THE PHILOSOPHERS' PROCESS ON LADDER GRAPHS

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ABSTRACT

The Philosophers' Process is a Markovian version of the famous Dining Philosophers Problem introduced by Dijkstra as a model for resource sharing. It can be viewed as a reversible interacting particle system. This paper deals with the problem of computing its reversible measure in the case of a large but finite number of sites. This equilibrium measure is a Markov field on the underlying graph of interactions. When this graph has a linear structure (ladder graph), the reversible measure can be interpreted as the distribution of a stationary Markov chain. Its computation reduces to matrix products. This leads to explicit algorithms of computation in most cases of practical interest (grids, hypercubes, trees). Some examples of explicit formulae illustrate the results.

Keywords: Dining Philosophers Problem, Markov field, reversible measure, partition function.

1 INTRODUCTION.

Many different ways of modelling resource sharing situations have been proposed. Dijkstra's Dining Philosophers Problem [2] is one of the first systematic attempts in the area. It describes a situation where the users (philosophers) are the vertices of a graph the edges of which (chopsticks) represent resources. When one of the users starts working, he needs all available resources (adjacent edges) and therefore none of his neighbors can work. The dining philosophers problem is a deterministic situation for which the main question was to find an algorithm that would ensure fairness, *i.e.* give each user a correct amount of working time. One way of solving the problem of fairness is to introduce a stochastic evolution of the situation, namely a Markov process with values in the set of configurations of busy and idle users. The Philosophers Process (PP) is defined as follows. The philosophers are the vertices of an undirected countable graph $G = (S, E)$ with bounded degree. The edges of the graph represent the philosophers' interactions. If $x \in S$ is a philosopher we denote by $N(x)$ the set of his neighbors on the graph.

$$N(x) = \{y \in S \ s.t. \ \{x, y\} \in E\}.$$

Each philosopher can be in one of two states: 0 if he is thinking, 1 if he is eating. A configuration is a mapping from S into $\{0, 1\}$. When the configuration is η, the state of philosopher x is $\eta(x)$ (0 or 1). The PP is a Feller process $\{\eta_t \ ; \ t \geq 0\}$ on the set of configurations $X = \{0, 1\}^S$. It is a particular case of Spin System (*cf.* Liggett's book [10] for a general reference). The system evolves in time according to the following local dynamics: if the current configuration is $\eta \in X$, then a philosopher $x \in S$ changes its value from $\eta(x)$ to $1 - \eta(x)$ at rate $c(x, \eta)$.

$$\begin{aligned} c(x,\eta) &= 1 \ \text{ if } \eta(x) = 1 \\ &= \lambda \ \text{ if } \eta(x) = 0 \text{ and } \eta(y) = 0 \ \forall y \in N(x) \\ &= 0 \ \text{ in any other case.} \end{aligned}$$

A philosopher eating flips from 1 to 0 after an exponential time with parameter 1. A thinking philosopher flips from 0 to 1 after an exponential time with parameter λ if none of his neighbors is already eating. Note that the rates

$c(x, \eta)$ depend only on $\eta(x)$ and the values of the adjacent sites of x (nearest neighbor interaction). Actually the evolution of the process is restricted by its dynamics to a subset of X that we shall call the set of admissible configurations and denote by A_G. In such configurations, adjacent philosophers cannot eat simultaneously.

In the past ten years the PP has been introduced by several authors in different contexts (*cf.* [19] for detailed references). The first article where it is related with Dijkstra's Dining Philosophers Problem is that of Zenie [20], but it had appeared before as a model for communications networks (*cf.* Kelly [7]). The study of the PP is motivated by its possible use as a model for resource sharing in computer science, in particular shared memory systems or database locking. Consider a database that consists of several items (resources). A transaction (philosopher) is associated to a list of items that are needed for processing. Requests for transactions arrive independently at Poisson rate λ. Upon arrival of a request, if any of the corresponding items is already locked the request is blocked and discarded. If the items are available, the request is accepted for processing. The items are locked for an exponential time until the processing is complete. An interaction graph can be associated to such a system. Its vertices are the types of transactions. There is an edge between two of them if they use one or more common items requiring exclusive lock. Therefore, no two adjacent vertices on the graph can be simultaneously active. The process of the active transactions in the database can be modelled by a PP. We refer to [7] or [12] for further details. The PP can be also used in the modelling of communication networks. Consider several places connected by a communication network. When a place wants to emit towards an other, it makes a call (request) for a route (transaction) that will require several communication links (items). It may be of interest to study the performances of such a system for different routing strategies in order to determine which routing strategy is the best. The PP is a particular case of the Loss Networks introduced by Kelly [8].

The underlying interaction graph G depends on the structure of the system and some are classical in computer science. The cyclic graph which was studied in [20], typically appears when considering a multiprocessor interconnection network called tandem. In such a network, a processor can be involved in a transmission with one of its two neighbors but not with both (think of a telephone network). Another simple interconnection network is that in which n processors are connected to m memories via a single bus. The interaction graph is the complete graph (any two vertices are connected) K_{nm} with nm vertices. In a $n \times m$ crossbar network, switches connect n processors to m memories. Two processors cannot simultaneously access the same memory module. The interaction graph is the Cartesian product (see definition 2.1) $K_n \times K_m$. Other graphs may also be of interest in statistical physics. The PP is a particular case of the so-called hard-core lattice gas model. It is a simple model of gas whose particles have a non-negligible size. The particles centers are associated to the vertices of a graph. A vertex takes the value 1 if it is occupied by a particle and 0 otherwise. The particles cannot overlap so that two adjacent vertices cannot be both occupied. In this model the rate λ may depend on the vertex (or philosopher) x and is referred to as the activity a_x (see for instance [17]).

The aim of this paper is to show that the equilibrium measure of the PP can be computed, at least numerically, for a wide class of graphs that we shall call ladder graphs. They are obtained by taking the Cartesian product of a cycle, a line, or a tree by a fixed subgraph (*cf.* section 2 for a more precise definition). This class of graphs includes all the classical structures considered in computer science and statistical mechanics such as grids, toruses, hypercubes (a usual structure in parallel systems), *etc.* The key observation is that the equilibrium measure of the PP is a Markov field on the underlying graph G which is factorizable on the cliques of G. In the case of ladder graphs this leads to an algorithm involving mainly matrix products.

For an infinite number of philosophers, the main problem is that of ergodicity or the possible existence of phase transitions (*cf.* [7], [12] and [19]). For a finite number of sites the problem is different. It is easy to see that the PP is irreducible on the set A_G of admissible configurations, and therefore has a unique stationary distribution. Moreover, this equilibrium measure is reversible and has a product form (*cf.* [9], [19]). Let μ_G be this measure. One has

$$\forall \eta \in A_G \ , \ \ \mu_G[\, \eta \,] \ = \ \mu_G[\, 0_G \,] \ . \ \lambda^{\sum_{x \in S} \eta(x)}, \tag{1}$$

where 0_G denotes the empty configuration (all philosophers thinking). The reversible measure is known up to the normalizing constant $\mu_G[\, 0_G \,]$. The inverse of that constant is the partition function of the system and will be denoted by Z_G. It is a polynomial in λ, the coefficient of order k being the number of admissible configurations with k eating philosophers exactly. These admissible configurations correspond to independent sets of G with k vertices. An independent set of vertices (or coclique) in a graph G is a set of vertices no two of which are adjacent. Independent sets have been widely investigated in the graph theory literature (see [1] for a general reference). However it seems that most of the work in the area concerns the number of maximal independent sets in a graph G and the largest possible value of this number when G belongs to various classes of graphs (see [11] and references therein). Another quantity of interest in graph theory is the independence number of a graph G. It is the largest cardinality of an independent set in G (see [18] chapter 31 for some illustrations). In our model it is the degree of Z_G as a polynomial in λ, while the number of maximal independent sets is the coefficient of highest order in Z_G. Using our results it is possible to recover some properties of these numbers. As an example, lemma 1 in [11] is an immediate consequence of equation (11) in proposition 2.6 below. Thus the problem of determining μ_G is essentially a combinatorial one. However, the size of the state space A_G prohibits in general direct enumeration, even by computer. In the case of a cyclic graph, an explicit expression of the partition function is given in [19], proposition 2.2 p. 361. Notice that the value of the partition function for $\lambda = 1$ is the cardinality of A_G. For instance, for 100 philosophers around a round table (cyclic graph) this number is

$$792,070,839,848,372,253,127.$$

In section 2 we define ladder graphs and describe our general algorithm of computation for the partition function and various other quantities. We also present some material on Markov fields to show how the previous results fit into a more general context. Section 3 is devoted to explicit results obtained using the algorithm of the previous section. Examples of grids, toruses and products by cliques are treated. In section 4, we use again the ideas of section 2 to propose a recursive algorithm for the computation of the reversible measure of the PP on the hypercube in dimension n. Finally section 5 treats the case of trees and their Cartesian products by other finite graphs. In the particular case of the product of an infinite tree by a clique, we give a sufficient condition for the existence of a phase transition, thus generalizing some results of Kelly [7].

2 LADDER GRAPHS.

We define a class of finite graphs for which the computation of the reversible measure of the PP will reduce to matrix powers. Then the connection between our main results (propositions 2.4 and 2.5) and classical results on Markov fields and chains will be presented. We will require some notation and definitions from graph theory (*cf.* [1] for a general reference). Here is first the definition of a Cartesian product (also called the Cartesian sum in some references).

Definition 2.1 *Chartrand and Lesniak [1] p. 11. Let $G_1 = (S_1, E_1)$ and $G_2 = (S_2, E_2)$ be two undirected graphs. The Cartesian product, denoted by $G_1 \times G_2$ is the graph $G = (S, E)$ defined by*

$$\begin{aligned} S &= S_1 \times S_2 \\ E &= \{\ \{(x,y),(x',y')\}\ ;\ (x,y) \text{ and } (x',y') \in S_1 \times S_2 \qquad s.t. \\ &\quad x = x' \quad \text{and} \quad \{y,y'\} \in E_2 \\ &\quad \text{or} \\ &\quad y = y' \quad \text{and} \quad \{x,x'\} \in E_1\ \} \quad . \end{aligned}$$

Let C_m be the cycle graph with m vertices and L_m be the line (also called path graph) obtained by removing an edge from C_m.

Definition 2.2 *Let $g = (s, e)$ be a finite undirected graph. We call cycle ladder (respectively line ladder) with rung g and length m the Cartesian product $g \times C_m$ (respectively $g \times L_m$).*

Examples of such graphs are given in figure 1.

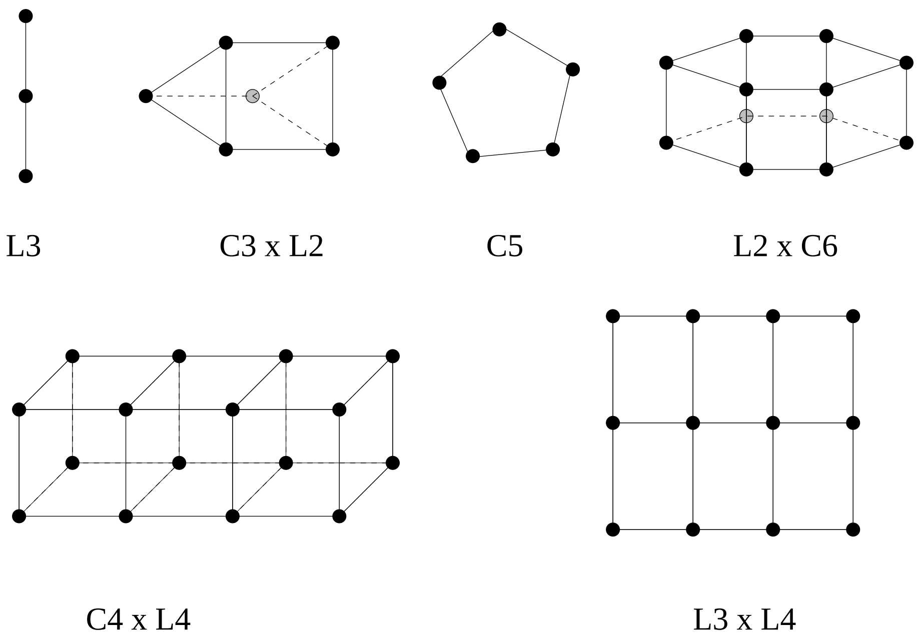

FIG. 1: Examples of ladder graphs.

If $G = g \times L_m$ or $g \times C_m$ is a ladder graph, it is formed with m 'rungs' (subgraphs) all isomorphic to g, connected together. They can be indexed as $g_1, \ldots, g_m$. Let η be a configuration on G. For all $i = 1, \ldots, m$, we denote by η_i its restriction to g_i. Any configuration can be written

$$\eta = (\eta_1, \ldots, \eta_m) \quad .$$

Thus it is interpreted as a configuration on L_m or C_m with state space $\{0,1\}^g$. Actually in the PP case, we shall deal only with admissible configurations, and the state space for configurations on L_m or C_m will be A_g. Our idea is to treat the reversible measure μ_G of the PP on G as a Markov field (see definition 2.7) on L_m or C_m, with state space A_g. This Markov field on G can be interpreted as a Markov chain with state space A_g and time set $\mathbb{Z}$. The compatibility matrix, to be defined below, can be regarded as the combinatorial version of the transition matrix for this chain.

Definition 2.3 *Let $g = (s, e)$ be a finite graph. Two configurations ξ and ζ of A_g are said to be compatible if*

$$\forall x \in s \qquad \xi(x)\,\zeta(x) = 0 \quad .$$

The compatibility matrix P_g of a finite graph g is the square matrix indexed by the elements of A_g defined by

$$\forall \xi, \zeta \in A_g, \qquad P_g(\xi, \zeta) \;=\; \lambda^{\sum_{x \in s} \xi(x)} \quad \textit{if } \xi \textit{ and } \zeta \textit{ are compatible}$$
$$= \; 0 \;\; \textit{otherwise.} \tag{2}$$

There is no natural ordering of the elements of A_g. However we shall agree that the first line of P_g corresponds to the empty configuration, denoted by 0_g. Hence all its elements are equal to 1. The main results concerning line and cycle ladders are given in the two following propositions. They express the reversible measure in terms of powers of the matrix P_g.

Proposition 2.4 *Let $G = g \times L_m$ be a line ladder graph. Then for any configuration $(\eta_1, \dots, \eta_m)$*

$$\mu_G[(\eta_1, \dots, \eta_m)] \;=\; {Z_G}^{-1}\, P_g(0_g, \eta_1)\, \Big(\prod_{i=1}^{m-1} P_g(\eta_i, \eta_{i+1}) \Big)\, P_g(\eta_m, 0_g)\;. \tag{3}$$

The partition function is

$$Z_G \;=\; P_g^{m+1}(0_g, 0_g)\;. \tag{4}$$

The probability of having a given configuration ξ on the i-th rung g_i is

$$\mu_G[\,\eta_i = \xi\,] \;=\; Z_G^{-1}\, P_g^i(0_g, \xi)\, P_g^{m+1-i}(\xi, 0_g)\;. \tag{5}$$

Proposition 2.5 *Let $G = g \times C_m$ be a cycle ladder graph with $m > 1$. Then for any configuration $(\eta_1, \dots, \eta_m)$*

$$\mu_G[(\eta_1, \dots, \eta_m)] \;=\; {Z_G}^{-1} \prod_{i=1}^{m} P_g(\eta_i, \eta_{i+1}) \;\;, \tag{6}$$

with the convention $m + 1 = 1$. The partition function is

$$Z_G \;=\; trace[P_g^m]\;. \tag{7}$$

The probability of having a given configuration ξ on the i-th rung g_i is

$$\mu_G[\,\eta_i = \xi\,] \;=\; Z_G^{-1}\, P_g^m(\xi, \xi)\;. \tag{8}$$

Proof. The first identities in both propositions are easy consequences of definition 2.3, equation (2) and the product form (1). The expressions of Z_G in both cases follow by summing up the previous ones over all configurations. The last assertions are obtained in a similar way.

The compatibility matrix P_g of definition 2.3 is well adapted to algorithmic calculations. Other matrices could yield the same results. Indeed, let us define an equivalence relation $\mathcal{E}$ between matrices indexed by A_g as follows.

$$P' \; \mathcal{E} \; P \quad \Leftrightarrow \quad \forall \xi, \zeta \in A_g, \;\; P'(\xi,\zeta) = P(\xi,\zeta) \, \frac{v(\zeta)}{\alpha \; v(\xi)} \quad ,$$

where α is a positive constant, v a positive function on A_g.
It is easy to see that any matrix P' with $P' \; \mathcal{E} \; P_g$ can replace P_g in propositions 2.4 and 2.5. For instance, it is possible to replace P_g by a symmetric matrix. Another choice is more natural. Remark that P_g is an irreducible non-negative matrix and apply the Perron-Frobenius theorem. Let α be the maximal eigenvalue of P_g and v be a corresponding positive eigenvector. It follows that

$$\begin{aligned} \forall \xi, \zeta \in A_g, \;\; P'(\xi,\zeta) \;\; &= \;\; \frac{v(\zeta)}{\sum v(\eta)} \;\; \text{if } \xi \text{ and } \zeta \text{ are compatible} \\ &= \;\; 0 \qquad\quad\;\; \text{otherwise,} \end{aligned}$$

where the sum in the denominator is over all η compatible with ξ . Then P' is a stochastic matrix, actually the only stochastic matrix satisfying $P' \; \mathcal{E} \; P_g$. It is the transition matrix of a Markov chain with state space A_g.

Propositions 2.4 and 2.5 will be the main tools in the applications to be developed in sections 3 and 4. In section 5, the occurence of a phase transition for the PP on tree ladders will be discussed. For that, we shall use the following identities.

Proposition 2.6 *Let $G = (S, E)$ be a finite undirected graph. Let x be an element of S and C a complete subgraph of G. We denote by $G-x$ (respectively $G-\overline{x}$, $G-C$) the graph obtained by removing x (respectively $\{x\} \cup N(x)$, C) and all adjacent edges from G.*

$$\mu_G[\, \eta(x) = 0 \,] \;\; = \;\; \frac{Z_{G-x}}{Z_G} \tag{9}$$

$$\mu_G[\, \eta(x) = 1 \,] \;\; = \;\; \lambda \, \frac{Z_{G-\overline{x}}}{Z_G} \tag{10}$$

$$
\begin{aligned}
Z_G &= Z_{G-x} + \lambda\, Z_{G-\overline{x}} && (11)\\
Z_G &= Z_{G-C} + \lambda \sum_{x\in C} Z_{G-\overline{x}} \quad . && (12)
\end{aligned}
$$

As for propositions 2.4 and 2.5, the above results can be checked directly from the product form (1).

It may be of interest to replace the above results in the general context of Markov fields. Many definitions of a Markov field can be given (*cf.* Georgii's book [5] for a general reference). The term Markov refers in general to some local dependence. A measure μ can be required for instance to satisfy that the probability of 1 at one given site $x \in S$ should depend only on the values of the neighbors of x (Nearest Neighbor System, *cf.* for instance Suomela [16] definition 2.3 p. 13 or Georgii [5]). One can ask also that the same property of local dependence should be true for any finite subset of S (and not only for singletons). Here is the definition that we shall consider.

Definition 2.7 *Let V be a finite subset of S. We denote by $N(V)$ the set of all neighbors of the elements of V.*

$$N(V) \;=\; \{y \notin V \;s.t.\; \exists x \in V,\; \{x,y\} \in E\} \quad .$$

If η is an element of X, we denote by η^V the restriction of η to V.
A probability measure μ on X is a Markov Random Field (MRF) if and only if, for all finite $V \subset S$, for all finite U such that $V \cup N(V) \subset U$, for all $\eta \in X$,

$$\mu[\,\eta^U\,] \;=\; \mu[\,\eta^{U-V}\,] \quad \mu[\,\eta^V \mid \eta^{N(V)}\,] \quad . \tag{13}$$

This definition is equivalent to that of Georgii [5] p. 157. The notion of a MRF is obviously stronger than that of a Nearest Neighbor System. When all local specifications of μ are positive (in the sense of Preston [13] p. 35) the two notions are equivalent (*cf.* [13] theorem 4.1 p. 27). For a Spin System having strictly positive, nearest neighbor rates, it is well known that any reversible measure is a Nearest Neighbor System. This is essentially the statement of theorem 2.13 chap IV p. 194 of Liggett [10]. See also Kelly [6], theorem 9.3 p. 189. It concerns finite graphs and Spin Systems for which condition (iii) of the theorem is actually equivalent to the strict positivity of the rates. However it is easily checked that Liggett's proof for this result still holds if the flip rates, instead of being all positive, satisfy only, for all $x \in S$ and all $\eta \in X$,

$$c(x,\eta) + c(x,\eta_x) \;\; > \;\; 0 \;\; , \tag{14}$$

where η_x denotes the configuration deduced from η by changing its value at site x. This is obviously the case for the PP. Moreover, the local specifications can be easily deduced from the rates.
One has, for all $x \in S$ and all $\eta \in X$,

$$\mu[\, \eta(x) \mid \eta^{N(x)} \,] \;\; = \;\; \frac{c(x,\eta_x)}{c(x,\eta) + c(x,\eta_x)} \quad . \tag{15}$$

For the PP, this expression is

$$\begin{array}{rcll} \mu[\, \eta(x) \mid \eta^{N(x)} \,] & = & 1 & \text{if } \eta(x) = 0 \text{ and } \Sigma_{y \in N(x)}\, \eta(y) > 0, \\ & = & 0 & \text{if } \eta(x) = 1 \text{ and } \Sigma_{y \in N(x)}\, \eta(y) > 0, \\ & = & (1+\lambda)^{-1} & \text{if } \eta(x) = 0 \text{ and } \Sigma_{y \in N(x)}\, \eta(y) = 0, \\ & = & \lambda(1+\lambda)^{-1} & \text{if } \eta(x) = 1 \text{ and } \Sigma_{y \in N(x)}\, \eta(y) = 0. \end{array} \tag{16}$$

The reversible measures of the PP satisfy a stronger factorization property, which corresponds to the notion of a Clique System (CS) (*cf.* Suomela [16] definition 2.10 p. 17).

Proposition 2.8 *Let μ_G be a reversible measure for the PP on G. For all finite $U \subset S$ and all $\eta \in X$,*

$$\mu_G[\, \eta^U \,] \;\; = \;\; Z^{-1} Q_0(\eta^{N(S-U)}) \prod_C Q_C(\eta^C) \;, \tag{17}$$

where the product extends over all non empty C's which are complete subsets of U (cliques). The terms Q_C are functions on $\{0,1\}^C$ that can be chosen as follows.

$$\begin{array}{rcll} Q_{\{x,y\}}(\eta^{\{x,y\}}) & = & 0 & \textit{if } \{x,y\} \in E \textit{ and } \eta(x) = \eta(y) = 1 \;, \\ Q_{\{x\}}(\eta^{\{x\}}) & = & \lambda & \textit{if } x \in S \textit{ and } \eta(x) = 1 \;, \\ Q_C(\eta^C) & = & 1 & \textit{for any other clique } C \textit{ of } G \;. \end{array}$$

Proof. When G is finite the term Q_0 can be forgotten and (17) is clear from the product form (1). For infinite G, the product form (1) cannot be used. However, (2.8), proposition 2.7 p. 192 in [10] (reversibility for Spin Systems) yields the analogous expression (18) below. If μ_G is a reversible measure for the PP on G then for all finite subsets U and V such that $V \cup N(V) \subset U$, for all $\eta \in X$,

$$\begin{aligned} \mu_G[\, \eta^U \,] &= \lambda^{\sum_{x\in V} \eta(x)}\, \mu_G[\, \eta^{U-V} 0^V \,] \quad \text{if } \eta^U \in A_U \\ &= 0 \qquad\qquad\qquad\qquad\qquad\quad \text{otherwise,} \end{aligned} \tag{18}$$

where A_U is the set of admissible configurations on the subgraph induced by U and $\eta^{U-V}0^V$ is the configuration that coincides with η on $U-V$ and with 0_G on V. Using (18), when $V = U - N(S-U)$ leads to (17).

A Clique System is clearly a Markov field in the sense of definition 2.7. When all local specifications of a measure μ are positive the two notions of MRF and CS are equivalent. This comes for instance from the well known equivalence between positive Markov Random Fields and Gibbs Measures (see Preston [13] theorem 4.1 p. 27 for details). In the general case, the two definitions do not coincide (*cf.* [16] example 2.13 p. 18). Reversible measures for the PP do not have positive local specifications, since only configurations in the set A_G are allowed. However Suomela ([16] theorem 2.15 p. 19) shows that the equivalence still holds under a weaker condition, which is trivially satisfied in the case of the PP. The combinatorial expressions of proposition 2.6 are direct consequences of the factorisation property (17). For a line ladder $G = g \times L_\infty$, the reversible measures of the PP should be viewed as Clique Systems on $\mathbb{Z}$ with state space A_g. Such a CS on $\mathbb{Z}$, is also the probability distribution of a stationary Markov chain with time set $\mathbb{Z}$, and state space A_g (see proposition 4.52 p. 41 in [3]). The transition matrix of this chain is precisely the stochastic matrix P' introduced above (for more details on this relation between Markov fields on $\mathbb{Z}$ and Markov chains, see Georgii [5] p.45 or Spitzer [15]). Markov chains can also be defined on trees (*cf* chapter 12 of [5] or section 3.3 of [3]). For an infinite tree T and $G = g \times T$, not all reversible measures of the PP on G are Markov chains on A_g, indexed by T (see proposition 4.51 p. 40 in [3]).

3 APPLICATIONS.

Obviously propositions 2.4 and 2.5 are of most interest when the compatibility matrix P_g is not too large and can be computed easily. Then taking its m-th power for large m's will be much less expensive than a direct enumeration of the admissible configurations on $g \times L_m$ or $g \times C_m$. In this section we particularize the results of section 2 to some rungs g, namely cliques, lines and cycles.

Let K_n denote the clique with n vertices. For $g = K_n$, the set of admissible configurations A_g contains only $n+1$ elements namely 0_g and the n configurations with exactly one philosopher eating. For the purposes of calculations all non empty configurations are equivalent. Let ξ be any of them. Then the probability that a philosopher belonging to rung g_i is eating (occupation rate) does not depend on which philosopher is eating and is given by $\mu_G[\ \eta_i = \xi\ ]$. The compatibility matrix P_g also has a very particular structure. We first state the results for the line and cycle cases.

Proposition 3.9 *Let $G = K_n \times L_m$ be a line ladder graph. The partition function of μ_G is*

$$\begin{aligned} Z_G &= \frac{1}{\sqrt{\Delta}} \Big[\ \frac{(1+n\lambda)}{2^m} \big[\ (1+(n-1)\lambda+\sqrt{\Delta})^m - (1+(n-1)\lambda-\sqrt{\Delta})^m\ \big] \\ &+ \frac{\lambda}{2^{m-1}} \big[\ (1+(n-1)\lambda+\sqrt{\Delta})^{m-1} - (1+(n-1)\lambda-\sqrt{\Delta})^{m-1}\ \big]\ \Big] \end{aligned}$$

$$\textit{where} \quad \Delta \; = \; 1+2(n+1)\lambda+(n-1)^2\lambda^2\ . \tag{19}$$

Let ξ be any of the non empty configurations in A_g. The probability that a philosopher belonging to rung g_i is eating (occupation rate) is given by

$$\begin{aligned} \mu_G[\ \eta_i = \xi\ ] &= {Z_G}^{-1}\ P_g^i(0_g,\xi)\ P_g^{m+1-i}(\xi,0_g) \\ &= {Z_G}^{-1}\ \lambda\ P_g^i(0_g,\xi)\ P_g^{m+1-i}(0_g,\xi) \quad . \end{aligned} \tag{20}$$

Proposition 3.10 *Let $G = K_n \times C_m$ be a cycle ladder graph with $m > 1$. The partition function of μ_G is*

$$Z_G \;=\; \frac{(n-1)(-2)^m\lambda^m + [1+(n-1)\lambda+\sqrt{\Delta}]^m + [1+(n-1)\lambda-\sqrt{\Delta}]^m}{2^m}$$

$$with \quad \Delta \; = \; 1 + 2(n+1)\lambda + (n-1)^2\lambda^2 \; . \tag{21}$$

Let ξ be any of the non empty configurations in A_g. The probability that a philosopher belonging to rung g_i is eating (occupation rate) is given by

$$\mu_G[\, \eta_i = \xi \,] \;=\; (n \, Z_{K_n \times C_m})^{-1} \left(Z_{K_n \times C_m} - Z_{K_n \times L_{m-1}} \right) \quad . \tag{22}$$

Proofs. The matrix P_g and its powers have a very particular structure. Recall that the first index corresponds to 0_g. Remark that

$$P_g^k = \begin{pmatrix} \alpha_k & \beta_k & \dots & \dots & \beta_k \\ \gamma_k & \delta_k & \epsilon_k & \dots & \epsilon_k \\ \vdots & \epsilon_k & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \epsilon_k \\ \gamma_k & \epsilon_k & \dots & \epsilon_k & \delta_k \end{pmatrix}$$

with $\alpha_1 = \beta_1 = 1$, $\delta_1 = 0$, $\gamma_1 = \epsilon_1 = \lambda$. We first look at the partition functions. They are polynomials in λ whose coefficients depend on the number of rungs m and the clique order n. Thus we need only to compute $P_g^{k+1}(0_g, 0_g) = \alpha_{k+1}$ in the line case and $P_g^k(0_g, 0_g) = \alpha_k$ and $P_g^k(\xi, \xi) = \delta_k$ in the cycle case since $trace[P_g^k] = \alpha_k + n\delta_k$. This can be done recursively.

$$\begin{aligned} \begin{bmatrix} P_g^{k+1}(0_g, 0_g) \\ P_g^{k+1}(0_g, \xi) \end{bmatrix} &= \begin{bmatrix} 1 & n\lambda \\ 1 & (n-1)\lambda \end{bmatrix} \begin{bmatrix} P_g^{k}(0_g, 0_g) \\ P_g^{k}(0_g, \xi) \end{bmatrix} \\ &= \begin{bmatrix} 1 & n\lambda \\ 1 & (n-1)\lambda \end{bmatrix}^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned} \tag{23}$$

and

$$\begin{aligned} \begin{bmatrix} P_g^{k+1}(0_g, \xi) \\ P_g^{k+1}(\xi, \xi) \\ P_g^{k+1}(\zeta, \xi) \end{bmatrix} &= \begin{bmatrix} 1 & 1 & (n-1) \\ \lambda & 0 & (n-1)\lambda \\ \lambda & \lambda & (n-2)\lambda \end{bmatrix} \begin{bmatrix} P_g^{k}(0_g, \xi) \\ P_g^{k}(\xi, \xi) \\ P_g^{k}(\zeta, \xi) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & (n-1) \\ \lambda & 0 & (n-1)\lambda \\ \lambda & \lambda & (n-2)\lambda \end{bmatrix}^k \begin{bmatrix} 1 \\ 0 \\ \lambda \end{bmatrix} \end{aligned} \tag{24}$$

where ξ and ζ are any two distinct non empty configurations.

In the line case solving (23) leads to (19) and in the cycle case (24) leads to (21). Notice that $Z_{K_n \times L_m}$ and $Z_{K_n \times C_m}$ are indeed polynomials in λ since odd powers of $\sqrt{\Delta}$ vanish in expressions (19) and (21).

The partition function is not the only quantity of interest that can be computed explicitly. Occupation rates can also be determined using expressions (5) and (8). When g is the clique K_n, the probability that a philosopher belonging to g_i is equal to 1 is given by $\mu_G[\ \eta_i = \xi\ ]$.

For line ladders $G = K_n \times L_m$, one has

$$\begin{aligned} \mu_G[\ \eta_i = \xi\ ] &= {Z_G}^{-1}\ P_g^i(0_g, \xi)\ P_g^{m+1-i}(\xi, 0_g) \\ &= {Z_G}^{-1}\ \lambda\ P_g^i(0_g, \xi)\ P_g^{m+1-i}(0_g, \xi) \quad . \end{aligned}$$

This expression is invariant under the transformation $i \to m+1-i$, which is consistent with the symmetry in L_m. Expression for $P_g^k(0_g, \xi)$ is already given by (23).

For cycle ladders $G = K_n \times C_m$,

$$\begin{aligned} \mu_G[\ \eta_i = \xi\ ] &= {Z_G}^{-1}\ P_g^m(\xi, \xi) \\ &= (n\ Z_G)^{-1}\ (trace[P_g^m] - P_g^m(0_g, 0_g)) \\ &= (n\ Z_{K_n \times C_m})^{-1}\ (Z_{K_n \times C_m} - Z_{K_n \times L_{m-1}}) \quad . \end{aligned}$$

The expressions in table 1 were computed using formula (19). For simple cases, their validity is checked by directly counting admissible configurations. For $n = 1$, graphs $K_n \times L_m$ are simply line graphs L_m. Table 2 illustrates applications of formula (21). Some graphs can be either interpreted as line or cycle ladders. For instance, $K_2 \times C_2 = K_1 \times C_4$ and $K_2 \times C_3 = K_3 \times C_2$ with $K_2 = C_2 = L_2$. Therefore some results in tables 1 and 2 are redundant. Notice that replacing m by 2 in (19) and (21) leads to the same expression as expected. For $n = 1$ and arbitrary m, expression (21) simplifies into the formula given in Ycart [19], proposition 2.2 p. 36.

We turn now to grids, toruses and cylinders. They can indeed be interpreted as line or cycle ladders by setting $g = L_n$ or $g = C_n$. To determine

$n=1$	$m=2$	1	$+2\lambda$			
	$m=3$	1	$+3\lambda$	$+\lambda^2$		
	$m=4$	1	$+4\lambda$	$+3\lambda^2$		
$n=2$	$m=2$	1	$+4\lambda$	$+2\lambda^2$		
	$m=3$	1	$+6\lambda$	$+8\lambda^2$	$+2\lambda^3$	
	$m=4$	1	$+8\lambda$	$+18\lambda^2$	$+12\lambda^3$	$+2\lambda^4$
$n=3$	$m=2$	1	$+6\lambda$	$+6\lambda^2$		
	$m=3$	1	$+9\lambda$	$+21\lambda^2$	$+12\lambda^3$	
	$m=4$	1	$+12\lambda$	$+45\lambda^2$	$+60\lambda^3$	$+24\lambda^4$
$n=4$	$m=2$	1	$+8\lambda$	$+12\lambda^2$		
	$m=3$	1	$+12\lambda$	$+40\lambda^2$	$+36\lambda^3$	
	$m=4$	1	$+16\lambda$	$+84\lambda^2$	$+168\lambda^3$	$+108\lambda^4$

TABLE 1: Partition functions for line ladders $K_n \times L_m$.

$n=1$	$m=2$	1	$+2\lambda$							
	$m=3$	1	$+3\lambda$							
	$m=4$	1	$+4\lambda$	$+2\lambda^2$						
	$m=5$	1	$+5\lambda$	$+5\lambda^2$						
	$m=6$	1	$+6\lambda$	$+9\lambda^2$	$+2\lambda^3$					
$n=2$	$m=2$	1	$+4\lambda$	$+2\lambda^2$						
	$m=3$	1	$+6\lambda$	$+6\lambda^2$						
	$m=4$	1	$+8\lambda$	$+16\lambda^2$	$+8\lambda^3$	$+2\lambda^4$				
	$m=5$	1	$+10\lambda$	$+30\lambda^2$	$+30\lambda^3$	$+10\lambda^4$				
	$m=6$	1	$+12\lambda$	$+48\lambda^2$	$+76\lambda^3$	$+48\lambda^4$	$+12\lambda^5$	$+2\lambda^6$		
	$m=7$	1	$+14\lambda$	$+70\lambda^2$	$+154\lambda^3$	$+154\lambda^4$	$+70\lambda^5$	$+14\lambda^6$		
	$m=9$	1	$+18\lambda$	$+126\lambda^2$	$+438\lambda^3$	$+810\lambda^4$	$+810\lambda^5$	$+438\lambda^6$	$+126\lambda^7$	$+18\lambda^8$
$n=3$	$m=2$	1	$+6\lambda$	$+6\lambda^2$						
	$m=3$	1	$+9\lambda$	$+18\lambda^2$	$+6\lambda^3$					
	$m=4$	1	$+12\lambda$	$+42\lambda^2$	$+48\lambda^3$	$+18\lambda^4$				
	$m=5$	1	$+15\lambda$	$+75\lambda^2$	$+150\lambda^3$	$+120\lambda^4$	$+30\lambda^5$			
$n=4$	$m=2$	1	$+8\lambda$	$+12\lambda^2$						
	$m=3$	1	$+12\lambda$	$+36\lambda^2$	$+24\lambda^3$					
	$m=4$	1	$+16\lambda$	$+80\lambda^2$	$+144\lambda^3$	$+84\lambda^4$				

TABLE 2: Partition functions for cycle ladders $K_n \times C_m$.

their partition functions, one has to compute the compatibility matrices P_{L_n} and P_{C_n} for lines and cycles. They can be constructed recursively.

Proposition 3.11 *Let k_n be the dimension of P_{L_n} (cardinality of A_{L_n}). One has $k_n = k_{n-1} + k_{n-2}$.*

Let D_n be the $k_{n-2} \times k_{n-1}$ matrix made of the first k_{n-2} rows of $P_{L_{n-1}}$ multiplied by the rate λ.

Let F_n be the $k_{n-1} \times k_{n-2}$ matrix made of the first k_{n-2} columns of $P_{L_{n-1}}$.

Then,

$$P_{L_n} = \left(\begin{array}{c|c} P_{L_{n-1}} & F_n \\ \hline D_n & 0 \end{array} \right) .$$

Remark that this construction induces a natural ordering of the elements of A_{L_n} that will be described in the proof.
The construction of P_{C_n} is related to that of P_{L_n}.

Proposition 3.12 *Let k_n be the dimension of P_{L_n}.*

Let v_n be a vector constructed recursively as follows: the recursion starts at $n = 3$ with $v_3 = (\ 1\ )$. To obtain v_n from v_{n-1}, take numbers in v_{n-1} which are lower or equal to k_{n-4} and add k_{n-3} to them. Then join these values to those in v_{n-1}.

Let D'_n be the $k_{n-3} \times k_{n-1}$ matrix made of k_{n-3} rows of $P_{L_{n-1}}$ multiplied by λ and corresponding to numbers in v_n.

Let F'_n be the $k_{n-1} \times k_{n-3}$ matrix made of k_{n-3} columns of $P_{L_{n-1}}$ given by numbers in v_n.

Then,

$$P_{C_n} = \left(\begin{array}{c|c} P_{L_{n-1}} & F'_n \\ \hline D'_n & 0 \end{array} \right) .$$

Proof. We first examine the respective dimensions of matrices P_{L_n} and P_{C_n}. The partition functions for L_n and C_n are Z_{L_n} and Z_{C_n}. Equation (11) yields the following.

$$\begin{aligned} Z_{L_n} &= Z_{L_{n-1}} + \lambda\, Z_{L_{n-2}} \\ Z_{C_n} &= Z_{L_{n-1}} + \lambda\, Z_{L_{n-3}} \quad . \end{aligned} \tag{25}$$

If λ is set to 1, it appears that the number of admissible configurations on L_n, denoted by k_n, is the n-th term of a Fibonacci sequence starting from $k_0 = 1$ and $k_1 = 2$. It is also the dimension of the matrix P_{L_n}. In addition, the number k'_n of admissible configurations on C_n (and also the dimension of P_{C_n}) is given by $k'_n = k_{n-1} + k_{n-3}$.

A configuration on L_n consists of a value for a single site x_1 and a configuration on L_{n-1}. Configurations on L_n are ordered recursively as described now. Assume an order is given for configurations on L_{n-1}. Then, configurations on L_n which give value 0 to x_1 are put in a first block of size k_{n-1}. Those with $x_1 = 1$ in a second block of size k_{n-2}. Inside the blocks the order of L_{n-1} is respected. So that P_{L_n} consists of four blocks :

$$P_{L_n} = \left(\begin{array}{c|c} P_{L_{n-1}} & F_n \\ \hline D_n & 0 \end{array}\right) \quad .$$

The structure of $P_{L_{n-1}}$ is identical,

$$P_{L_{n-1}} = \left(\begin{array}{c|c} P_{L_{n-2}} & F_{n-1} \\ \hline D_{n-1} & 0 \end{array}\right)$$

and it is easy to see that

$$F_n = \left(\begin{array}{c} P_{L_{n-2}} \\ \hline D_{n-1} \end{array}\right) \quad \text{and } D_n = \lambda \left(\begin{array}{c|c} P_{L_{n-2}} & F_{n-1} \end{array}\right) \quad .$$

So that P_{L_n} can be deduced from $P_{L_{n-1}}$.

The same idea holds for P_{C_n}.

$$P_{C_n} = \left(\begin{array}{c|c} P_{L_{n-1}} & F'_n \\ \hline D'_n & 0 \end{array} \right) \quad .$$

A site x_0 of C_n is chosen. Configurations on C_n with $x_0 = 0$ are ordered before those with $x_0 = 1$. Then the first block in matrix P_{C_n} is $P_{L_{n-1}}$. The block F'_n is made from k_{n-3} columns of $P_{L_{n-1}}$ and D'_n from k_{n-3} rows of $P_{L_{n-1}}$ after multiplication by λ. Let us determine first which columns of $P_{L_{n-1}}$ must be used to construct F'_n. Let v_n be a vector containing these numbers. It can be constructed recursively from v_{n-1}. Columns in F'_n (i.e. numbers in v_n) correspond to admissible configurations on C_n with $x_0 = 1$. Therefore they correspond to configurations on L_{n-1} with extremal sites $x_1 = 0$ and $x_{n-1} = 0$. Configurations with $x_1 = 0$ are in the first block of matrix $P_{L_{n-1}}$ (equal to $P_{L_{n-2}}$). Among them we must find those also ending with $x_{n-1} = 0$. But each configuration on L_{n-1} with $x_1 = 0$ corresponds to a configuration on L_{n-2}. Thus we are looking for configurations on L_{n-2} ending with $x_{n-2} = 0$. We already have numbers in v_{n-1} which are related to configurations on L_{n-2} starting and ending with 0. The numbers in v_{n-1} give the first columns of F'_n. The last ones are configurations on L_{n-2} starting with $x_1 = 1$ and ending with $x_{n-2} = 0$. They are located in the second block of matrix $P_{L_{n-2}}$ and satisfy $x_2 = 0$. The required numbers are determined by taking numbers in v_{n-1} which are lower or equal to k_{n-4} and add k_{n-3} to them. It suffices to join these values to those in v_{n-1} to obtain v_n. The recursion starts at $n = 3$. The matrices $P_{C_1} = P_{L_1}$, $P_{C_2} = P_{L_2}$ and $P_{C_3} = P_{K_3}$ are easily calculated and v_3 is set to $(\ 1\ )$. The first values of v_n are given below.

$$\begin{array}{llll} v_3 = (\ 1\ ) & & k_0 = 1 & k_1 = 2 \\ v_4 = (\ 1, 1+2\ ) & = (\ 1, 3\ ) & k_1 = 2 & k_2 = 3 \\ v_5 = (\ 1, 3, 1+3\ ) & = (\ 1, 3, 4\ ) & k_2 = 3 & k_3 = 5 \\ v_6 = (\ 1, 3, 4, 1+5, 3+5\ ) & = (\ 1, 3, 4, 6, 8\ ) & k_3 = 5 & k_4 = 8. \end{array} \qquad (26)$$

As an illustration we calculated the number of admissible configurations for various graphs. This number is given by the partition function taken at $\lambda = 1$. Some ladders with 100 philosophers are illustrated in table 3. The number of edges is given for each graph, in order to illustrate the intuitive idea that the more edges, the less admissible configurations.

Grids and toruses in three dimensions can also be seen as ladders. It suffices to set g to a 2-dimensional grid or torus. To calculate the partition function, the compatibility matrix for 2-dimensional grids and toruses is required. As for L_n and C_n (1-dimensional grids and toruses), it is possible to construct this matrix recursively. Equations are analogous to those for P_{L_n} and P_{C_n} above but more complex. Details will not be given.

4 HYPERCUBES.

In this section we consider n-dimensional cubes as defined in [1] p. 12. We propose a recursive construction of their compatibility matrix in order to calculate their partition function. Our algorithm is faster than a direct counting of admissible configurations but still has a complexity which is exponential in n. We observe first that cubes Q_n can be constructed recursively.

$$
\begin{aligned}
Q_0 &= L_1 \\
Q_n &= Q_{n-1} \times C_2 && (27) \\
&= Q_{n-2} \times C_2 \times C_2 \quad = \quad Q_{n-2} \times C_4 \quad . && (28)
\end{aligned}
$$

Thus Q_n is a particular case of line ladder with $g = Q_{n-1}$ and also a cycle ladder with $g = Q_{n-2}$. Let P_n be the compatibility matrix of Q_n as defined in (2). The partition function Z_{Q_n} for the Philosophers' Process on Q_n can either be calculated using (27) or (28). Let 0_n be the empty configuration on Q_n. From propositions 2.4 and 2.5, one deduces

$$
\begin{aligned}
Z_{Q_n} &= P_{n-1}^3(0_{n-1}, 0_{n-1}) \\
&= trace[\ P_{n-2}^4\ ] \quad .
\end{aligned}
$$

Our main problem is to calculate the matrix P_n. Its size increases very quickly with n. For instance, P_3 is of dimension 35, P_4 is of dimension 743 and P_5 of dimension 254475. Here is the recursive calculation of P_n.

$L100$	$(99\ edges)$	$927,372,692,193,078,999,176$
$C100$	$(100\ edges)$	$792,070,839,848,372,253,127$
$L2 \times L50$	$(148\ edges)$	$16,616,132,878,186,749,607$
$L2 \times C50$	$(150\ edges)$	$13,765,255,184,676,885,127$
$L4 \times L25$	$(171\ edges)$	$3,706,188,077,591,198,432$
$L4 \times C25$	$(175\ edges)$	$2,662,270,770,083,033,841$
$C4 \times L25$	$(196\ edges)$	$849,258,745,421,311,327$
$C4 \times C25$	$(200\ edges)$	$643,439,082,960,504,561$
$L5 \times L20$	$(175\ edges)$	$2,840,321,953,949,094,447$
$L5 \times C20$	$(180\ edges)$	$1,904,270,890,157,965,061$
$C5 \times L20$	$(195\ edges)$	$635,511,992,964,231,701$
$C5 \times C20$	$(200\ edges)$	$456,085,875,187,977,011$
$L10 \times L10$	$(180\ edges)$	$2,030,049,051,145,980,050$
$L10 \times C10$	$(190\ edges)$	$974,602,314,570,939,359$
$C10 \times C10$	$(200\ edges)$	$498,819,827,260,367,617$
$K4 \times L25$	$(246\ edges)$	$5,527,939,700,884,757$
$K4 \times C25$	$(250\ edges)$	$4,721,424,167,835,361$
$K5 \times L20$	$(295\ edges)$	$233,528,957,822,051$
$K5 \times C20$	$(300\ edges)$	$203,080,369,893,131$
$K10 \times L10$	$(540\ edges)$	$12,010,471,551$
$K10 \times C10$	$(550\ edges)$	$11,035,502,511$
$L5 \times K20$	$(1030\ edges)$	$3,393,261$
$C5 \times K20$	$(1050\ edges)$	$3,240,081$
$K5 \times K20$	$(1150\ edges)$	$2,514,181$
$L4 \times K25$	$(1275\ edges)$	$408,176$
$C4 \times K25$	$(1300\ edges)$	$393,151$
$K4 \times K25$	$(1350\ edges)$	$362,501$
$L2 \times K50$	$(2500\ edges)$	$2,551$
$K100$	$(4950\ edges)$	101

TABLE 3: Numbers of admissible configurations.

Proposition 4.13 *Let e_1, e_2 be configurations on Q_{n-1}. Notation $e_1 \times e_2$ represents the configuration on $Q_n = Q_{n-1} \times L_2$ which coincides with e_1 on the first version of Q_{n-1} and with e_2 on the second.*

If $P_{n-1}(e_1, e_2) \neq 0$ then $e_1 \times e_2 \in A_{Q_n}$ otherwise $e_1 \times e_2$ is not admissible and can be omitted.

Then for admissible states $e_1 \times e_2$ and $e_3 \times e_4$,

$$P_n(e_1 \times e_2, e_3 \times e_4) = P_{n-1}(e_1, e_3) P_{n-1}(e_2, e_4) \quad .$$

Proof. The relationship between P_n and P_{n-1} is based on the equality $g \times C_4 = g \times C_2 \times L_2$ that holds for all g. The graph $g \times C_2$ is constructed by joining two identical versions of g. If e_1, e_2 are configurations on g, $e_1 \times e_2$ represents the configuration on $g \times C_2$ which coincides with e_1 on the first version of g and with e_2 on the second. For instance, $0_n = 0_{n-1} \times 0_{n-1}$. Taking $g = Q_{n-1}$, we have $g \times C_2 = Q_n$. If f_1 and f_2 are configurations on $g \times C_2$, they can be identified to $f_1 = e_1 \times e_2$ and $f_2 = e_3 \times e_4$. Therefore, to construct P_n we first need to list admissible states $e_1 \times e_2$ on Q_n. They consist of products of compatible states of Q_{n-1} and thus satisfy $P_{n-1}(e_1, e_2) \neq 0$. Then we shall look at incompatible states on Q_n. $e_1 \times e_2$ and $e_3 \times e_4$ are not compatible on Q_n if e_1 and e_3 or e_2 and e_4 are not compatible on Q_{n-1}. Proposition 4.13 follows.

Examples. The partition functions for Q_3, Q_4 and Q_5 are given in table 4. For Q_6 we only calculated the number of admissible configurations, which is

$$19,768,832,143.$$

Q_3	1	$+8\lambda$	$+16\lambda^2$	$+8\lambda^3$	$+2\lambda^4$	
Q_4	1	$+16\lambda$	$+88\lambda^2$	$+208\lambda^3$	$+228\lambda^4$	$+128\lambda^5$
	$+56\lambda^6$	$+16\lambda^7$	$+2\lambda^8$			
Q_5	1	$+32\lambda$	$+416\lambda^2$	$+2880\lambda^3$	$+11760\lambda^4$	$+29856\lambda^5$
	$+48960\lambda^6$	$+54304\lambda^7$	$+44240\lambda^8$	$+29920\lambda^9$	$+17952\lambda^{10}$	$+9088\lambda^{11}$
	$+3672\lambda^{12}$	$+1120\lambda^{13}$	$+240\lambda^{14}$	$+32\lambda^{15}$	$+2\lambda^{16}$	

TABLE 4: Partition functions for Q_3, Q_4 and Q_5

5 TREE LADDERS.

Reversible measures for the PP on trees have been investigated by Kelly [7] as a model for conflicts among transactions in a database. That work dealt with finite subgraphs of infinite regular trees (with the same number of edges connected to each vertex). Kelly showed that reversible measures can be constructed using a non-homogeneous Markov chain along the branches of the finite trees. Regarding infinite trees the occurence of a phase transition was demonstrated and a critical value of rate λ was given. In this section tree ladders are considered. They consist of a graph g repeated along a tree. A recursion is first given for the partition function on such graphs for arbitrary g. Then we concentrate on the particular case $g = K_n$ and study what may occur as the tree size increases to infinity. For $n = 1$ (g is a single vertex) Kelly's result on phase transition is recovered. This result is generalized to $g = K_n$ with $n \geq 2$.

Let T_m^d be a finite tree consisting of a distinguished vertex O called root with d edges emanating from it. Each neighbor of O also has d more edges emanating from it and so on. Index m refers to the height of the tree, which is the maximal length of a path from O to a vertex in T_m^d. Such trees can be constructed recursively : T_m^d is obtained by considering d versions of T_{m-1}^d and connecting their roots to a new vertex, the root of T_m^d. Note that this differs slightly from the definition given in Kelly [7]. Kelly considered Cayley trees with the same number r of edges connected to each vertex. In our definition, each vertex has degree $d + 1$ except for the root, which has only degree d. For results concerning infinite trees this difference appears to be negligible. However, it is essential for our calculations that the tree T_m^d can be constructed from copies of T_{m-1}^d by adding one single vertex. This is not satisfied by Cayley trees.

As for lines and cycles (*cf.* definition 2.2) it is possible to define tree ladders.

Definition 5.14 *Let $g = (s, e)$ be a finite undirected graph. We shall refer to the Cartesian product $g \times T_m^d$ as a tree ladder with rung g, degree d and height m.*

The reversible measure μ_G of the PP on such a graph G is a Markov Random Field (or equivalently a Clique System). It is also a MRF on T_m^d provided the state space $B = \{0, 1\}$ is changed into $B = A_g$, the set of admissible configurations on g. The maximum number of vertices in a complete set of a tree is 2. It follows that μ_G admits a factorization such as (3) or (6) in terms of the compatibility matrix P_g indexed on A_g. However for a tree ladder with $d > 1$, the expressions are no longer linear.

Proposition 5.15 *Let $G = g \times T_m^d$ be a tree ladder. Let Z_G be the partition function for the reversible measure on G. Then,*

$$Z_G \;=\; \sum_{\eta_g \in A_g} Z_m(\eta_g) \quad ,$$

where $Z_m(\eta_g)$ are the components of a vector Z_m indexed on A_g and defined recursively by

$$\begin{aligned} \forall m \geq 0, \quad Z_m \;&=\; [\, [\, P_g \,]^{(1/d)}.Z_{m-1} \,]^{(d)} \\ \textit{and} \\ Z_{-1} \;&=\; (\, 1, 0, \ldots, 0 \,)^t \quad , \end{aligned} \tag{29}$$

where for any vector or matrix v, $[\ v\ ]^{(d)}$ denotes the vector or matrix with coefficients equal to those of v elevated to the power d.

Proof. Let us call g_m^r the version of g in $g \times T_m^d$ corresponding to the root and η_m^r the restriction to g_m^r of a configuration η. For any configuration $\eta_g \in A_g$, define

$$Z_m(\eta_g) \;=\; \sum_{\eta \in A_G, \eta_m^r = \eta_g} \lambda^{\sum_{x \in S} \eta(x)} \quad .$$

Then the partition function is given by

$$Z_G = \sum_{\eta_g \in A_g} Z_m(\eta_g) \quad .$$

If P_g is the compatibility matrix defined in (2), it is easy to see that for all $\eta_g \in A_g$

$$Z_m(\eta_g) = (\sum_{\zeta_g \in A_g} (P_g(\eta_g, \zeta_g))^{1/d} \, . \, Z_{m-1}(\zeta_g))^d \quad .$$

The sum above is taken over configurations ζ_g which are compatible with η_g since otherwise $P_g(\eta_g, \zeta_g) = 0$. The power d comes from the fact that once the configuration on the root is fixed to η_g the tree can be separated into d independent parts which are all isomorphic to T^d_{m-1}. This is a way to express the Markov property on trees. Hence (29). For $d = 1$, the line case (formula (4)) is recovered.

From now on, notation referring to g and d will be omitted, so that T_m actually stands for $g \times T^d_m$ and Z_{T_m} for its partition function. We concentrate on $g = K_n$. Since in this case matrix P_g has a special form, (29) leads to a recursion on Z_{T_m}.

Proposition 5.16 *For $m \geq 1$,*

$$Z_{T_m} = (Z_{T_{m-1}})^d + \frac{\lambda}{n^{d-1}} \left[(Z_{T_{m-2}})^d + (n-1) Z_{T_{m-1}} \right]^d \quad , \qquad (30)$$

with

$$\begin{aligned} Z_{T_0} &= Z_{K_n} = 1 + n\lambda \\ Z_{T_{-1}} &= Z_{\emptyset} = 1 \quad . \end{aligned}$$

Proof. It follows from the particular form of P_{K_n} (see section 3) that Z_m has only two different coefficients denoted by $Z_m(0_g)$ and $Z_m(\xi)$ and thus $Z_{T_m} = Z_m(0_g) + n Z_m(\xi)$. Applying (29) leads to

$$\begin{aligned} Z_{T_m} &= (Z_{m-1}(0_g) + n Z_{m-1}(\xi))^d + n\lambda(Z_{m-1}(0_g) + (n-1) Z_{m-1}(\xi))^d \\ &= (Z_{T_{m-1}})^d + n\lambda(Z_{m-1}(0_g) + (n-1) Z_{m-1}(\xi))^d \quad . \end{aligned}$$

On the other hand, as K_n is a complete graph, equation (12) leads to

$$Z_{T_m} = (Z_{T_{m-1}})^d + n\lambda Z_{T_m - \overline{x}} \quad , \qquad (31)$$

where x is an arbitrary vertex of g^r_m.

For $m = 0$, T^d_m reduces to K_n so that $Z_0 = 1 + n\lambda$. This can be recovered from equation (31) using the convention that the partition function of the empty graph is 1. $N(x)$ consists of all vertices in g^r_m and one vertex in g^r_{m-1} for each of the d trees T_{m-1} emanating from g^r_m. Thus,

$$Z_{T_m} = (Z_{T_{m-1}})^d + n\lambda(Z_{T_{m-1}-x})^d \quad ,$$

where x is an arbitrary vertex of g^r_{m-1}.

Hence $Z_{T_{m-1}-x}$ can be identified with $Z_{m-1}(0_g) + (n-1)Z_{m-1}(\xi)$.

Then, it follows from equation (11) that

$$\begin{aligned} Z_{T_m} &= Z_{T_m-x} + \lambda Z_{T_m-\overline{x}} \\ &= Z_{T_m-x} + \lambda(Z_{T_{m-1}-x})^d \quad . \end{aligned}$$

Let us write V_m for Z_{T_m-x}. It follows that

$$\begin{aligned} Z_{T_m} &= (Z_{T_{m-1}})^d + n\lambda(V_{m-1})^d \\ V_m &= Z_{T_m} - \lambda(V_{m-1})^d \quad . \end{aligned}$$

Eliminating V_m and V_{m-1} leads to (30).

Let us turn to the problem of phase transitions on infinite tree ladders. Let T be the infinite tree obtained by letting m tend to infinity in T_m: $T = K_n \times T^d_\infty$. The PP on T may have more than one reversible measure (phase transition). Here is a sufficient condition for this to happen.

Proposition 5.17 *For $d \geq n \geq 2$ and $\lambda > \lambda_c$, with*

$$\lambda_c = (d+1)^d/(d^d\,(d-n+1)) \quad , \tag{32}$$

the PP on T has more than one reversible measure.

Proof. The general idea to study the reversible measures of the PP on T is the same as for the stochastic Ising model on an infinite set of sites (*cf.* Liggett [10] theorem 1.16 p. 184). For a fixed m, one has to put boundary conditions on the leaves of T_m. Each possible boundary condition leads to a probability measure on T, which is the reversible measure of the PP on T_m conditioned to that boundary condition. Let $\mathcal{R}_m$ be their convex hull. Let $\mathcal{R}$ be the set of reversible measures for the PP on T. One has

$$\mathcal{R} \;=\; \bigcap_m \mathcal{R}_m \quad .$$

If $\mathcal{R}$ is reduced to a single measure μ_T then any sequence μ_m of reversible measures on T_m has to converge to μ_T, whatever the boundary conditions. In order to prove that there is a phase transition, it is enough to exhibit one set of boundary conditions leading to a sequence μ_m which does not converge. And to prove that μ_m does not converge, it will be enough to look at probabilities of configurations on the root of the tree.

The most natural choice for a boundary condition is to set it to the empty configuration. But if the leaves of T_m are bound to be empty, the induced measure is the reversible measure of the PP on T_{m-1}. Hence there will be a phase transition once it is shown that $\mu_m[\ \eta^r_m = 0_g\ ]$ has no limit when m tends to infinity. Therefore we first look at a_m defined as

$$a_m \quad = \quad \mu_m[\ \eta^r_m = 0_g\ ] \;=\; \frac{(Z_{m-1})^d}{Z_m} \quad .$$

The last equality comes from equation (9) in proposition 2.6.

Using recursion (30), it follows that

$$a_m \;=\; \frac{1}{1+\frac{\lambda}{n^{d-1}}\,[a_{m-1}+(n-1)]^d} \quad . \tag{33}$$

Setting n to 1 in (33) gives equation (2.9) p. 383 in Kelly [7]. Recursion (33) has a unique fixed point in $]0,1[$. There is a phase transition if this fixed point is not stable. This corresponds to some critical value for the rate λ. For $n=1$ Kelly's result [7] p. 383 is recovered. For $n \geq 2$ the empty boundary condition does not lead to a good sufficient condition. Better results are obtained with a non symmetric condition. The tree ladder T can be viewed as n copies of T^d_∞ connected together. We distinguish one of these copies and label by x all its vertices so that in each of the cliques K_n composing T, one philosopher has label x and all philosophers labelled x are connected together along the distinguished copy of T^d_∞. As a boundary condition, we decide that on the leaves of T_m all philosophers labelled x are eating. We want to know whether or not this affects the probability for the philosopher x in the root to be eating. The probabilities of configurations on the root can be described by three quantities. Let α_m be the probability of the empty configuration. It is distinct from a_m. Let β_m be the probability that philosopher x is eating and γ_m the probability that some philosopher different from x is eating. Similarly to (33) one obtains the following equations:

$$\alpha_m + \beta_m + (n-1)\gamma_m \;=\; 1 \tag{34}$$

$$\beta_m \;=\; \lambda\, \alpha_m\, (1-\beta_{m-1})^d \tag{35}$$

$$\gamma_m \;=\; \lambda\, \alpha_m\, (1-\gamma_{m-1})^d \quad . \tag{36}$$

Equations (35) and (36) can be rewritten in the form of a two dimensional discrete dynamical system.

$$\begin{aligned} \beta_m &= \lambda\,(1-\beta_{m-1})^d\,/(1+\lambda(1-\beta_{m-1})^d+(n-1)\lambda(1-\gamma_{m-1})^d) \\ \gamma_m &= \lambda\,(1-\gamma_{m-1})^d\,/(1+\lambda(1-\beta_{m-1})^d+(n-1)\lambda(1-\gamma_{m-1})^d) \quad . \end{aligned} \tag{37}$$

If the reversible measure is unique then limits for β_m and γ_m exist and are equal. Thus a sufficient condition for the occurence of a phase transition can be found by studying the fixed point stability for this system. This can be done by considering the eigenvalues of the Jacobian matrix of the system (37). Applying theorem 7.1 p. 369 of Sandefur [14], one obtains that for $d \geq n \geq 2$ and $\lambda > \lambda_c$ given by (32) the fixed point is not stable, hence the PP has more than one reversible measure. Details of calculations are not given here (*cf.* [4]).

Note that for $n = 1$ system (37) simplifies and the stability condition leads to the critical value given in Kelly [7] p. 383.

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