- maxima :

$$\max_{i=1,\ldots,n} n^{-1}\mathbf{X}_i^* \to \mathbf{Z}^* \sim G^*, \quad n \to \infty.$$

- threshold exceedances :

$$\mathbf{X}^*/u \mid \left(\max_{j=1,\ldots,D} X_j^* \geq u\right) \to \mathbf{Y}^* \sim H^*, \quad u \to \infty.$$

G^* and H^* are characterized by a so-called exponent measure η on $[0,\infty]^D$:

- $G^*(\mathbf{z}) = \exp\left\{-\eta\left([\mathbf{0},\mathbf{z}]^C\right)\right\}$;
- $H^*(\cdot) = \frac{\eta\left\{(\cdot)\cap[\mathbf{0},\mathbf{1}]^C\right\}}{\eta\left([\mathbf{0},\mathbf{1}]^C\right)}$;
- The index α is now a dependence parameter.

Construction of the limit distributions

[Opitz, 2013, Thibaud and Opitz, 2013]

Let $\mathbf{\Sigma} = \mathbf{A}\mathbf{A}^T$ the correlation matrix.

- Max-stable vectors $\mathbf{Z}^* \sim G^*$ are constructed as

$$\mathbf{Z}^* = [\mathbb{E}(U_{1,1})_+^\alpha]^{-1} \times \max_{i=1,2,\ldots} (\mathbf{A}\mathbf{U}_i)_+^\alpha / V_i$$

  with

  - $\mathbf{U}_i \sim \mathrm{Unif}\{\mathbf{x} : \mathbf{x}^{\mathrm{T}}\mathbf{x} = 1\}$ i.i.d. ;
  - $V_1 < V_2 < \cdots$ a unit rate Poisson process on $[0, \infty)$.

- Peaks-over-threshold stable vectors $\mathbf{Y}^* \sim H^*$ are constructed as

$$\mathbf{Y}^* = \left\{ R\,(\mathbf{A}\mathbf{U})_+^\alpha \mid \left[\|R\,(\mathbf{A}\mathbf{U})_+^\alpha\|_\infty \geq 1\right] \right\} \quad \text{with} \quad R \sim \mathrm{Par}(1).$$

The elliptical structure persists in the limit.
The exponent measure η has positive mass on $\{\mathbf{x} \in [0, \infty)^D \mid \min_j x_j = 0\}$.

The role of the shape parameter α

- α is a concentration parameter.
- Convergence to asymptotic independence when $\alpha \to \infty$ with fixed $\mathbf{\Sigma}$.
- Convergence to the Hüsler-Reiss dependence when $\alpha \to \infty$ and

$$\alpha \left[\mathbf{1}\mathbf{1}^T - \mathbf{\Sigma}(\alpha) \right]$$

has a nontrivial limit for $\alpha \to \infty$.
[Hüsler and Reiss, 1989, Hashorva, 2005, Nikoloulopoulos et al., 2009]

Inference

We intend to use max-stable and multivariate Pareto distribution for modelling multivariate extremal behavior.

Asymptotically, joint tail behavior is the same for max-stable and Pareto distributions. However, due to the componentwise maximum operation, we find stronger dependence in the max-stable tail.

It can be useful to decouple the marginal behavior from the dependence structure.

Univariate extreme value theory suggests marginal tail parameters μ_j (position), $\sigma_j > 0$ (scale) and ξ_j (shape). The literature on their estimation is vast.

We focus on estimation of the dependence structure.

Likelihood inference with partial censoring of exceedances

Let $V(\mathbf{u}) = \eta\left\{[\mathbf{0},\mathbf{u}]^C\right\}$ the dependence function. Then $\Pr(\mathbf{X}^* \not\leq \mathbf{u}) \approx \mathrm{V}(\mathbf{u})$.

Principle

- An event $\mathbf{X}^*$ is considered as extreme when a threshold vector $\mathbf{u}$ is exceeded in at least one component, i.e. when $\max_j X_j^*/u_j \geq 1$.
- Components $X_j^* < u_j$ are censored.
- Likelihood contribution of $\mathbf{X}^*$:
  - when none of the components exceeds its thresholds : $1 - V(\mathbf{u})$;
  - when w.l.o.g. components $X_1^* = x_1, ..., X_{j_0}^* = x_{j_0}$ are exceedances :

$$-\frac{\partial^{j_0}}{\partial x_1 \times \ldots \times \partial x_{j_0}} V(x_1, \ldots, x_{j_0}, u_{j_0+1}, ..., u_D).$$

The main difficulty typically lies in the calculation of partial derivatives when D is large (≥ 3).

Partial derivates can be calculated for the **extremal elliptical model even in large dimension** [Thibaud and Opitz, 2013].

The dependence function V

V can be expressed in terms of multivariate t probabilities :

$$V_{\alpha,\mathbf{\Sigma}}(\mathbf{u}) = \sum_{j=1}^{D} u_j^{-1} t_{\alpha+1} \left\{ (\mathbf{u}_{-j}/u_j)^{1/\alpha} \mid \mathbf{\Sigma}_{-j,j}, (\alpha+1)^{-1} \left(\mathbf{\Sigma}_{-j,-j} - \mathbf{\Sigma}_{-j,j} \mathbf{\Sigma}_{-j,j}^{T} \right) \right\}$$

[Nikoloulopoulos et al., 2009]

Algorithms beyond plain Monte-Carlo exist for the calculation of $V(\mathbf{u})$ with integer-valued α ([Genz and Bretz, 2009] ; package `mvtnorm` in `R`).

Modelling of dependence in extreme financial losses (1998-2013)

(**work in progress**)

- 13 stocks from **financial institutions** in the European union : Allianz, Banco Bilbao, BNP Paribas, AXA, Deutsche Bank, Generali, Société Générale, Ing Groep, Munich Re, Banco Santander, Unicredit, Commerzbank, Crédit Agricole ;
- we model dependence in $\mathrm{GARCH}(1,1)$-**residual daily losses**, considered as stationary ;
- We estimate a *t*-copula from all data : $\widehat{\mathrm{df}}_{\mathrm{glob}}$, $\hat{\boldsymbol{\Sigma}}_{\mathrm{glob}}$.
- We estimate the extremal elliptical model from exceedances in a Bayesian framework :
  - partial censoring ;
  - nonparametric correlation matrix ;
  - uniform prior for $\alpha \in \{1, 2, \ldots, 20\}$.

Interpretation of ellipticity :
the variable R in an elliptical random vector $R\mathbf{AU}$ captures systemic risk.

Robust estimation of t-copulae from all data

Cf. [Klüppelberg et al., 2007, Wang and Peng, 2013].

If $\boldsymbol{\Sigma}$ is the correlation matrix associated to an elliptical random vector $\mathbf{X}$, then Kendall's τ for two components X_{j_1}, X_{j_2} is

$$\tau_{j_1 j_2} = 2\pi^{-1}\mathrm{arcsin}(\sigma_{\mathrm{j_1 j_2}})$$

Hence we can define the estimator $\hat{\boldsymbol{\Sigma}}_{\mathrm{glob}}$ of $\boldsymbol{\Sigma}_{\mathrm{glob}}$ with entries

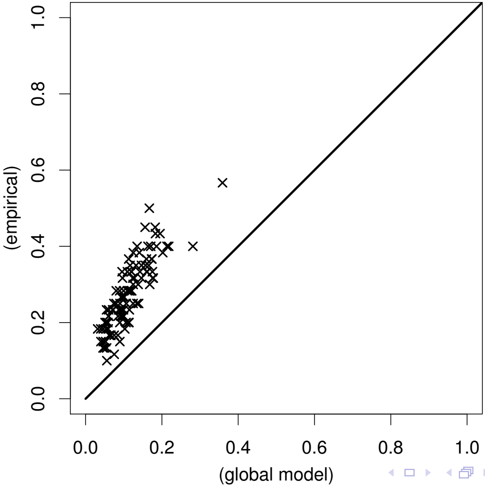
$$\hat{\sigma}_{j_1 j_2} = \sin(0.5\pi\hat{\tau}_{j_1 j_2}),$$

where $\hat{\tau}_{j_1 j_2}$ is the empirical version of Kendall's τ.

Given $\hat{\boldsymbol{\Sigma}}_{\mathrm{glob}}$, we can use the maximum-likelihood estimator $\widehat{\mathrm{df}}$ of the degree of freedom based on the empirical copula.

Exploring the data : global vs. extremal bivariate behavior

Empirical estimates of the tail correlation are based on 98%-exceedances.
The global t-copula model underestimates tail correlation (with $\alpha = \widehat{\mathrm{df}} = 13$).



Prior distribution for the correlation matrix $\boldsymbol{\Sigma}$

It is difficult to define non-informative priors on correlation matrices ; e.g., a uniform prior on all correlation matrices leads to $\sigma_{j_1 j_2}$ concentrated around 0 for $j_1 \neq j_2$.

Here we aim at centering $\boldsymbol{\Sigma}$ on $\hat{\boldsymbol{\Sigma}}_{\text{glob}}$:

- if $\mathbf{A}\mathbf{A}^T = \boldsymbol{\Sigma}$, the row vectors of $\mathbf{A}$ lie on the Euclidean unit sphere ;
- we can use von Mises-Fisher priors for the rows $\mathbf{a}_j$ of $\mathbf{A}$;
- we center $\mathbf{a}_j$ on $\hat{\mathbf{a}}_{\text{glob,j}}$, with prior density

$$c_\kappa \exp\left(\kappa\, \mathbf{a}_j^T \hat{\mathbf{a}}_{\text{glob,j}}\right);$$

- the prior of the concentration parameter κ is uniform over $[0, 100]$.

Note : The matrix root $\mathbf{A}$ is not unique, but this is not really a problem in the Bayesian context.

Modified partial censoring

We consider an event $\mathbf{X}^*$ as extreme when at least one of the individual losses exceeds its 99% quantile, i.e. $u_j = 100$ for $j = 1, \ldots, 13$ (219 extreme events).

For improved estimation efficiency, we apply partial censoring with a lower threshold $\tilde{u}_j = 10$, $j = 1, \ldots, 13$.

Hence the likelihood contribution of an observation $\mathbf{x}$ of $\mathbf{X}^*$ is as follows :

- when $\max x_j/u_j < 1$:
$$1 - V(\mathbf{u});$$
- when $\max x_j/u_j \geq 1$ and
w.l.o.g. components $x_1 \geq \tilde{u}_1$,...,$x_{j_0} \geq \tilde{u}_{j_0}$ are exceedances :
$$-\frac{\partial^{j_0}}{\partial x_1 \times \ldots \times \partial x_{j_0}} V(x_1, \ldots, x_{j_0}, \tilde{u}_{j_0+1}, \ldots, \tilde{u}_D).$$

A technical difficulty : different shape parameters α

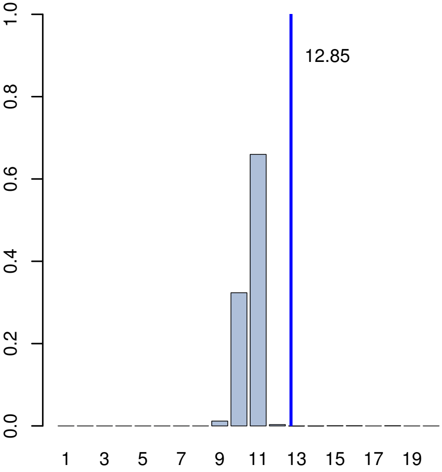
We use MCMC with the Metropolis-Hastings algorithm to simulate the posterior distribution.

If α increases, values $\sigma_{j_1 j_2}$ must also increase to maintain the same degree of tail dependence. Hence, finding a good Metropolis-Hastings proposal for $\mathbf{\Sigma}$ is complicated when $\alpha \in \{1, 2, \ldots, 20\}$ changes.

Instead, we propose :

- First, run MCMC chains independently for each value α.
- Then apply **Bayesian model averaging** with respect to the parameter α.

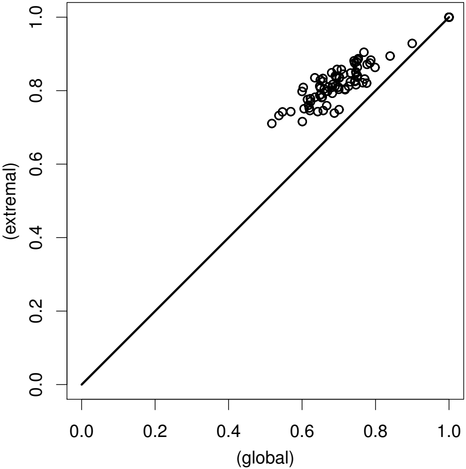
Results : Posterior distribution of α



Values concentrate below the estimation of the global model, leading to stronger dependence.

Results : Posterior mean correlation matrix

For the mode $\alpha = 11$.



We obtain higher correlation coefficients in the extremal model.

Motivation

Limit distributions for elliptical extremes

Tail dependence in elliptical distributions

Likelihood inference with partial censoring

Application : Bayesian modelling of financial extremes

Conclusions

Conclusions

- Since mixtures of dependence functions V are still dependence functions, the Bayesian model defines a valid dependence function.
- In financial data, global models may tend to underestimate tail dependence.
- Results without volatility filtering are similar, although estimated values of α are smaller ($\widehat{\mathrm{df}}_{\mathrm{glob}} = 5$).

Some perspectives

- Consider other censoring schemes that avoid the heavy calculation of $V(\mathbf{u})$.
- Refine the model to take account of extreme events that affect only a single component (operational risk).
- Model other than financial data :
  the comparison of the global t copula model and the extremal elliptical model for extremes can be useful in other contexts.

— Thank you —

Anderson, T. W. and Fang, K. T. (1990).
On the theory of multivariate elliptically contoured distributions and their applications.
In *Statistical Inference in Elliptically Contoured and Related Distributions*, pages 1–24. Allerton Press.

Cambanis, S., Huang, S., and Simons, G. (1981).
On the theory of elliptically contoured distributions.
J. Multivar. Anal., 11(3) :368–385.

Genz, A. and Bretz, F. (2009).
Computation of multivariate normal and t probabilities.
Springer.

Hashorva, E. (2005).
Elliptical triangular arrays in the max-domain of attraction of Hüsler-Reiss distribution.
Stat. & Probab. Lett., 72(2) :125–135.

Hashorva, E. (2006).
On the regular variation of elliptical random vectors.
Stat. & Probab. Lett., 76(14) :1427–1434.

Hult, H. and Lindskog, F. (2002).
Multivariate extremes, aggregation and dependence in elliptical distributions.
Adv. Appl. Probab., 34(3) :587–608.

Hüsler, J. and Reiss, R. D. (1989).
Maxima of normal random vectors : between independence and complete dependence.
Stat. & Probab. Lett., 7(4) :283–286.

Klüppelberg, C., Kuhn, G., and Peng, L. (2007).
Estimating the tail dependence function of an elliptical distribution.
Bernoulli, 13(1) :229–251.

Nikoloulopoulos, A. K., Joe, H., and Li, H. (2009).
Extreme value properties of multivariate t copulas.
Extremes, 12(2) :129–148.

Opitz, T. (2013).
Extremal t processes : Elliptical domain of attraction and a spectral representation.
J. Multivar. Anal.
To appear.

Thibaud, E. and Opitz, T. (2013).
Efficient inference and simulation for elliptical pareto processes.
In preparation.

Wang, R. and Peng, L. (2013).
Estimating bivariate t-copula via Kendall's tau.
Submitted paper.