

Finite-Sample Analysis in Reinforcement Learning

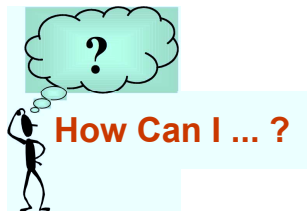
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Outline

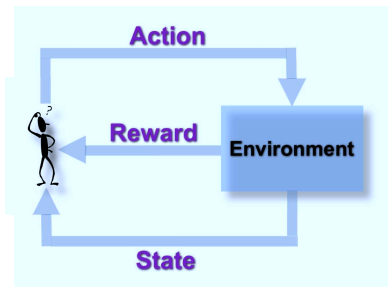
- 1 Introduction to RL and DP
- 2 Approximate Dynamic Programming (AVI & API)
- 3 How does Statistical Learning Theory come to the picture?
- 4 Error Propagation (AVI & API Error Propagation)
- 5 An AVI Algorithm (Fitted Q-Iteration)
 - FQI: error at each iteration
 - Final performance bound of FQI
- 6 An API Algorithm (Least-Squares Policy Iteration)
 - Error at each iteration (LSTD error)
 - Final performance bound of LSPI
- 7 Discussion

Sequential Decision-Making under Uncertainty



- Move around in the physical world (e.g. driving, navigation)
- Play and win a game
- Retrieve information over the web
- Medical diagnosis and treatment
- Maximize the throughput of a factory
- Optimize the performance of a rescue team

Reinforcement Learning (RL)



- **RL:** A class of learning problems in which an agent interacts with a dynamic, stochastic, and incompletely known environment
- **Goal:** Learn an action-selection strategy, or policy, to optimize some measure of its long-term performance
- **Interaction:** Modeled as a MDP or a POMDP

Markov Decision Process

MDP

- An MDP $\mathcal{M}$ is a tuple $\langle \mathcal{X}, \mathcal{A}, r, p, \gamma \rangle$.
- The state space $\mathcal{X}$ is a **bounded closed** subset of $\mathbb{R}^d$.
- The set of actions $\mathcal{A}$ is **finite** ($|\mathcal{A}| < \infty$).
- The reward function $r : \mathcal{X} \times \mathcal{A} \rightarrow \mathbb{R}$ is **bounded by $R_{\max}$** .
- The transition model $p(\cdot | x, a)$ is a **distribution** over $\mathcal{X}$.
- $\gamma \in (0, 1)$ is a **discount** factor.

- **Policy:** a mapping from states to actions $\pi(x) \in \mathcal{A}$

Value Function

For a policy π

- **Value function** $V^\pi : \mathcal{X} \rightarrow \mathbb{R}$

$$V^\pi(x) = \mathbb{E} \left[\sum_{t=0}^{\infty} \gamma^t r(X_t, \pi(X_t)) | X_0 = x \right]$$

- **Action-value function** $Q^\pi : \mathcal{X} \times \mathcal{A} \rightarrow \mathbb{R}$

$$Q^\pi(x, a) = \mathbb{E} \left[\sum_{t=0}^{\infty} \gamma^t r(X_t, A_t) | X_0 = x, A_0 = a \right]$$

Bellman Operator

- Bellman operator for policy π

$$\mathcal{T}^\pi : \mathcal{B}^V(\mathcal{X}; V_{\max}) \rightarrow \mathcal{B}^V(\mathcal{X}; V_{\max})$$

- V^π is the unique **fixed-point** of the Bellman operator

$$(\mathcal{T}^\pi V)(x) = r(x, \pi(x)) + \gamma \int_{\mathcal{X}} p(dy|x, \pi(x)) V(y)$$

- The action-value function Q^π is defined as

$$Q^\pi(x, a) = r(x, a) + \gamma \int_{\mathcal{X}} p(dy|x, a) V^\pi(y)$$

Optimal Value Function and Optimal Policy

- **Optimal value function**

$$V^*(x) = \sup_{\pi} V^{\pi}(x) \quad \forall x \in \mathcal{X}$$

- **Optimal action-value function**

$$Q^*(x, a) = \sup_{\pi} Q^{\pi}(x, a) \quad \forall x \in \mathcal{X}, \forall a \in \mathcal{A}$$

- A policy π is **optimal** if

$$V^{\pi}(x) = V^*(x) \quad \forall x \in \mathcal{X}$$

Bellman Optimality Operator

- Bellman optimality operator

$$\mathcal{T} : \mathcal{B}^V(\mathcal{X}; V_{\max}) \rightarrow \mathcal{B}^V(\mathcal{X}; V_{\max})$$

- V^* is the unique **fixed-point** of the Bellman optimality operator

$$(\mathcal{T}V)(x) = \max_{a \in \mathcal{A}} \left[r(x, a) + \gamma \int_{\mathcal{X}} p(dy|x, a) V(y) \right]$$

- Optimal action-value function Q^* is defined as

$$Q^*(x, a) = r(x, a) + \gamma \int_{\mathcal{X}} p(dy|x, a) V^*(y)$$

Properties of Bellman Operators

- **Monotonicity:** if $V_1 \leq V_2$ component-wise

$$\mathcal{T}^\pi V_1 \leq \mathcal{T}^\pi V_2 \quad \text{and} \quad \mathcal{T} V_1 \leq \mathcal{T} V_2$$

- **Max-Norm Contraction:** $\forall V_1, V_2 \in \mathcal{B}^V(\mathcal{X}; V_{\max})$

$$\|\mathcal{T}^\pi V_1 - \mathcal{T}^\pi V_2\|_\infty \leq \gamma \|V_1 - V_2\|_\infty$$

$$\|\mathcal{T} V_1 - \mathcal{T} V_2\|_\infty \leq \gamma \|V_1 - V_2\|_\infty$$

Dynamic Programming Algorithms

Value Iteration

- start with an arbitrary action-value function Q_0
- at each iteration k $Q_{k+1} = \mathcal{T}Q_k$

Convergence

- $\lim_{k \rightarrow \infty} V_k = V^*$.

$$\|V^* - V_{k+1}\|_{\infty} = \|\mathcal{T}V^* - \mathcal{T}V_k\|_{\infty} \leq \gamma \|V^* - V_k\|_{\infty} \leq \gamma^{k+1} \|V^* - V_0\|_{\infty} \xrightarrow{k \rightarrow \infty} 0$$

Dynamic Programming Algorithms

Policy Iteration

- start with an arbitrary policy π_0
- at each iteration k
 - **Policy Evaluation:** Compute Q^{π_k}
 - **Policy Improvement:** Compute the *greedy* policy w.r.t. Q^{π_k}

$$\pi_{k+1}(x) = (\mathcal{G}\pi_k)(x) = \arg \max_{a \in \mathcal{A}} Q^{\pi_k}(x, a)$$

Convergence

PI generates a sequence of policies with increasing performance ($V^{\pi_{k+1}} \geq V^{\pi_k}$) and stops after a finite number of iterations with the optimal policy π^* .

$$V^{\pi_k} = \mathcal{T}^{\pi_k} V^{\pi_k} \leq \mathcal{T} V^{\pi_k} = \mathcal{T}^{\pi_{k+1}} V^{\pi_k} \leq \lim_{n \rightarrow \infty} (\mathcal{T}^{\pi_{k+1}})^n V^{\pi_k} = V^{\pi_{k+1}}$$

Approximate Dynamic Programming

Approximate Dynamic Programming Algorithms

Value Iteration

- start with an arbitrary action-value function Q_0
- at each iteration k $Q_{k+1} = \mathcal{T}Q_k$

What if $Q_{k+1} \approx \mathcal{T}Q_k$?

$$\|Q^* - Q_{k+1}\| \stackrel{?}{\leq} \gamma \|Q^* - Q_k\|$$



Approximate Dynamic Programming Algorithms

Policy Iteration

- start with an arbitrary policy π_0
- at each iteration k
 - **Policy Evaluation:** Compute Q^{π_k}
 - **Policy Improvement:** Compute the *greedy* policy w.r.t. Q^{π_k}

$$\pi_{k+1}(x) = (\mathcal{G}\pi_k)(x) = \arg \max_{a \in \mathcal{A}} Q^{\pi_k}(x, a)$$

What if we cannot compute Q^{π_k} exactly? (Compute $\hat{Q}^{\pi_k} \approx Q^{\pi_k}$ instead)

$$\pi_{k+1}(x) = \arg \max_{a \in \mathcal{A}} \hat{Q}^{\pi_k}(x, a) \neq (\mathcal{G}\pi_k)(x) \longrightarrow V^{\pi_{k+1}} \overset{?}{\geq} V^{\pi_k}$$



Statistical Learning Theory in RL & ADP

Approximate Value Iteration (AVI)

$$Q_{k+1} \approx \mathcal{T}Q_k$$

- finding a function that best approximates $\mathcal{T}Q_k$ $Q = \min_f \|f - \mathcal{T}Q_k\|_\mu$

- only noisy observations of $\mathcal{T}Q_k$ are available $\hat{\mathcal{T}}Q_k$

Target Function = $\mathcal{T}Q_k$

Noisy Observation = $\hat{\mathcal{T}}Q_k$

- we minimize the **empirical error** $Q_{k+1} = \hat{Q} = \min_f \|f - \hat{\mathcal{T}}Q_k\|_{\hat{\mu}}$

with the target of minimizing the **true error** $Q = \min_f \|f - \mathcal{T}Q_k\|_\mu$

- Objective:** $\|\hat{Q} - \mathcal{T}Q_k\|_\mu \leq \underbrace{\|\hat{Q} - Q\|_\mu}_{\text{estimation error}} + \underbrace{\|Q - \mathcal{T}Q_k\|_\mu}_{\text{approximation error}}$ to be small

regression

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regression

Statistical Learning Theory in RL & ADP

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- **Objective:** $\|\hat{Q} - \mathcal{T}Q_k\|_\mu \leq \underbrace{\|\hat{Q} - Q\|_\mu}_{\text{estimation error}} + \underbrace{\|Q - \mathcal{T}Q_k\|_\mu}_{\text{approximation error}}$ to be small

regression

Statistical Learning Theory in RL & ADP

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$$Q_{k+1} \approx \mathcal{T}Q_k$$

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Target Function = $\mathcal{T}Q_k$

Noisy Observation = $\hat{\mathcal{T}}Q_k$

- we minimize the **empirical error** $Q_{k+1} = \hat{Q} = \min_f \|f - \hat{\mathcal{T}}Q_k\|_{\hat{\mu}}$

with the target of minimizing the **true error** $Q = \min_f \|f - \mathcal{T}Q_k\|_\mu$

- Objective:** $\|\hat{Q} - \mathcal{T}Q_k\|_\mu \leq \underbrace{\|\hat{Q} - Q\|_\mu}_{\text{estimation error}} + \underbrace{\|Q - \mathcal{T}Q_k\|_\mu}_{\text{approximation error}}$ to be small

regression

Statistical Learning Theory in RL & ADP

Approximate Policy Iteration (API) - policy evaluation

- finding a function that best approximates Q^{π_k} $Q = \min_f \|f - Q^{\pi_k}\|_{\mu}$

- only noisy observations of Q^{π_k} are available $\hat{Q}^{\pi_k}$

Target Function = Q^{π_k}

Noisy Observation = $\hat{Q}^{\pi_k}$

- we minimize the **empirical error** $\hat{Q} = \min_f \|f - \hat{Q}^{\pi_k}\|_{\hat{\mu}}$

with the target of minimizing the **true error** $Q = \min_f \|f - Q^{\pi_k}\|_{\mu}$

- Objective:** $\|\hat{Q} - Q^{\pi_k}\|_{\mu} \leq \underbrace{\|\hat{Q} - Q\|_{\mu}}_{\text{estimation error}} + \underbrace{\|Q - Q^{\pi_k}\|_{\mu}}_{\text{approximation error}}$ to be small

regression

Statistical Learning Theory in RL & ADP

Approximate Policy Iteration (API)

$$\pi_{k+1} \approx \mathcal{G}\pi_k$$

- finding a policy that best approximates $\mathcal{G}\pi_k$

$$\pi = \min_f \mathcal{L}(f, \pi_k; \mu)$$

- we minimize the **empirical error**

$$\pi_{k+1} = \hat{\pi} = \min_f \hat{\mathcal{L}}(f, \pi_k; \hat{\mu})$$

with the target of minimizing the **true error**

$$\pi = \min_f \mathcal{L}(f, \pi_k; \mu)$$

- Objective:** $\mathcal{L}(\hat{\pi}, \pi_k; \mu) \leq \underbrace{\mathcal{L}(\hat{\pi}, \pi; \mu)}_{\text{estimation error}} + \underbrace{\mathcal{L}(\pi, \pi_k; \mu)}_{\text{approximation error}}$ to be small

classification (we do not discuss it in this talk)

Statistical Learning Theory in RL & ADP

Approximate Policy Iteration (API) - policy evaluation

- finding the fixed-point of $\mathcal{T}^{\pi_k}$
- only noisy observations of $\mathcal{T}^{\pi_k}$ are available

$\hat{\mathcal{T}}^{\pi_k}$

a fixed-point problem

SLT in RL & ADP

- supervised learning methods (regression, classification) appear in the inner-loop of ADP algorithms (performance at each iteration)
- tools from SLT that are used to analyze supervised learning methods can be used in RL and ADP (e.g., how many samples are required to achieve a certain performance)

What makes RL more challenging?

- the objective is not always to recover a target function from its noisy observations (fixed-point vs. regression)
- the target sometimes has to be approximated given sample trajectories (non i.i.d. samples)
- propagation of error (control problem)

is there any hope?

Approximate Value Iteration (AVI)

$$V_{k+1} = \mathcal{T}V_k + \epsilon_k \quad \text{or} \quad \|V_{k+1} - \mathcal{T}V_k\|_\infty = \epsilon_k$$

Proposition (AVI Error Propagation)

We run AVI for K iterations and $\pi_K = \mathcal{G}V_K$

$$\|V^* - V^{\pi_K}\|_\infty \leq \frac{2\gamma}{(1-\gamma)^2} \max_{0 \leq k < K} \epsilon_k + \frac{2\gamma^{K+1}}{1-\gamma} \|V^* - V_0\|_\infty.$$

Proof

$$\|V^* - V_{k+1}\|_\infty \leq \|\mathcal{T}V^* - \mathcal{T}V_k\|_\infty + \|\mathcal{T}V_k - V_{k+1}\|_\infty = \gamma\|V^* - V_k\|_\infty + \epsilon_k$$

so

$$\|V^* - V_K\|_\infty \leq \sum_{k=0}^{K-1} \gamma^{K-1-k} \epsilon_k + \gamma^K \|V^* - V_0\|_\infty \leq \frac{1}{1-\gamma} \max_{0 \leq k < K} \epsilon_k + \gamma^K \|V^* - V_0\|_\infty$$

the result follows by the fact that $\|V^* - V^{\pi_K}\|_\infty \leq \frac{2\gamma}{1-\gamma} \|V^* - V_K\|_\infty$.

Approximate Policy Iteration (API)

① $V_k = V^{\pi_k} + \epsilon_k$ or $\|V_k - V^{\pi_k}\|_\infty = \epsilon_k$ (*Policy Evaluation Error*)

② $V_k = \mathcal{T}^{\pi_k} V_k + \epsilon_k$ or $\|V_k - \mathcal{T}^{\pi_k} V_k\|_\infty = \epsilon_k$ (*Bellman Residual*)

Proposition (API Asymptotic Performance)

$$(1) \quad \limsup_{k \rightarrow \infty} \|V^* - V^{\pi_k}\|_\infty \leq \frac{2\gamma}{(1-\gamma)^2} \limsup_{k \rightarrow \infty} \underbrace{\|V_k - V^{\pi_k}\|_\infty}_{\epsilon_k}$$

$$(2) \quad \limsup_{k \rightarrow \infty} \|V^* - V^{\pi_k}\|_\infty \leq \frac{2\gamma}{(1-\gamma)^2} \limsup_{k \rightarrow \infty} \underbrace{\|V_k - \mathcal{T}^{\pi_k} V_k\|_\infty}_{\epsilon_k}$$

Approximate Dynamic Programming (ADP)

Proposition (AVI Asymptotic Performance)

$$\limsup_{k \rightarrow \infty} \|V^* - V^{\pi_k}\|_{\infty} \leq \frac{2\gamma}{(1-\gamma)^2} \limsup_{k \rightarrow \infty} \underbrace{\|V_{k+1} - \mathcal{T}V_k\|_{\infty}}_{\epsilon_k}$$

Proposition (API Asymptotic Performance)

$$(1) \quad \limsup_{k \rightarrow \infty} \|V^* - V^{\pi_k}\|_{\infty} \leq \frac{2\gamma}{(1-\gamma)^2} \limsup_{k \rightarrow \infty} \underbrace{\|V_k - V^{\pi_k}\|_{\infty}}_{\epsilon_k}$$

$$(2) \quad \limsup_{k \rightarrow \infty} \|V^* - V^{\pi_k}\|_{\infty} \leq \frac{2\gamma}{(1-\gamma)^2} \limsup_{k \rightarrow \infty} \underbrace{\|V_k - \mathcal{T}^{\pi_k} V_k\|_{\infty}}_{\epsilon_k}$$

Error Propagation

AVI Error Propagation

Error at each iteration k :

$$\epsilon_k = \mathcal{T}V_k - V_{k+1}$$

π_K is a greedy policy w.r.t. V_{K-1}

$$\pi_K = \mathcal{G}(V_{K-1})$$

Proposition (AVI Pointwise Error Bound)

$$\begin{aligned} V^* - V^{\pi_K} \leq & (I - \gamma P^{\pi_K})^{-1} \left\{ \sum_{k=0}^{K-1} \gamma^{K-k} [(P^{\pi^*})^{K-k} + P^{\pi_K} P^{\pi_K-1} \dots P^{\pi_{k+1}}] |\epsilon_k| \right. \\ & \left. + \gamma^{K+1} [(P^{\pi^*})^{K+1} + (P^{\pi_K} P^{\pi_K-1} \dots P^{\pi_0})] |V^* - V_0| \right\} \end{aligned}$$

AVI Error Propagation

Proposition (AVI L_p Error Bound)

$$\epsilon_k = \mathcal{T}V_k - V_{k+1}$$

$$\|V^* - V^{\pi_K}\|_{p,\rho} \leq \frac{2\gamma}{(1-\gamma)^2} \left[C_{\rho,\mu}^{1/p} \max_{0 \leq k < K} \|\epsilon_k\|_{p,\mu} + 2\gamma^{K/p} V_{\max} \right] \quad (\text{A1})$$

$$\|V^* - V^{\pi_K}\|_{\infty} \leq \frac{2\gamma}{(1-\gamma)^2} \left[C_{\mu}^{1/p} \max_{0 \leq k < K} \|\epsilon_k\|_{p,\mu} + 2\gamma^{K/p} V_{\max} \right] \quad (\text{A2})$$

AVI Error Propagation

Proposition (AVI L_p Error Bound)

$$\epsilon_k = \mathcal{T}V_k - V_{k+1}$$

$$\|V^* - V^{\pi_K}\|_{p,\rho} \leq \frac{2\gamma}{(1-\gamma)^2} \left[C_{\rho,\mu}^{1/p} \max_{0 \leq k < K} \|\epsilon_k\|_{p,\mu} + 2\gamma^{K/p} V_{\max} \right] \quad (\text{A1})$$

$$\|V^* - V^{\pi_K}\|_{\infty} \leq \frac{2\gamma}{(1-\gamma)^2} \left[C_{\mu}^{1/p} \max_{0 \leq k < K} \|\epsilon_k\|_{p,\mu} + 2\gamma^{K/p} V_{\max} \right] \quad (\text{A2})$$

- $\|\epsilon_k\|_{p,\mu}$: error at each iteration k , **note that** $\epsilon_k = \mathcal{T}V_k - V_{k+1}$

AVI Error Propagation

Proposition (AVI L_p Error Bound)

$$\epsilon_k = \mathcal{T}V_k - V_{k+1}$$

$$\|V^* - V^{\pi_K}\|_{p,\rho} \leq \frac{2\gamma}{(1-\gamma)^2} \left[C_{\rho,\mu}^{1/p} \max_{0 \leq k < K} \|\epsilon_k\|_{p,\mu} + 2\gamma^{K/p} V_{\max} \right] \quad (\text{A1})$$

$$\|V^* - V^{\pi_K}\|_{\infty} \leq \frac{2\gamma}{(1-\gamma)^2} \left[C_{\mu}^{1/p} \max_{0 \leq k < K} \|\epsilon_k\|_{p,\mu} + 2\gamma^{K/p} V_{\max} \right] \quad (\text{A2})$$

- $\|\epsilon_k\|_{p,\mu}$: error at each iteration k , **note that** $\epsilon_k = \mathcal{T}V_k - V_{k+1}$
- $2\gamma^{K/p} V_{\max}$: initialization error $|V^* - V_0|$

AVI Error Propagation

Proposition (AVI L_p Error Bound)

$$\epsilon_k = \mathcal{T}V_k - V_{k+1}$$

$$\|V^* - V^{\pi_K}\|_{p,\rho} \leq \frac{2\gamma}{(1-\gamma)^2} \left[C_{\rho,\mu}^{1/p} \max_{0 \leq k < K} \|\epsilon_k\|_{p,\mu} + 2\gamma^{K/p} V_{\max} \right] \quad (\text{A1})$$

$$\|V^* - V^{\pi_K}\|_{\infty} \leq \frac{2\gamma}{(1-\gamma)^2} \left[C_{\mu}^{1/p} \max_{0 \leq k < K} \|\epsilon_k\|_{p,\mu} + 2\gamma^{K/p} V_{\max} \right] \quad (\text{A2})$$

- $\|\epsilon_k\|_{p,\mu}$: error at each iteration k , **note that** $\epsilon_k = \mathcal{T}V_k - V_{k+1}$
- $2\gamma^{K/p} V_{\max}$: initialization error $|V^* - V_0|$
- $C_{\rho,\mu}, C_{\mu}$: final performance is evaluated w.r.t. a measure $\rho \neq \mu$

AVI Error Propagation (Concentrability Coefficients)

Final performance is evaluated w.r.t. a measure $\rho \neq \mu$, $\|V^* - V^{\pi_K}\|_{\rho, \rho}$

Assumption 1. (Uniformly Stochastic Transitions)

For all $x \in \mathcal{X}$ and $a \in \mathcal{A}$, there exists a constant $C_\mu < \infty$ such that $P(\cdot|x, a) \leq C_\mu \mu(\cdot)$.

Assumption 2. (Discounted-Average Concentrability of Future-State Distribution)

For any sequence of policies $\{\pi_m\}_{m \geq 1}$, there exists a constant $c_{\rho, \mu}(m) < \infty$ such that $\rho P^{\pi_1} P^{\pi_2} \dots P^{\pi_m} \leq c_{\rho, \mu}(m) \mu$. We define

$$C_{\rho, \mu} = (1 - \gamma)^2 \sum_{m \geq 1} m \gamma^{m-1} c_{\rho, \mu}(m)$$

- Note that $C_{\rho, \mu} \leq C_\mu$.

API Error Propagation

Error at each iteration k :

$$\epsilon_k = V_k - \mathcal{T}^{\pi_k} V_k$$

π_K is a greedy policy w.r.t. V_{K-1}

$$\pi_K = \mathcal{G}(V_{K-1})$$

Proposition (API Pointwise Error Bound)

$$V^* - V^{\pi_K} \leq \gamma \sum_{k=0}^{K-1} (\gamma P^{\pi^*})^{K-k-1} E_k |\epsilon_k| + (\gamma P^{\pi^*})^K |V^* - V^{\pi_0}|$$

where $E_k = P^{\pi_{k+1}}(I - \gamma P^{\pi_{k+1}})^{-1} - P^{\pi^*}(I - \gamma P^{\pi_k})^{-1}$

API Error Propagation

Proposition (API L_p Error Bound)

$$\epsilon_k = V_k - \mathcal{T}^{\pi_k} V_k$$

$$\|V^* - V^{\pi_K}\|_{p,\rho} \leq \frac{2\gamma}{(1-\gamma)^2} \left[C_{\rho,\mu}^{1/p} \max_{0 \leq k < K} \|\epsilon_k\|_{p,\mu} + 2\gamma^{K/p} V_{\max} \right] \quad (\text{A1})$$

$$\|V^* - V^{\pi_K}\|_{\infty} \leq \frac{2\gamma}{(1-\gamma)^2} \left[C_{\mu}^{1/p} \max_{0 \leq k < K} \|\epsilon_k\|_{p,\mu} + 2\gamma^{K/p} V_{\max} \right] \quad (\text{A2})$$

API Error Propagation

Proposition (API L_p Error Bound)

$$\epsilon_k = V_k - \mathcal{T}^{\pi_k} V_k$$

$$\|V^* - V^{\pi_K}\|_{p,\rho} \leq \frac{2\gamma}{(1-\gamma)^2} \left[C_{\rho,\mu}^{1/p} \max_{0 \leq k < K} \|\epsilon_k\|_{p,\mu} + 2\gamma^{K/p} V_{\max} \right] \quad (\text{A1})$$

$$\|V^* - V^{\pi_K}\|_{\infty} \leq \frac{2\gamma}{(1-\gamma)^2} \left[C_{\mu}^{1/p} \max_{0 \leq k < K} \|\epsilon_k\|_{p,\mu} + 2\gamma^{K/p} V_{\max} \right] \quad (\text{A2})$$

- $\|\epsilon_k\|_{p,\mu}$: error at each iteration k , **note that** $\epsilon_k = V_k - \mathcal{T}^{\pi_k} V_k$
- $2\gamma^{K/p} V_{\max}$: initialization error $|V^* - V^{\pi_0}|$
- $C_{\rho,\mu}, C_{\mu}$: final performance is evaluated w.r.t. a measure $\rho \neq \mu$

API Error Propagation (Concentrability Coefficients)

Final performance is evaluated w.r.t. a measure $\rho \neq \mu$, $\|V^* - V^{\pi_K}\|_{\rho, \rho}$

Assumption 1. (Uniformly Stochastic Transitions)

For all $x \in \mathcal{X}$ and $a \in \mathcal{A}$, there exists a constant $C_\mu < \infty$ such that $P(\cdot|x, a) \leq C_\mu \mu(\cdot)$.

Assumption 2. (Discounted-Average Concentrability of Future-State Distribution)

For any policy π and any non-negative integers s and t , there exists a constant $c_{\rho, \mu}(s, t) < \infty$ such that $\rho(P^*)^s (P^\pi)^t \leq c_{\rho, \mu}(s, t) \mu$. We define

$$C_{\rho, \mu} = (1 - \gamma)^2 \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \gamma^{s+t} c_{\rho, \mu}(s, t)$$

- Note that $C_{\rho, \mu} \leq C_\mu$.

Finite-Sample Performance Bound of an AVI Algorithm

Approximate Value Iteration (AVI)

- if $\mathcal{F}$ is a function space, then V_{k+1} can be defined as

$$V_{k+1} = \inf_{V \in \mathcal{F}} \|V - \mathcal{T}V_k\|_{\mathcal{L}} = \Pi_{\mathcal{L}} \mathcal{T}V_k$$

(projection of $\mathcal{T}V_k$ into $\mathcal{F}$ according to the norm $\mathcal{L}$)

- if $V_{k+1} = \Pi_{\infty} \mathcal{T}V_k$ then AVI converges to the **unique fixed-point of $\Pi_{\infty} \mathcal{T}$** , i.e. $\hat{V} \in \mathcal{F} : \hat{V} = \Pi_{\infty} \mathcal{T}\hat{V}$

($\mathcal{T}$ is a contraction in L_{∞} -norm and Π_{∞} is non-expansive)

- if we consider another norm, e.g. $L_2(\mu)$, then AVI does **not** necessarily converge

($\Pi_{2,\mu} \mathcal{T}$ is **not** necessarily a contraction)

An Approximate Value Iteration Algorithm

- Linear function space $\mathcal{F} = \{f : f(\cdot) = \sum_{j=1}^d \alpha_j \varphi_j(\cdot)\}$

$$\{\varphi_j\}_{j=1}^d \in \mathcal{B}((\mathcal{X}, \mathcal{A}); L) \quad , \quad \phi : (\mathcal{X}, \mathcal{A}) \rightarrow \mathbb{R}^d, \quad \phi(\cdot) = (\varphi_1(\cdot), \dots, \varphi_d(\cdot))^\top$$

Fitted Q-Iteration (FQI)

At each iteration k :

- Generate N samples of the form (X_i, A_i, X'_i, R_i) , where $(X_i, A_i) \sim \mu$, $X'_i \sim p(\cdot | X_i, A_i)$, $R_i \sim r(X_i, A_i)$
- Build the training set $\mathcal{D}_k = \left\{ ((X_i, A_i), \hat{T}Q_k(X_i, A_i)) \right\}_{i=1}^N$, where $\hat{T}Q_k(X_i, A_i) = R_i + \gamma \max_{a \in \mathcal{A}} Q_k(X'_i, a)$
- $Q_{k+1} = \arg \min_{f \in \mathcal{F}} \|f - \hat{T}Q_k\|_N^2 = \arg \min_{f \in \mathcal{F}} \frac{1}{N} \sum_{i=1}^N [f(X_i, A_i) - \hat{T}Q_k(X_i, A_i)]^2$
(regression)

FQI - Error at Each Iteration

Theorem (FQI - Error at Iteration k)

Let $\mathcal{F}$ be a d -dim linear space, $\mathcal{D}_k = \{(X_i, A_i, X'_i, R_i)\}_{i=1}^N$, $(X_i, A_i) \stackrel{\text{iid}}{\sim} \mu$, $X'_i \sim p(\cdot | X_i, A_i)$, $R_i = r(X_i, A_i)$, and $\tilde{Q}$ be the *training set* and the *truncated solution* at the k 'th iteration of FQI. Then with probability $1 - \delta$, we have

$$\|\tilde{Q} - \mathcal{T}Q_k\|_{\mu} \leq 4 \inf_{f \in \mathcal{F}} \|f - \mathcal{T}Q_k\|_{\mu} + O\left(\|\alpha_k^*\| \sqrt{\frac{\log(1/\delta)}{N}}\right) + O\left(\sqrt{\frac{d \log(N/\delta)}{N}}\right).$$

- Note that $Q_{k+1} = \tilde{Q}$.

FQI - Error at Each Iteration

Theorem (FQI - Error at Iteration k)

Let $\mathcal{F}$ be a d -dim linear space, $\mathcal{D}_k = \{(X_i, A_i, X'_i, R_i)\}_{i=1}^N$, $(X_i, A_i) \stackrel{\text{iid}}{\sim} \mu$, $X'_i \sim p(\cdot | X_i, A_i)$, $R_i = r(X_i, A_i)$, and $\tilde{Q}$ be the *training set* and the *truncated solution* at the k 'th iteration of FQI. Then with probability $1 - \delta$, we have

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- Note that $Q_{k+1} = \tilde{Q}$.
- $N = \#$ of samples , $d =$ dimension of the linear function space $\mathcal{F}$

FQI - Error at Each Iteration

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- Note that $Q_{k+1} = \tilde{Q}$.
- $N = \#$ of samples , $d =$ dimension of the linear function space $\mathcal{F}$
- $\alpha_k^* \longrightarrow f_{\alpha_k^*} = \Pi_{2,\mu} \mathcal{T}Q_k$: the best approximation of $\mathcal{T}Q_k$ in $\mathcal{F}$ w.r.t. μ

FQI - Error at Each Iteration

FQI - Error at Iteration k

$$\|\tilde{Q} - \mathcal{T}Q_k\|_\mu \leq \underbrace{4 \inf_{f \in \mathcal{F}} \|f - \mathcal{T}Q_k\|_\mu}_{\text{approximation error}} + \underbrace{O\left(\|\alpha_k^*\| \sqrt{\frac{\log(1/\delta)}{N}}\right) + O\left(\sqrt{\frac{d \log(N/\delta)}{N}}\right)}_{\text{estimation error}}$$

- **Approximation error:** it depends on how well the function space $\mathcal{F}$ can approximate $\mathcal{T}Q_k$
- **Estimation error:** it depends on the number of samples N , the dim of the function space d , and $\|\alpha_k^*\|$

FQI Error Bound

Theorem (FQI Error Bound)

Let $Q_{-1} \in \tilde{\mathcal{F}}$ be an arbitrary initial value function, $\tilde{Q}_0, \dots, \tilde{Q}_{K-1}$ be the sequence of truncated action-value functions generated by FQI after K iterations, and π_K be the greedy policy w.r.t. $\tilde{Q}_{K-1}$. Then with probability $1 - \delta$, we have

$$\begin{aligned} \|V^* - V^{\pi_K}\|_\rho \leq & \frac{2\gamma}{(1-\gamma)^2} \left\{ \sqrt{C_{\rho,\mu}} \left[d_\mu(\mathcal{T}\mathcal{F}, \mathcal{F}) + O\left(Q_{\max} \sqrt{\frac{\log(K/\delta)}{N \nu_\mu}} \right) \right. \right. \\ & \left. \left. + O\left(\sqrt{\frac{d \log(NK/\delta)}{N}} \right) \right] + 2\gamma^{K/2} Q_{\max} \right\} \end{aligned}$$

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- **Approximation error:** $d_\mu(\mathcal{T}\mathcal{F}, \mathcal{F}) = \sup_{f \in \tilde{\mathcal{F}}} \inf_{g \in \mathcal{F}} \|g - \mathcal{T}f\|_\mu$

FQI Error Bound

Theorem (FQI Error Bound)

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$$\|V^* - V^{\pi_K}\|_\rho \leq \frac{2\gamma}{(1-\gamma)^2} \left\{ \sqrt{C_{\rho,\mu}} \left[d_\mu(\mathcal{T}\mathcal{F}, \mathcal{F}) + O\left(Q_{\max} \sqrt{\frac{\log(K/\delta)}{N \nu_\mu}}\right) \right. \right. \\ \left. \left. + O\left(\sqrt{\frac{d \log(NK/\delta)}{N}}\right) \right] + 2\gamma^{K/2} Q_{\max} \right\}$$

- **Approximation error:** $d_\mu(\mathcal{T}\mathcal{F}, \mathcal{F}) = \sup_{f \in \tilde{\mathcal{F}}} \inf_{g \in \mathcal{F}} \|g - \mathcal{T}f\|_\mu$
- **Estimation error:** depends on N, d, ν_μ, K . Note that $\|\alpha_k^*\| \leq \frac{Q_{\max}}{\nu_\mu}$
 ν_μ = the smallest eigenvalue of the Gram matrix $(\int \varphi_i \varphi_j d\mu)_{i,j}$

FQI Error Bound

Theorem (FQI Error Bound)

Let $Q_{-1} \in \tilde{\mathcal{F}}$ be an arbitrary initial value function, $\tilde{Q}_0, \dots, \tilde{Q}_{K-1}$ be the sequence of truncated action-value functions generated by FQI after K iterations, and π_K be the greedy policy w.r.t. $\tilde{Q}_{K-1}$. Then with probability $1 - \delta$, we have

$$\|V^* - V^{\pi_K}\|_\rho \leq \frac{2\gamma}{(1-\gamma)^2} \left\{ \sqrt{C_{\rho,\mu}} \left[d_\mu(\mathcal{T}\mathcal{F}, \mathcal{F}) + O\left(Q_{\max} \sqrt{\frac{\log(K/\delta)}{N \nu_\mu}}\right) \right] + O\left(\sqrt{\frac{d \log(NK/\delta)}{N}}\right) \right\} + 2\gamma^{K/2} Q_{\max}$$

- **Approximation error:** $d_\mu(\mathcal{T}\mathcal{F}, \mathcal{F}) = \sup_{f \in \tilde{\mathcal{F}}} \inf_{g \in \mathcal{F}} \|g - \mathcal{T}f\|_\mu$
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 ν_μ = the smallest eigenvalue of the Gram matrix $(\int \varphi_i \varphi_j d\mu)_{i,j}$
- **Initialization error:** error due to the choice of the initial action-value function $|Q^* - Q_0|$

Finite-Sample Performance Bound of an API Algorithm

Least-Squares Temporal-Difference Learning (LSTD)

- Linear function space $\mathcal{F} = \{f : f(\cdot) = \sum_{j=1}^d \alpha_j \varphi_j(\cdot)\}$

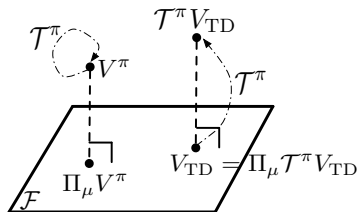
$$\{\varphi_j\}_{j=1}^d \in \mathcal{B}(\mathcal{X}; L) \quad , \quad \phi : \mathcal{X} \rightarrow \mathbb{R}^d, \phi(\cdot) = (\varphi_1(\cdot), \dots, \varphi_d(\cdot))^\top$$

- V^π is the fixed-point of $\mathcal{T}^\pi$ $\mathcal{T}^\pi V^\pi = V^\pi$
- V^π may not belong to $\mathcal{F}$ $V^\pi \notin \mathcal{F}$
- LSTD searches for the fixed-point of $\Pi_{\mathcal{F}} \mathcal{T}^\pi$ instead ($\Pi_{\mathcal{F}}$ is a projection into $\mathcal{F}$ w.r.t. L_2 -norm)
- $\Pi_{\infty} \mathcal{T}^\pi$ is a **contraction** in L_∞ -norm
 - L_∞ -projection is numerically expensive when the number of states is large or infinite
- LSTD searches for the fixed-point of $\Pi_{2,\mu} \mathcal{T}^\pi$
 $\Pi_{2,\mu} g = \arg \min_{f \in \mathcal{F}} \|f - g\|_{2,\mu}$

Least-Squares Temporal-Difference Learning (LSTD)

When the fixed-point of $\Pi_\mu \mathcal{T}^\pi$ exists, we call it the LSTD solution

$$V_{TD} = \Pi_\mu \mathcal{T}^\pi V_{TD}$$



$$\langle \mathcal{T}^\pi V_{TD} - V_{TD}, \varphi_i \rangle_\mu = 0, \quad i = 1, \dots, d$$

$$\langle r^\pi + \gamma P^\pi V_{TD} - V_{TD}, \varphi_i \rangle_\mu = 0$$

$$\underbrace{\langle r^\pi, \varphi_i \rangle_\mu}_{b_i} - \sum_{j=1}^d \underbrace{\langle \varphi_j - \gamma P^\pi \varphi_j, \varphi_i \rangle_\mu}_{A_{ij}} \cdot \alpha_{TD}^{(j)} = 0 \quad \rightarrow \quad \textcolor{red}{A} \alpha_{TD} = \textcolor{red}{b}$$

- In general, $\Pi_\mu \mathcal{T}^\pi$ is not a contraction and does not have a fixed-point.
- If $\mu = \mu^\pi$, the stationary dist. of π , then $\Pi_{\mu^\pi} \mathcal{T}^\pi$ has a unique fixed-point. 

LSTD Algorithm

Proposition (LSTD Performance)

$$\|V^\pi - V_{TD}\|_{\mu^\pi} \leq \frac{1}{\sqrt{1-\gamma^2}} \inf_{V \in \mathcal{F}} \|V^\pi - V\|_{\mu^\pi}$$

LSTD Algorithm

- We observe a trajectory generated by following the policy π ($X_0, R_0, X_1, R_1, \dots, X_N$) where $X_{t+1} \sim P(\cdot | X_t, \pi(X_t))$ and $R_t = r(X_t, \pi(X_t))$
- We build estimators of the matrix A and vector b

$$\hat{A}_{ij} = \frac{1}{N} \sum_{t=0}^{N-1} \varphi_i(X_t) [\varphi_j(X_t) - \gamma \varphi_j(X_{t+1})] \quad , \quad \hat{b}_i = \frac{1}{N} \sum_{t=0}^{N-1} \varphi_i(X_t) R_t$$

$$\bullet \quad \hat{A} \hat{\alpha}_{TD} = \hat{b} \quad , \quad \hat{V}_{TD}(\cdot) = \phi(\cdot)^\top \hat{\alpha}_{TD}$$

when $n \rightarrow \infty$ then $\hat{A} \rightarrow A$ and $\hat{b} \rightarrow b$, and thus, $\hat{\alpha}_{TD} \rightarrow \alpha_{TD}$ and $\hat{V}_{TD} \rightarrow V_{TD}$.

LSTD Error Bound

When the Markov chain induced by the policy under evaluation π has a stationary distribution μ^π (Markov chain is ergodic - e.g. β -mixing), then

Theorem (LSTD Error Bound)

Let $\tilde{V}$ be the truncated LSTD solution computed using n samples along a trajectory generated by following the policy π . Then with probability $1 - \delta$, we have

$$\|V^\pi - \tilde{V}\|_{\mu^\pi} \leq \frac{c}{\sqrt{1 - \gamma^2}} \inf_{f \in \mathcal{F}} \|V^\pi - f\|_{\mu^\pi} + O\left(\sqrt{\frac{d \log(d/\delta)}{n \nu}}\right)$$

- n = # of samples , d = dimension of the linear function space $\mathcal{F}$
- ν = the smallest eigenvalue of the Gram matrix $(\int \varphi_i \varphi_j d\mu^\pi)_{i,j}$
(**Assume:** eigenvalues of the Gram matrix are strictly positive - existence of the model-based LSTD solution)
- β -mixing coefficients are hidden in O notation

LSTD Error Bound

LSTD Error Bound

$$\|V^\pi - \tilde{V}\|_{\mu^\pi} \leq \frac{c}{\sqrt{1-\gamma^2}} \underbrace{\inf_{f \in \mathcal{F}} \|V^\pi - f\|_{\mu^\pi}}_{\text{approximation error}} + \underbrace{O\left(\sqrt{\frac{d \log(d/\delta)}{n \nu}}\right)}_{\text{estimation error}}$$

- **Approximation error:** it depends on how well the function space $\mathcal{F}$ can approximate the value function V^π
- **Estimation error:** it depends on the number of samples n , the dim of the function space d , the smallest eigenvalue of the Gram matrix ν , the mixing properties of the Markov chain (hidden in O)

LSPI Error Bound

Theorem (LSPI Error Bound)

Let $V_{-1} \in \tilde{\mathcal{F}}$ be an arbitrary initial value function, $\tilde{V}_0, \dots, \tilde{V}_{K-1}$ be the sequence of truncated value functions generated by LSPI after K iterations, and π_K be the greedy policy w.r.t. $\tilde{V}_{K-1}$. Then with probability $1 - \delta$, we have

$$\|V^* - V^{\pi_K}\|_{\rho} \leq \frac{4\gamma}{(1-\gamma)^2} \left\{ \sqrt{CC_{\rho,\mu}} \left[cE_0(\mathcal{F}) + O\left(\sqrt{\frac{d \log(dK/\delta)}{n \nu_{\mu}}}\right) \right] + \gamma^{\frac{K-1}{2}} R_{\max} \right\}$$

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- **Approximation error:** $E_0(\mathcal{F}) = \sup_{\pi \in \mathcal{G}(\tilde{\mathcal{F}})} \inf_{f \in \mathcal{F}} \|V^{\pi} - f\|_{\mu^{\pi}}$

LSPI Error Bound

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$$\|V^* - V^{\pi_K}\|_{\rho} \leq \frac{4\gamma}{(1-\gamma)^2} \left\{ \sqrt{CC_{\rho,\mu}} \left[cE_0(\mathcal{F}) + \mathcal{O} \left(\sqrt{\frac{d \log(dK/\delta)}{n \nu_{\mu}}} \right) \right] + \gamma^{\frac{K-1}{2}} R_{\max} \right\}$$

- **Approximation error:** $E_0(\mathcal{F}) = \sup_{\pi \in \mathcal{G}(\tilde{\mathcal{F}})} \inf_{f \in \mathcal{F}} \|V^{\pi} - f\|_{\mu^{\pi}}$
- **Estimation error:** depends on n, d, ν_{μ}, K

LSPI Error Bound

Theorem (LSPI Error Bound)

Let $V_{-1} \in \tilde{\mathcal{F}}$ be an arbitrary initial value function, $\tilde{V}_0, \dots, \tilde{V}_{K-1}$ be the sequence of truncated value functions generated by LSPI after K iterations, and π_K be the greedy policy w.r.t. $\tilde{V}_{K-1}$. Then with probability $1 - \delta$, we have

$$\|V^* - V^{\pi_K}\|_\rho \leq \frac{4\gamma}{(1-\gamma)^2} \left\{ \sqrt{CC_{\rho,\mu}} \left[cE_0(\mathcal{F}) + O\left(\sqrt{\frac{d \log(dK/\delta)}{n \nu_\mu}}\right) \right] + \gamma^{\frac{K-1}{2}} R_{\max} \right\}$$

- **Approximation error:** $E_0(\mathcal{F}) = \sup_{\pi \in \mathcal{G}(\tilde{\mathcal{F}})} \inf_{f \in \mathcal{F}} \|V^\pi - f\|_{\mu^\pi}$
- **Estimation error:** depends on n, d, ν_μ, K
- **Initialization error:** error due to the choice of the initial value function or initial policy $|V^* - V^{\pi_0}|$

LSPI Error Bound

LSPI Error Bound

$$\|V^* - V^{\pi_K}\|_{\rho} \leq \frac{4\gamma}{(1-\gamma)^2} \left\{ \sqrt{CC_{\rho,\mu}} \left[cE_0(\mathcal{F}) + O\left(\sqrt{\frac{d \log(dK/\delta)}{n \nu_{\mu}}}\right) \right] + \gamma^{\frac{K-1}{2}} R_{\max} \right\}$$

Lower-Bounding Distribution

There exists a distribution μ such that for any policy $\pi \in \mathcal{G}(\tilde{\mathcal{F}})$, we have $\mu \leq C\mu^{\pi}$, where $C < \infty$ is a constant and μ^{π} is the stationary distribution of π . Furthermore, we can define the **concentrability** coefficient $C_{\rho,\mu}$ as before.

LSPI Error Bound

LSPI Error Bound

$$\|V^* - V^{\pi_K}\|_{\rho} \leq \frac{4\gamma}{(1-\gamma)^2} \left\{ \sqrt{CC_{\rho,\mu}} \left[cE_0(\mathcal{F}) + O\left(\sqrt{\frac{d \log(dK/\delta)}{n \nu_{\mu}}}\right) \right] + \gamma^{\frac{K-1}{2}} R_{\max} \right\}$$

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There exists a distribution μ such that for any policy $\pi \in \mathcal{G}(\tilde{\mathcal{F}})$, we have $\mu \leq C\mu^{\pi}$, where $C < \infty$ is a constant and μ^{π} is the stationary distribution of π . Furthermore, we can define the **concentrability** coefficient $C_{\rho,\mu}$ as before.

- ν_{μ} = the smallest eigenvalue of the Gram matrix $(\int \varphi_i \varphi_j d\mu)_{i,j}$

Discussion

we obtain the optimal rate of regression and classification for RL (ADP) algorithms

What makes RL more challenging then?

- the propagation of error (**control problem**)
- the approximation error is more complex
- the sampling problem (how to choose μ - **exploration problem**)

Other Finite-Sample Analysis Results in RL

- Approximate Value Iteration [MS08]
- Approximate Policy Iteration
 - LSTD and LSPI [LGM10, LGM11]
 - Bellman Residual Minimization [MMLG10]
 - Modified Bellman Residual Minimization [ASM08]
 - Classification-based Policy Iteration [FYG06, LGM10, GLGS11]
- Regularized Approximate Dynamic Programming
 - L_2 -Regularization
 - L_2 -Regularized Policy Iteration [FGSM08]
 - L_2 -Regularized Fitted Q-Iteration [FGSM09]
 - L_1 -Regularization and High-Dimensional RL
 - Lasso-TD [GLMH11]
 - LSTD (LSPI) with Random Projections [GLMM10]

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