On the estimation of the second order parameter in extreme-value theory

by

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Exemples of Applications

- Historically : meteorological events (rains, winds....).
- Finance : stock market crash and financial crises.
- Reinsurance.
- Aviation.
- Biology : Epidemiology, security food.

Main results on extreme value theory

Let $X_1, \ldots, X_n$ be a sequence of independent copies of a real random variable (r.v.) X with cumulative distribution function F. The order statistics associated to this sample are denoted by : $X_{1,n} \leq \cdots \leq X_{n,n}$.

Fisher-Tippett-Gnedenko theorem

Under some conditions of regularity on F, there exists a real parameter γ and two sequences $(a_n)_{n\geq 1} > 0$ and $(b_n)_{n\geq 1} \in \mathbb{R}$ such that for all $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\left(a_n^{-1}(X_{n,n} - b_n) \leq x\right) = EV_\gamma(x),$$

with $$EV_\gamma(x) = \begin{cases} \exp\left(-(1+\gamma x)_+^{-1/\gamma}\right) & \text{if } \gamma \neq 0, \\ \exp\left(-e^{-x}\right) & \text{if } \gamma = 0, \end{cases}$$

where $y_+ = \max(y, 0)$.

Definition

◇ The parameter γ is the tail index, the primary parameter of extreme events.
◇ EV_γ is called the extreme value distribution and F is then said to belong to the domain of attraction of EV_γ ($F \in DA(EV_\gamma)$).

Heavy-tailed model

◇ In statistics of Extremes, a model F is said to be heavy-tailed whenever, for some $\gamma > 0$, its survival function is of the forme :

$$1 - F(x) = x^{-1/\gamma}\ell_F(x) \Leftrightarrow \mathbb{U}(x) = x^{\gamma}\ell_{\mathbb{U}}(x)$$

where $\mathbb{U}(x) = \inf\{y : F(y) \geq 1 - 1/x\}$ and $\ell_\bullet(\cdot)$ is a slowly varying function *i.e.* $\ell_\bullet(\lambda x)/\ell_\bullet(x) \to 1$ as $x \to \infty$ for all $\lambda \geq 1$.
The present model is now often restated as the asumption of F regulary variation at infinity with index $-1/\gamma$ (denoted $1 - F(x) \in RV_{-1/\gamma}$.

Inference statistic of γ for heavy-tailed model

In Statistics of extremes, inference is often based

$\diamond W_{i,k} = (\log X_{n-i+1,n} - \log X_{n-k,n})$	the log-excesses

$\diamond Z_{i,k} = i\,(\log X_{n-i+1,n} - \log X_{n-i,n})$	the rescaled log-spacings

Exemple of γ estimator

- $H_{n,k} = \frac{1}{k}\sum_{i=1}^{k} W_{i,k} = \frac{1}{k}\sum_{i=1}^{k} Z_{i,k}$, Hill Estimator Hill (1975)
- $\mathcal{M}_{n,k}^{\alpha} = \frac{1}{k}\sum_{i=1}^{k} W_{i,k}^{\alpha}$, Moment Estmator, Deker *et al (1989)*
- $\mathcal{W}_{n,k} = \frac{1}{k}\sum_{i=1}^{k} W\left(\frac{i}{k+1}\right) Z_{i,k}$, Kernel estimator of γ, W is a positive Kernel function .
- $\mathcal{S}_{n,k} = \frac{1}{k}\sum_{j=1}^{k} G_{\alpha}\left(\frac{i}{k+1}\right) W_{i,k}^{\alpha}$, G_{α} is a positive weight function and α a positive real number.

Second Order Condition

The asymptotic distribution of estimators of γ is obtained under a second order condition.

Second Order Condition (S.O.C)

There exist a function $A(x) \to 0$ and a second order parameter $\rho \leq 0$ such that, for every $\lambda > 0$,

$$\lim_{x\to\infty} \frac{\log \mathbb{U}(\lambda x) - \log \mathbb{U}(x) - \gamma \log \lambda}{A(x)} = K_\rho(\lambda),$$

where $K_\rho(\lambda) = \int_1^\lambda u^{\rho-1} du$.

Remarks

- $|A|$ is regularly varying with index ρ.
- If ρ decreases, the rate of convergence of $\log \mathbb{U}(tx) - \log \mathbb{U}(t)$ to $\gamma \log$ increases.
- ρ controls the bias of the estimators of γ.
- A third order condition is needed to deal with the asymptotic distribution of ρ estimators.

Third Order Condition

Third Order Condition (T.O.C)

There exist functions $A(x) \to 0$ and $B(x) \to 0$, a second order parameter $\rho \leq 0$ and a third order parameter $\beta \leq 0$ such that, for every $\lambda > 0$,

$$\lim_{x\to\infty} \frac{(\log \mathbb{U}(\lambda x) - \log \mathbb{U}(x) - \gamma \log \lambda)/A(x) - K_\rho(\lambda)}{B(x)} = L_{\rho,\beta}(\lambda),$$

where

$$L_{(\rho,\beta)}(\lambda) = \int_1^\lambda s^{\rho-1} \int_1^s u^{\beta-1} duds,$$

and the functions $|A|$ and $|B|$ are regularly varying with index ρ and β respectively.

Objective

Propose a new class of estimators for the second order parameter and to study their asymptotic properties.

New family of estimators for the second order parameter

The model

- $T_n = T_n(X_1, ..., X_n)$: a random vector in $\mathbb{R}^d$ drawn from the sample $X_1, \ldots, X_n$.
- The statistics can always be expanded as :

$$\omega_n^{-1}(T_n - \chi_n \mathbb{I}) \xrightarrow{\mathbb{P}} f(\rho)$$

where

- $\mathbb{I} = {}^t(1, \ldots, 1) \in \mathbb{R}^d$,
- χ_n and ω_n : random variables,
- $\xi_n \in \mathbb{R}^d$: a random vector,
- $f : \mathbb{R}^- \to \mathbb{R}^d$: a function continuously differentiable in a neighborhood of ρ (independent of γ).

General approach

Notations

$\psi : \mathbb{R}^d \to \mathbb{R}$ such that

◇ Invariance properties (Inv-prop)

$\psi(x + \lambda\mathbb{I}) = \psi(x)$ for all $x \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$,

$\psi(\lambda x) = \psi(x)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$,

◇ Regularity properties (Reg-prop)

ψ is continuously differentiable in a neighborhood of $f(\rho)$,

$\varphi := \psi \circ f$ is continuous in a neighborhood of ρ

◇ Bijection property (Bij-prop)

there exist $J_0 \subseteq \mathbb{R}^-$ and an open interval $J \subset \mathbb{R}$ such that φ is a bijection from J_0 to J.

The estimator

Clearly

- by the invariance and the regularity properties

  ◇ $\psi(\omega_n^{-1}(T_n - \chi_n \mathbb{I})) = \psi(T_n) \xrightarrow{\mathbb{P}} \psi(f(\rho))$.

  ◇ $Z_n = \psi(T_n) \approx \varphi(\rho)$.

- Under the bijection property, our family of estimators of the second order parameter is thus defined by :

$$\hat{\rho}_n = \varphi^{-1}(Z_n) \mathbb{1}\{Z_n \in J\}.$$

$\mathbb{1}_A$ is the indicator function of the set A

Asymptotic properties

Theorems

Suppose that Inv-prop, Reg-prop and Bij-prop hold then

- $\hat{\rho}_n \xrightarrow{\mathbb{P}} \rho$ as $n \to \infty$
- if there exists two a sequence $v_n \to \infty$, a function, $m : \mathbb{R}^- \to \mathbb{R}^d$ and a $d \times d$ matrix Σ such that

$$v_n(\omega_n^{-1}(T_n - \chi_n \mathbb{I}) - f(\rho)) \xrightarrow{d} \mathcal{N}_d(m(\rho), \Sigma).$$

then

$$v_n(\hat{\rho}_n - \rho) \xrightarrow{d} \mathcal{N}\left(\frac{m_\psi(\rho)}{\varphi'(\rho)}, \frac{\sigma_\psi^2(\rho)}{(\varphi'(\rho))^2}\right)$$ with,

◇ $\varphi'(\rho) = {}^t f'(\rho) \nabla\psi(f(\rho))$

◇ $m_\psi(\rho) = {}^t m(\rho) \nabla\psi(f(\rho))$

◇ $\sigma_\psi^2(\rho) = {}^t \nabla\psi(f(\rho))\, \Sigma\, \nabla\psi(f(\rho)).$

Link with existing estimators

Existing estimators

In the literature, at least two ways of estimating the second order parameter can be found :

- Estimators based on rescaled log-spacings, $j(\log X_{n-j+1} - \log X_{n-j})$:

  ⇒ Goegebeur *et al.*, *(JSPI, 2010)*.
- Estimators based on log-spacings, $\log X_{n-j+1} - \log X_{n-k}$:

  ⇒ Gomes*et al.* ,*(Extremes, 2002)*,

  ⇒ Fraga-Alves *et al.*, *(Portugaliae Mathematica, 2003)*,

  ⇒ Ciuperca and Mercadier , *(Extremes, 2010)*.

Link with existing estimators

1. Estimators based on rescaled log-spacings : $j(\log X_{n-j+1} - \log X_{n-j})$

$$R_k(\tau) = \frac{1}{k}\sum_{j=1}^{k} H_\tau\left(\frac{j}{k+1}\right) j \log \frac{X_{n-j+1,n}}{X_{n-j,n}},$$

- H_τ is a kernel function integrating to one.
- This statistic is used for instance by Beirlant *et al.*, *(Extremes, 1999)* to estimate the extreme value index γ and by Goegebeur *et al.* *(JSPI, 2010)* to estimate the second order parameter ρ.
- They proved asymptotic normality of these estimators under a technical condition on the kernel, denoted by **(C1)** hereafter.

Links with existing estimators

Link with our framework

Suppose the third order condition and **(C1)** hold. If the sequence k satisfies

$$k \to \infty, \ n/k \to \infty, \ k^{1/2}A(n/k) \to \infty,$$

$$k^{1/2}A^2(n/k) \to \lambda_A \text{ and } k^{1/2}A(n/k)B(n/k) \to \lambda_B,$$

then the random vector

$$T_n^{(R)} := \left((R_k(\tau_i)/\gamma)^{\theta_i}, \ i = 1, \dots, d \right),$$

satisfies the model i.e. $\omega_n^{-1}(T_n^{(R)} - \chi_n \mathbb{I}) \xrightarrow{\mathbb{P}} f^{(R)}(\rho)$ with $\chi_n = 1$,

$\omega_n = A(n/k)/\gamma(1 + o_{\mathbb{P}}(1))$,

$$f^{(R)}(\rho) = \left(\theta_i \int_0^1 H_{\tau_i}(u) u^{-\rho} du, \ i = 1, \dots, d \right)$$

Link with existing estimators

Choice of the kernel and of the function ψ

- $d = 8$, $\psi_\delta : \mathcal{D} \mapsto \mathbb{R} \setminus \{0\}$

$$\psi_\delta(x_1, \dots, x_8) = \widetilde{\psi}_\delta(x_1 - x_2, x_3 - x_4, x_5 - x_6, x_7 - x_8), \text{ where } \delta \geq 0$$

$$\mathcal{D} = \{(x_1, \dots, x_8) \in \mathbb{R}^8;\ x_1 \neq x_2,\ x_3 \neq x_4,\ \text{and } (x_5 - x_6)(x_7 - x_8) > 0\}.$$

$$\widetilde{\psi}_\delta : \mathbb{R}^4 \mapsto \mathbb{R} \text{ textttis given by} : \widetilde{\psi}_\delta(y_1, \dots, y_4) = \frac{y_1}{y_2}\left(\frac{y_4}{y_3}\right)^\delta.$$

- $H_\tau(u) = (\tau + 1)u^\tau$, the statistic $T_n^{(R)}$ depends on 16 parameters $\{(\theta_i, \tau_i) \in (0, \infty)^2,\ i = 1, \dots, 8\}$.
- Let $\tilde{\theta} = (\tilde{\theta}_1, \dots, \tilde{\theta}_4) \in (0, \infty)^4$ with $\tilde{\theta}_3 \neq \tilde{\theta}_4$ and consider
  - ◇ $\{\theta_i = \tilde{\theta}_{\lceil i/2 \rceil},\ i = 1, \dots, 8\}$ with $\delta = (\tilde{\theta}_1 - \tilde{\theta}_2)/(\tilde{\theta}_3 - \tilde{\theta}_4)$ with $\lceil x \rceil = \inf\{n \in \mathbb{N} | x \leq n\}$.
  - ◇ $\tau_1 < \tau_2 \leq \tau_3 < \tau_4$, $\tau_5 < \tau_6 \leq \tau_7 < \tau_8$
- $T_n^{(R)}$ involves 12 free parameters. $Z_n^{(R)} = \psi_\delta(T_n^{(R)})$ and $\varphi^{(R)} = \psi_\delta \circ f^{(R)}$.

Link with existing estimators

Our contributions

- We can thus define the following family of estimators :

$$\hat{\rho}_n^{(R)} = \varphi^{-1}(Z_n^{(R)})\mathbb{1}\{Z_n^{(R)} \in J\}.$$

- New estimators of ρ (explicit or not), with Consistency and Asymptotic normality (consequence of ours theorems)
- Examples of explicit estimators
  - ◇ $\delta = 1$ *i.e.* $\tilde{\theta}_1 - \tilde{\theta}_2 = \tilde{\theta}_3 - \tilde{\theta}_4$. The rv $Z_n^{(R)}$ is denoted by $Z_{n,1}^{(R)}$.

$$\hat{\rho}_{n,1}^{(R)} = \frac{\tau_5\omega(1,\tilde{\theta}) - \tau_1 Z_{n,1}^{(R)}}{\omega(1,\tilde{\theta}) - Z_{n,1}^{(R)}}\mathbb{1}\{Z_{n,1}^{(R)} \in \omega(1,\tilde{\theta}) \bullet (1, \widetilde{\psi}_1(\tau_4,\tau_1,\tau_4,\tau_5))\}.$$

  - ◇ $\delta = 0$ *i.e.* $\tilde{\theta}_1 = \tilde{\theta}_2$. The rv $Z_n^{(R)}$ is thus denoted by $Z_{n,2}^{(R)}$:

$$\hat{\rho}_{n,2}^{(R)} = \frac{\tau_4\omega(0,\tilde{\theta}) - \tau_1 Z_{n,2}^{(R)}}{\omega(0,\tilde{\theta}) - Z_{n,2}^{(R)}}\mathbb{1}\{Z_{n,2}^{(R)} \in \omega(0,\tilde{\theta}) \bullet (1, \widetilde{\psi}_0(\tau_4,\tau_1,\tau_4,\tau_5))\}.$$

with $\omega(\delta,\tilde{\theta}) = \widetilde{\psi}_\delta(\tilde{\theta}_1(\tau_1-\tau_2), \tilde{\theta}_2(\tau_2-\tau_4), \tilde{\theta}_2(\tau_2-\tau_4), \tilde{\theta}_4(\tau_6-\tau_4))$

Link with existing estimators

2. Estimators based on log-spacings : $\log X_{n-j+1} - \log X_{n-k}$

$$S_k(\tau,\alpha) = \frac{1}{k}\sum_{j=1}^{k} G_{\tau,\alpha}\left(\frac{j}{k+1}\right)\left(\log \frac{X_{n-j+1,n}}{X_{n-k,n}}\right)^{\alpha}, \ \alpha > 0,$$

- $G_{\tau,\alpha}$ is a positive function.
- Dekkers *et al.* (*Annals of statistics, 1989*) proposed an estimator of γ based on this statistic in the particular case where $G_{\tau,\alpha}$ is constant.
- Ciuperca and Mercadier (*Extremes, 2010*) used the general statistic to estimate the parameters γ and ρ.
- They proved the asymptotic normality under a technical condition on the function $G_{\tau,\alpha}$, denoted by **(C2)** hereafter.

Link with existing estimators

Link with our framework

Suppose the third order condition, **(C2)** hold. If the sequence k satisfies

$$k \to \infty, \quad n/k \to \infty, \quad k^{1/2}A(n/k) \to \infty,$$

$$k^{1/2}A^2(n/k) \to \lambda_A \text{ and } k^{1/2}A(n/k)B(n/k) \to \lambda_B,$$

then the random vector

$$T_n^{(S)} = \left(\left(\frac{S_k(\tau_i, \alpha_i)}{\gamma^{\alpha_i}} \right)^{\theta_i}, \; i = 1, \dots, d \right)$$

satisfies the model i.e. $\omega_n^{-1}(T_n^{(S)} - \chi_n \mathbb{I}) \xrightarrow{\mathbb{P}} f^{(S)}(\rho)$ with $\chi_n = 1$,

$\omega_n = A(n/k)/\gamma(1 + o_{\mathbb{P}}(1))$, and

$$f^{(S)}(\rho) = \left(-\theta_i \alpha_i \int_0^1 G_{\tau_i, \alpha_i}(u)(\log(1/u))^{\alpha_i - 1} K_{-\rho}(u) du; \; i = 1, \dots, d \right),$$

Link with existing estimators

Choice of the weighted function and of the function ψ

- $d = 8$, the weighted function $G_{\tau,\alpha}$ is given by :

$$G_{\tau,\alpha}(u) = \frac{g_\tau(u)}{\int_0^1 g_\tau(x)(-\log x)^\alpha dx}, \ \tau \geq 0, \alpha > 0$$

  where $g_0(x) = 1$ and $g_\tau(x) = (\tau+1)(1-x^\tau)/\tau$ if $\tau > 0$.

- 24 free parameters : $\{(\theta_i, \tau_i, \alpha_i) \in (0,\infty)^3, \ i = 1, \ldots, 8\}$

- Let $(\zeta_1, \ldots, \zeta_4) \in (0,\infty)^4$ with $\zeta_3 \neq \zeta_4$, such that

  $\{\theta_i \alpha_i = \zeta_{\lceil i/2 \rceil}, \ i = 1, \ldots, 8\}$ with $\delta = (\zeta_1 - \zeta_2)/(\zeta_3 - \zeta_4)$.

  $\lceil x \rceil = \inf\{n \in \mathbb{N} | x \leq n\}$.

- $(\tau_{2i-1}, \alpha_{2i-1}) \neq (\tau_{2i}, \alpha_{2i})$, for $i = 1, \ldots, 4$ and,

  for $i = 3, 4$, $(\tau_{2i-1}, \alpha_{2i-1}) \leq (\tau_{2i}, \alpha_{2i})$

Link with existing estimators

Our contributions

- We can thus define the following family of estimators :

$$\hat{\rho}_n^{(S)} = \varphi^{-1}(Z_n^{(S)})\mathbb{1}\{Z_n^{(S)} \in J\}.$$

- New estimators of ρ (not necessarily explicit) with Consistency and Asymptotic normality.

- Exemples of explicit estimators

  ◇ $\delta = 0$ (i.e. $\zeta_1 = \zeta_2$), $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$, $\tau_1 = \alpha_5 = \alpha_8 = 2$

  $\tau_4 = \alpha_6 = 3$. Denoting by $Z_{n,4}^{(S)}$ the rv $Z_n^{(S)}$, the estimator of ρ is given by :

$$\hat{\rho}_{n,4}^{(S)} = \frac{6(Z_{n,4}^{(S)} + 2)}{3Z_{n,4}^{(S)} + 4}\mathbb{1}\{Z_{n,4}^{(S)} \in (-2, -4/3)\}.$$

  Consider ω^*, a function depends only on δ and ζ

Link with existing estimators

Exemples of explicit estimators

◇ $\delta = 0$, $\alpha_1 = \alpha_3 = \alpha_4 = 1$, $\tau_1 = \tau_4 = \alpha_2 = \alpha_5 = \alpha_8 = 2$ and $\alpha_6 = 3$.
Denoting by $Z_{n,5}^{(S)}$ the rv $Z_n^{(S)}$,

$$\hat{\rho}_{n,5}^{(S)} = \frac{2(Z_{n,5}^{(S)} - 2)}{2Z_{n,5}^{(S)} - 1} \mathbb{1}\{Z_{n,5}^{(S)} \in (1/2, 2)\}.$$

$\hat{\rho}_{n,4}^{(S)}$ and $\hat{\rho}_{n,5}^{(S)}$ are estimators Ciuperca and Mercadier , *(Extremes, 2010)*.

◇ $\delta = 1$, $\alpha_1 = \alpha_3 = \alpha_4 = 1$, $\tau_1 = \tau_4 = \alpha_2 = \alpha_5 = \alpha_8 = 2$ and $\alpha_6 = 3$

$Z_{n,6}^{(S)}$ the rv $Z_n^{(S)}$, a new estimator of ρ is given by ρ is given by

$$\hat{\rho}_{n,6}^{(S)} = \frac{3Z_{n,6}^{(S)} - 4\omega^*(1,\zeta)}{Z_{n,8}^{(S)} - \omega^*(1,\zeta)} \mathbb{1}\{Z_{n,6}^{(S)} \in \omega^*(1,\zeta) \bullet (1/2, 2/3)\}.$$

◇ $\delta = 1$ (i.e. $\zeta_1 - \zeta_2 = \zeta_3 - \zeta_4$), $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$, denoting by $Z_{n,7}^{(S)}$ the rv $Z_n^{(S)}$, a nother new estimator of ρ is given by :

$$\hat{\rho}_{n,7}^{(S)} = \frac{Z_{n,7}^{(S)} + 4/3\omega^*(1,\zeta)}{2Z_{n,7}^{(S)} + 4/3\omega^*(1,\zeta)} \mathbb{1}\{Z_{n,7}^{(S)} \in \omega^*(1,\zeta) \bullet (-4/3, -2/3)\}.$$

Asymptotic comparaison

Choice of the parameters

- The estimator based on rescaled log-spacings.
- $H_{\tau_i}(u) = (\tau_i + 1)u^{\tau_i}, \ i = 1, \ldots, 8.$
- $\tau_1, \ldots, \tau_8$ and $\tilde{\theta}_1, \tilde{\theta}_3, \tilde{\theta}_4$, taken as in Goegebeur *et al.*.
- $\tilde{\theta}_2 = \tilde{\theta}_1 + \delta(\tilde{\theta}_4 - \tilde{\theta}_3)$, $\delta \geq 0$.
- Choice of δ,
  1. minimization of the $\mathcal{AMSE}$ is impossible (depends on unknown parameters).
  2. We use an upper bound on the $\mathcal{AMSE}$: $\mathcal{AMSE} \leq c(\gamma, \lambda_A, \lambda_B)\pi(\delta, \rho, \beta)$.
  3. $\rho = \beta$, we minimize the function π in delta and the optimal δ as a function of ρ.

Asymptotic comparaison

Choice of δ

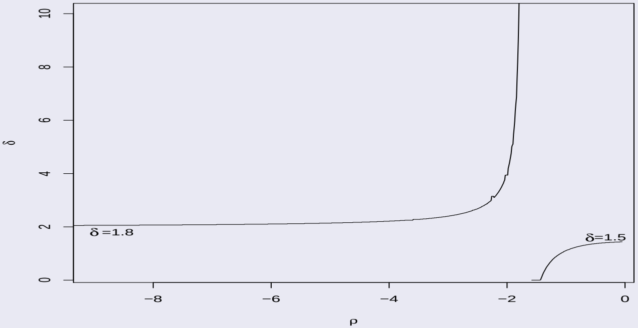

FIG.: Optimal δ as a function of ρ

- $\delta = 0,\ 1,\ 1.5,\ 1.8, +\infty$

Illustration on a Burr distribution

- Burr distribution, Burr(ζ,λ,η) : $1 - F(x) = (\zeta/(\zeta + x^\eta))^\lambda$, $x > 0$, $\zeta, \lambda, \eta > 0$, $\gamma = 1/\lambda\eta$ and $\rho = -1/\lambda$.
- The third order condition is hold with $\beta = \rho$, $A(x) = \gamma x^\rho/(1 - x^\rho)$ and $B(x) = \rho x^\rho/(1 - x^\rho)$.
- $n = 5000$, $\gamma = \zeta = 1$, $\eta = 1/\lambda$, $\rho = -\eta$ and $k = 1, \ldots, 4995$,

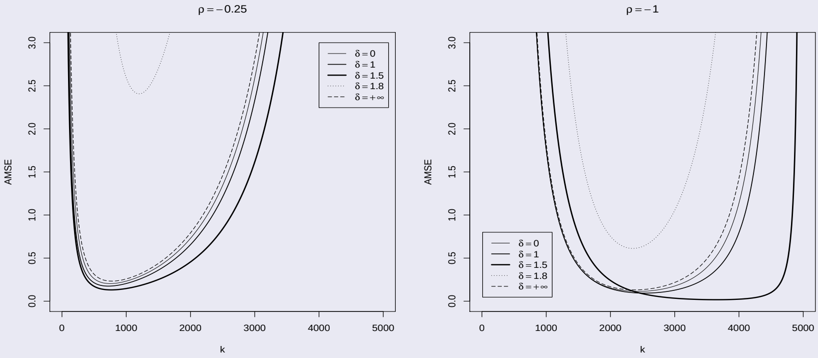

FIG.: Asymptotic mean squared error of $\hat{\rho}_n^{(R)}$, $\rho = -0.25; -1$

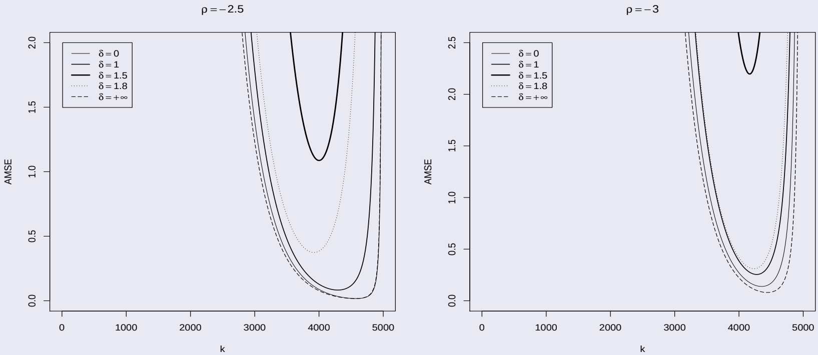

FIG.: Asymptotic mean squared error of $\hat{\rho}_n^{(R)}$, $\rho = -2.5; -3$

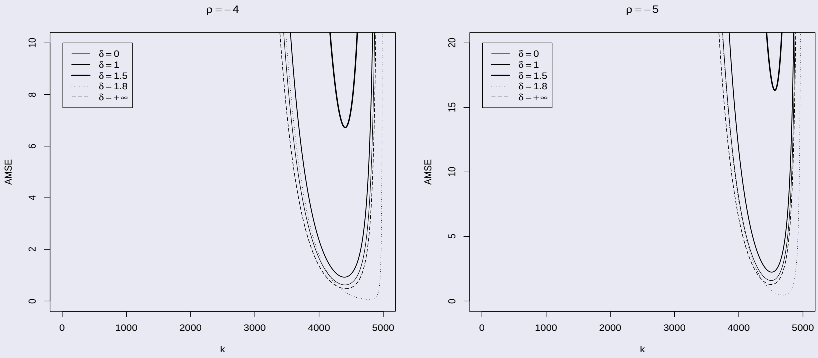

FIG.: Asymptotic mean squared error of $\hat{\rho}_n^{(R)}$, $\rho = -4; -5$

Concluding Remarks

- If $\rho \leq -4$, the smallest $\mathcal{AMSE}$ is obtained with $\delta = 1.8$.
- If $-3 \leq \rho \leq -2.5$, the best $\mathcal{AMSE}$ is given by $\delta = +\infty$.
- If $\rho \geq -1$, the smallest $\mathcal{AMSE}$ is given by $\delta = 1.5$.

◇ The values $\{1.5, 1.8, +\infty\}$ obtained by minimizing the function π are also of interest to minimize the asymptotic mean-squared error.

◇ More generally, the minimization of π should permit to determine optimal values for the parameters of any estimator of ρ.

AND THAT'S ALL . . .