Stochastic comparison for Markov processes on a product of partially ordered sets.

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Abstract

Classical stochastic comparison results for spin systems are generalized to more complex interacting particle systems. The state space of each site is a finite partially ordered set instead of $\{0,1\}$ and several sites may change status at a time.

Key Words : Interacting particle system, Spin systems with migration, Stochastic comparison, coupling

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1 Introduction.

We consider interacting particle systems with a finite phase space and a general type of interaction. In many large systems, the global dynamics are defined as a superposition of local dynamics involving only few components of the system. Such situations arise, for instance, in the modelling of systems of repairable components (Cocozza-Thivent and Roussignol, 1995) and of resource sharing systems (Forbes, François and Ycart, 1996). Stochastic comparison is a useful tool in the study of such systems. It is the subject of many other works for which an overview can be found in (Stoyan, 1983), (Liggett, 1985) or (Lindvall, 1992). We shall borrow both notations and formalism on interacting particle systems from the book by Liggett (1985). In that reference, section 1 of chapter II is devoted to applications of stochastic comparison and coupling techniques to spin systems for which configurations, with value in $\{0,1\}$, can change at one site at a time. Our objective is to extend the standard theory to more general interacting particle systems where many particles can change status at a time and where the state space for each particle is an arbitrary finite partially ordered set. In this article, we give new conditions for stochastic comparison which are relevant to various applications. Throughout the paper, S will denote a countable set of sites. For all x in S, W_x will be a finite set endowed with a partial order $\leq_x$. Interacting particle systems are Feller processes on the state space $X = \bigotimes_{x\in S} W_x$. For each subset T of S, the order naturally extends to a partial order $\leq_T$ on $X_T = \bigotimes_{x\in T} W_x$,

$$\forall \alpha,\beta \in X_T, \qquad \alpha \leq_T \beta \quad\Leftrightarrow\quad \forall x \in T, \quad \alpha(x) \leq_x \beta(x)\,.$$

The global dynamics of interacting particle systems are specified by a collection of local transition rates $\{c_T(\eta,\gamma),\ \eta\in X,\ T\in\mathcal{P}_f(S),\ \gamma\in X_T\}$. Here, the notation $\mathcal{P}_f(S)$ refers to the set of finite subsets of S. For all T in $\mathcal{P}_f(S)$ and all η in X, $c_T(\eta,\gamma)$ is a continuous function of η (X is endowed as usual with the product topology). The local dynamics are assumed to determine the evolution of the system as a whole in a unique way. Existence and uniqueness conditions can be found in (Liggett, 1985, chap. I p. 27). Under such conditions, an interacting particle system is associated to a Markov generator Ω . Let $\mathcal{D}$ be the set of functions on X depending on a finite number of coordinates. On $\mathcal{D}$, Ω is defined by

$$\forall f\in\mathcal{D},\ \forall \eta\in X, \quad \Omega f(\eta) = \sum_{T\in\mathcal{P}_f(S)} \Omega_T f(\eta)\ , \tag{1}$$

$$\text{with} \qquad \Omega_T f(\eta) = \sum_{\gamma\in X_T} c_T(\eta,\gamma)\left(f(\eta_T^\gamma) - f(\eta)\right)\ , \tag{2}$$

$$\text{and} \qquad \eta_T^\gamma(x) = \begin{cases} \gamma(x) & \text{if } x\in T \\ \eta(x) & \text{otherwise.}\end{cases}$$

The corresponding Feller semigroup acting on the set $\mathcal{C}(X)$ of continuous functions on X will be denoted by $S(t)$. The paper is organized as follows. Section 2 is devoted to basic material on stochastic comparison and coupling techniques. We shall then successively consider two ways of obtaining comparison conditions. The first one is based on a general result from Massey (1987) who studied stochastic comparison for Markov processes on countable partially ordered sets. In (Massey, 1987) necessary and sufficient conditions to compare two such processes are given. They involve increasing subsets (see definition 3.1 below) of the state space X and are not of great help when an infinite or a large state space is considered. They are not adapted to interacting particle systems since they do not take into account the locality of the rates. Therefore we shall give conditions for stochastic comparison that are local in the sense that they bear on the local rates and not on the rates of the processes as a whole. Our main results (theorems 3.2, 3.3 and 3.4) are presented in section 3. Theorem 3.2 gives general local sufficient conditions. Theorems 3.3 and 3.4 concern respectively spin-like systems and spin systems with migration. In the first case, a single site will be allowed to change status at a time. For such systems, we shall prove (theorem 3.3) that the sufficient conditions of theorem 3.2 are also necessary. In the second case, the phase space is $\{0,1\}$. The processes are built up by superposition of a spin system and a migration process. The sufficient local conditions of theorem 3.2 apply but do not lead to satisfactory results. However, in this particular case, it is possible to use Massey's result to find necessary and sufficient conditions of a different type (theorem 3.4). Although expressed in terms of the local rates, the conditions in theorems 3.2, 3.3 and 3.4 may remain difficult to check since they compare sums of rates and not individual rates. More explicit conditions can be given. They can be recovered using a second method that consists in constructing Markov increasing couplings via their generators on $X \times X$. This is illustrated in section 4 . We first study a generalization (introduced by Cocozza and al. (1995) or Francois (1992)) of the so-called basic coupling or Vasershtein coupling. We also give conditions (theorem 4.2) for the existence of Markov increasing couplings for spin systems with migration. The conditions in section 4 are more restrictive but they are generally easier to check. The points of view developed in sections 3 and 4 are not independent. As a conclusion we mention some cases for which they are equivalent.

2 Definitions and basics results.

Recall some basic tools for stochastic comparison and coupling. The notation $\alpha <_T \beta$ will often be used instead of $\alpha \leq_T \beta$ and $\alpha \neq \beta$. The notation $\alpha \sim_T \beta$ will be used when α and β are comparable with $\leq_T$ in the sense that $\alpha \leq_T \beta$ or $\beta \leq_T \alpha$. The opposite case will be denoted by $\alpha \not\sim_T \beta$. When no confusion arises, subscripts T will be dropped. The partial order on X induces a stochastic order $\leq_{st}$ on the set of probability measures on X. We denote by $\mathcal{M}(X)$ the set of continuous functions on X that are increasing for the partial order $\leq_X$ on X.

Definition 2.1 *Two measures μ_1 and μ_2 on X satisfy $\mu_1 \leq_{st} \mu_2$ whenever*

$$\forall f \in \mathcal{M}(X) \quad , \quad \int f d\mu_1 \leq \int f d\mu_2 \ .$$

A way to compare Feller processes on X, is to exhibit increasing couplings.

Definition 2.2 *Consider two Feller processes on X with respective semigroups $S^1(t)$ and $S^2(t)$. A coupling is a process (η_t, ζ_t) on $X \times X$ such that the marginal processes η_t and ζ_t are separately Markovian on X with respective semigroups $S^1(t)$ and $S^2(t)$. The coupling is said to be increasing if in addition, for all η, ζ in X, $\eta \leq \zeta$ implies*

$$\forall t \geq 0 \ , \qquad P[\ \eta_t \leq \zeta_t \mid (\eta_0, \zeta_0) = (\eta, \zeta)\ ] = 1 \ . \tag{3}$$

Feller processes on X can also be compared through their semigroups. If certain inequalities between the distributions of the processes hold initially, then they continue to hold at later times.

Definition 2.3 *Two Feller semigroups $S^1(t)$ and $S^2(t)$ satisfy $S^1(t) \leq_{st} S^2(t)$ iff for all probability measures μ_1 and μ_2 on X,*

$$\mu_1 \leq_{st} \mu_2 \quad \Rightarrow \quad \forall t \geq 0 \; , \quad \mu_1 S^1(t) \leq_{st} \mu_2 S^2(t) \; . \tag{4}$$

Definitions 2.2 and 2.3 are actually equivalent. Theorem 2.1 gives a third formulation.

Theorem 2.1 *Consider the Feller semigroups $S^1(t)$ and $S^2(t)$. The following statements are equivalent.*
- (a) *There exits an increasing coupling of $S^1(t)$ and $S^2(t)$.*
- (b) $S^1(t) \leq_{st} S^2(t)$.
- (c) $\eta \leq \zeta \Rightarrow \quad \forall t \geq 0, \, \forall f \in \mathcal{M}(X) \; , \quad S^1(t)f(\eta) \leq S^2(t)f(\zeta)$.

Proof. The equivalence between (a) and (b) comes from theorem 2.4 chap. I p. 72 in (Liggett, 1985). Assertion (b) implies (c) since for $\eta \leq \zeta$ the pointmasses δ_η and δ_ζ satisfy $\delta_\eta \leq_{st} \delta_\zeta$. For the other direction, assume that (c) holds and $\mu_1 \leq_{st} \mu_2$. By Strassen's theorem (see also theorem 2.4 chap. I p. 72 in (Liggett, 1985)), there exists a coupling measure ν on $X \times X$ for μ_1 and μ_2, which satisfies $\nu[\, (\eta, \zeta) : \eta \leq \zeta \,] = 1$.
For f in $\mathcal{M}(X)$, it comes

$$\begin{aligned}
\int_X f d[\mu_2 S^2(t)] - \int_X f d[\mu_1 S^1(t)] &= \int_X S^2(t) f d[\mu_2] - \int_X S^1(t) f d[\mu_1] \\
&= \int\int_{X \times X} [S^2(t) f(\zeta) - S^1(t) f(\eta)] \nu[d\eta, d\zeta] \\
&= \int\int_{\{\eta \leq \zeta\}} [S^2(t) f(\zeta) - S^1(t) f(\eta)] \nu[d\eta, d\zeta] \; ,
\end{aligned}$$

which is non negative since $S^2(t)f(\zeta) - S^1(t)f(\eta)$ is non negative for all $\eta \leq \zeta$. □

In our setting a coupling is not necessarily a Markov process on $X \times X$. Markov couplings can be constructed explicitly by giving their generators on $X \times X$. The rates for the coupling depend on the local dynamics of the processes to be compared. If the rates for the transitions that would take the process out of $K = \{(\eta, \zeta) \in X \times X, \eta \leq \zeta\}$ are zero, then the coupling is increasing. This is illustrated in section 4 . In the following section we shall give conditions (theorems 3.2, 3.3 and 3.4) to compare semigroups in the sense of definition 2.3 . These conditions will ensure the existence of an increasing coupling (not necessarily Markovian) without giving it explicitly.

3 Conditions for increasing couplings.

3.1 Local sufficient conditions.

Massey (1987) gives necessary and sufficient conditions to compare Markov processes on a countable partially ordered state space X. These conditions are written in terms of the rates of the processes and of increasing subsets of X.

Definition 3.1 *A subset Γ of X is increasing if it satisfies*

$$\eta \in \Gamma \text{ and } \eta \leq \zeta \quad \Rightarrow \quad \zeta \in \Gamma \; .$$

It is decreasing if it satisfies

$$\eta \in \Gamma \text{ and } \eta \geq \zeta \quad \Rightarrow \quad \zeta \in \Gamma \ .$$

Massey's theorem (5.3 p. 359 of (Massey, 1987)) is recalled below.

Theorem 3.1 *For $i = 1, 2$ consider Markov processes on a countable partially ordered state space X, with rates $\{c^i(\eta,\gamma), \eta, \gamma \in X\}$ and semigroups $S^i(t)$. They satisfy*

$$S^1(t) \leq_{st} S^2(t)$$

iff for all η, ζ in X with $\eta \leq \zeta$ and all increasing sets Γ such that $\eta \in \Gamma$ or $\zeta \notin \Gamma$,

$$\sum_{\gamma\in\Gamma} c^1(\eta,\gamma) \quad \leq \quad \sum_{\gamma\in\Gamma} c^2(\zeta,\gamma) \quad . \tag{5}$$

For large partially ordered X, increasing subsets may be difficult to enumerate and condition (5) is not easily checked. Massey's theorem is not of great help when an infinite or a finite but large S is considered. However, one can try to turn the fact that for such systems only particular transitions are allowed into account. The global dynamics is a superposition of local dynamics specified by the rates $c^i_T(\eta,\gamma)$. The subsets T involved are usually much smaller than S. In spin systems for instance they are all singletons. In theorem 3.2, sufficient conditions under which interacting particle systems can be compared are given. They take into account the locality of the rates. In the following developments, $\eta(T)$ will denote the restriction to T of configuration η.

Theorem 3.2 *For $i = 1, 2$ consider interacting particle systems with rates $c^i_T(\eta,\gamma)$ and semigroups $S^i(t)$. If for all T in $\mathcal{P}_f(S)$, for all η, ζ in X with $\eta \leq \zeta$,*

(*a*) *for all increasing subsets Γ_T of X_T such that $\eta(T) \notin \Gamma_T$ and $\zeta(T) \notin \Gamma_T$,*

$$\sum_{\gamma\in\Gamma_T} c^1_T(\eta,\gamma) \quad \leq \quad \sum_{\gamma\in\Gamma_T} c^2_T(\zeta,\gamma) \tag{6}$$

(*b*) *for all decreasing subsets Γ_T of X_T such that $\eta(T) \notin \Gamma_T$ and $\zeta(T) \notin \Gamma_T$,*

$$\sum_{\gamma\in\Gamma_T} c^1_T(\eta,\gamma) \quad \geq \quad \sum_{\gamma\in\Gamma_T} c^2_T(\zeta,\gamma) \tag{7}$$

then $S^1(t) \leq_{st} S^2(t)$.

Proof. In order to simplify formulations in the proof, we remark that conditions (6) and (7) can be written under the following more compact form (8).

For all T in $\mathcal{P}_f(S)$, for all η, ζ in X with $\eta \leq \zeta$ and for all increasing subsets Γ_T of X_T such that $\eta(T) \in \Gamma_T$ or $\zeta(T) \notin \Gamma_T$,

$$\sum_{\gamma\in\Gamma_T} c^1_T(\eta,\gamma) \quad \leq \quad \sum_{\gamma\in\Gamma_T} c^2_T(\zeta,\gamma) \ . \tag{8}$$

Theorem 3.2 is first proved for finite S using theorem 3.1 . Then it is extended to infinite S using a technique of finite approximations.

Let S be finite. Then X is also finite. Assume (8) holds and let T be a subset of S. For all increasing subsets Γ of X such that $\eta \in \Gamma$ or $\zeta \notin \Gamma$, consider the subsets Γ_T^η and Γ_T^ζ of X_T respectively defined by $\Gamma_T^\eta = \{\gamma \in X_T\,,\, \eta_T^\gamma \in \Gamma\}$ and $\Gamma_T^\zeta = \{\gamma \in X_T\,,\, \zeta_T^\gamma \in \Gamma\}$. They are both increasing subsets of X_T. Using (8) with $\Gamma_T = \Gamma_T^\eta$ and noticing that $\Gamma_T^\eta \subset \Gamma_T^\zeta$ yields

$$\sum_{\gamma \in \Gamma_T^\eta} c_T^1(\eta, \gamma) \quad \leq \quad \sum_{\gamma \in \Gamma_T^\zeta} c_T^2(\zeta, \gamma)\;, \tag{9}$$

for all η, ζ in X with $\eta \leq \zeta$ and all increasing subsets Γ of X such that $\eta \in \Gamma$ or $\zeta \notin \Gamma$. Since X is finite, by theorem 3.1 this last statement is equivalent to $S_T^1(t) \quad \leq_{st} \quad S_T^2(t)$, where $S_T^i(t)$ $(i = 1, 2)$ are the semigroups of the T-local processes. Since this is true for all subsets T of S, summing each side of (9) over T leads to (5) and thus to

$$S^1(t) \quad \leq_{st} \quad S^2(t)\;.$$

Therefore, for a finite S, to compare interacting particle systems on S it is enough to check conditions (8) for all subsets T of S. To show that this remains valid for infinite S, we shall introduce frozen processes (definition 3.2) analogous to finite approximations for spin systems, defined in (Liggett, 1985, chap. III p. 138).

Definition 3.2 *Consider an interacting particle system on S with rates*

$$\{c_T(\eta, \gamma),\, \eta \in X,\, T \in \mathcal{P}_f(S),\, \gamma \in X_T\}$$

and generator Ω. Let V be a finite subset of S and η in X. The process frozen to η on $S \backslash V$ is the interacting particle system on S with rates

$$\{c_T(\eta_V^{\alpha(V)}, \gamma)\,,\, \alpha \in X\,,\, T \subset V\,,\, \gamma \in X_T\}\;.$$

Lemma 3.1 below shows that it is sufficient to compare frozen processes for a collection of finite subsets increasing to S. Actually, condition (10) below is weaker than saying that the frozen processes are comparable.

Lemma 3.1 *For $i = 1, 2$ consider interacting particle systems on S with rates $c_T^i(\eta, \gamma)$. Let (T_n) be a sequence of finite sets increasing to S. For all n and all $\eta \in X$, denote by $S_{n,\eta}^i(t)$ the semigroup of the process frozen to η on $S \backslash T_n$. If for all $\eta \leq \zeta$ and for all n,*

$$\forall f \in \mathcal{M}(X)\;, \quad S_{n,\eta}^1(t) f(\eta) \quad \leq \quad S_{n,\zeta}^2(t) f(\zeta) \tag{10}$$

then $S^1(t) \leq_{st} S^2(t)$.

Proof. For all n and for all η in X, denote by $\Omega_{n,\eta}^i$ the generator of the process frozen to η on $S \backslash T_n$. It is easy to check that

$$\forall f \in \mathcal{D},\, \forall \eta \in X\;, \quad \lim_{n \to \infty} \Omega_{n,\eta}^i f \quad = \quad \Omega^i f\;.$$

By theorem 2.12 in (Liggett, 1985), this leads to

$$\forall f \in \mathcal{C}(X),\, \forall \eta \in X\;, \quad \lim_{n \to \infty} S_{n,\eta}^i(t) f \quad = \quad S^i(t) f\;, \tag{11}$$

uniformly for t in compact sets. Let $\eta \leq \zeta$. If for all n,

$$\forall f \in \mathcal{M}(X) \ , \quad S^1_{n,\eta}(t)f(\eta) \quad \leq \quad S^2_{n,\zeta}(t)f(\zeta) \ .$$

Taking the limit in n leads to

$$\forall f \in \mathcal{M}(X) \ , \quad S^1(t)f(\eta) \quad \leq \quad S^2(t)f(\zeta) \ ,$$

which by theorem 2.1 proves that $S^1(t) \ \leq_{st} \ S^2(t)$. □

Frozen processes are processes on S but they can be identified to processes on finite subsets of S. The process frozen to η on $S\backslash V$ can be associated to the interacting particle system on V with rates $\{c_T(\eta^\alpha_V, \gamma),\, \alpha \in X_V,\, T \subset V,\, \gamma \in X_T\}$. Moreover, it is easy to show that equation (10) between frozen processes can be obtained by comparing the associated processes on T_n. The proof of theorem 3.2 for infinite S follows. Take $\eta \leq \zeta$ and let T_n be finite sets increasing to S. For all T_n consider the frozen processes to η and ζ on T_n with semigroups $S^1_{n,\eta}(t)$ and $S^2_{n,\zeta}(t)$. Since T_n is finite and (8) is satisfied for all subsets T of T_n, we can use the statement of the theorem for $S = T_n$ to show that for all $\eta \leq \zeta$,

$$\forall f \in \mathcal{M}(X) \ , \quad S^1_{n,\eta}(t)f(\eta) \quad \leq \quad S^2_{n,\zeta}(t)f(\zeta) \ .$$

The conclusion is then given by lemma 3.1.

□

In the next subsection, we shall consider particular cases for which theorem 3.2 can be improved in the sense that the sufficient conditions (6) and (7) turn to be also necessary.

3.2 Necessary and sufficient conditions in the spin-like case.

In spin-like systems, $c_T(\eta, \gamma) = 0$ if $|T| \geq 2$. When a single site can change at a time, we rewrite the rates $c^i_{\{x\}}(\eta, \gamma)$ as $c^i(x, \eta, \gamma)$ to fit Liggett's notation. For such processes, the sufficient conditions of theorem 3.2 are also necessary.

Theorem 3.3 *For $i = 1, 2$ consider interacting particle systems with rates*

$$\{c^i(x, \eta, \gamma) \ , \ \eta \in X \ , \ x \in S \ , \ \gamma \in W_x\}$$

and semigroups $S^i(t)$. They satisfy $S^1(t) \leq_{st} S^2(t)$ iff for all x in S, for all η, ζ in X with $\eta \leq \zeta$,

(a) *for all increasing subsets Γ_x of W_x such that $\eta(x) \notin \Gamma_x$ and $\zeta(x) \notin \Gamma_x$,*

$$\sum_{\gamma \in \Gamma_x} c^1(x, \eta, \gamma) \quad \leq \quad \sum_{\gamma \in \Gamma_x} c^2(x, \zeta, \gamma)$$

(b) *for all decreasing subsets Γ_x of W_x such that $\eta(x) \notin \Gamma_x$ and $\zeta(x) \notin \Gamma_x$,*

$$\sum_{\gamma \in \Gamma_x} c^1(x, \eta, \gamma) \quad \geq \quad \sum_{\gamma \in \Gamma_x} c^2(x, \zeta, \gamma) \ .$$

Proof. Assume $S^1(t) \leq_{st} S^2(t)$. Let x be in S and η, ζ in X such that $\eta \leq \zeta$. Let Γ_x be an increasing subset of W_x such that $\eta(x) \notin \Gamma_x$ and $\zeta(x) \notin \Gamma_x$. Consider the indicator function on X, $g = \mathbb{1}_{\{\xi(x) \in \Gamma_x\}}$. The function g is in $\mathcal{M}(X)$ since Γ_x is increasing. It is in $\mathcal{D}$ since it depends only on the configuration at site x. For $i = 1, 2$ the generators of the processes satisfy for all f in $\mathcal{D}$ and all η in X,

$$\Omega^i f(\eta) \quad = \quad \lim_{t \searrow 0} (S^i(t) f(\eta) - f(\eta))/t \; .$$

Since $g(\eta) = g(\zeta) = 0$ and $S^1(t)g(\eta) \leq S^2(t)g(\zeta)$ it comes $\Omega^1 g(\eta) \leq \Omega^2 g(\zeta)$. Then, it is easy to check that $\Omega^1 g(\eta) = \sum_{\gamma \in \Gamma_x} c^1(x, \eta, \gamma)$ and $\Omega^2 g(\zeta) = \sum_{\gamma \in \Gamma_x} c^2(x, \zeta, \gamma)$. Thus (a) is satisfied. The proof is analogous for (b) with Γ_x decreasing and $g = \mathbb{1}_{\{\xi(x) \notin \Gamma_x\}}$.
The other direction is a straightforward application of theorem 3.2 .

□

Comment. More generally, the same holds as soon as the sets T on which the transitions occur do not overlap.
In the following subsection, we present a case for which theorem 3.2 applies but leads to too restrictive conditions. For this case, it is possible to give other necessary and sufficient conditions. They are of a different type and illustrate another way to use Massey's theorem.

3.3 Necessary and sufficient conditions for spin systems with migration.

A spin system with migration on S is an interacting particle system on $X = \{0,1\}^S$, obtained by superposition of a spin system and an exchange system. An exchange system can be seen as a generalization of the exclusion process as presented in chapter VIII of (Liggett, 1985). In an exchange system, only transitions on pairs of sites with different spins are allowed. The exchange of two identical spins has actually no effect on the current configuration. The rates of an exchange system satisfy

$$\begin{aligned} c_T(\eta, \gamma) &= 0 \quad \text{if } |T| \neq 2 \\ c_{\{x,y\}}(\eta, \gamma) &= 0 \quad \text{if } \eta(x) = \eta(y) \text{ or } \gamma(x) \neq \eta(y) \text{ or } \gamma(y) \neq \eta(x). \end{aligned}$$

The positive transition rates are written, for convenience of notation, $c(xy, \eta)$ instead of $c_{\{x,y\}}(\eta, (\eta(y), \eta(x))$. In addition, for each site x we denote by $\mathcal{S}(x)$ the set of sites that can exchange their spin with x, $\mathcal{S}(x) = \{y \in S \text{ s.t. } \sup_{\eta \in X} c(xy, \eta) > 0\}$. A spin system with rates $c(x, \eta)$ (following the notations of Liggett) is superposed to an exchange system. The rates of the corresponding spin system with migration are then denoted by $\{c(x, \eta), c(xy, \eta)\}$. The study of this case is motivated by interesting applications to contact processes with migration. However this would require further developments which are out of the scope of this paper. In this part, necessary and sufficient conditions are given to compare two spin systems with migration. For all x in S, the sets $\mathcal{S}(x)$ are assumed to be finite and equal for both processes. For these processes we can introduce the following notation, for $i = 0, 1$

$$\begin{aligned} \mathcal{V}_i(x, \eta, \zeta) &= \{y \in \mathcal{S}(x) \text{ with } \eta(y) = \zeta(y) = i\} \; , \\ \mathcal{V}_i(x, \eta) &= \{y \in \mathcal{S}(x) \text{ with } \eta(y) = i\} \; . \end{aligned}$$

As previously, the necessary and sufficient conditions of comparison can be deduced from Massey's theorem.

Theorem 3.4 *For $i = 1, 2$ consider spin systems with migration, with transition rates $\{c^i(x,\eta), c^i(xy,\eta)\}$ and semigroups $S^i(t)$. They satisfy $S^1(t) \leq_{st} S^2(t)$ iff for all η, ζ in X with $\eta \leq \zeta$,*

$\forall x \in S,\ \eta(x) = \zeta(x) = 0,\quad and\ \forall\ V \subset \mathcal{V}_1(x,\eta,\zeta),$

$$c^1(x,\eta) + \sum_{y \in \mathcal{V}_1(x,\eta)\setminus V} c^1(xy,\eta) \quad \leq \quad c^2(x,\zeta) + \sum_{y \in \mathcal{V}_1(x,\zeta)\setminus V} c^2(xy,\zeta) \tag{12}$$

$\forall x \in S,\ \eta(x) = \zeta(x) = 1,\quad and\ \forall\ V \subset \mathcal{V}_0(x,\eta,\zeta),$

$$c^1(x,\eta) + \sum_{y \in \mathcal{V}_0(x,\eta)\setminus V} c^1(xy,\eta) \quad \geq \quad c^2(x,\zeta) + \sum_{y \in \mathcal{V}_0(x,\zeta)\setminus V} c^2(xy,\zeta) \tag{13}$$

Proof. Let us first show that conditions (12) and (13) are necessary. Assume that $S^1(t) \leq_{st} S^2(t)$. Let η, ζ in X be such that $\eta \leq \zeta$ and x in S such that $\eta(x) = \zeta(x) = 0$. Let V be a subset of $\mathcal{V}_1(x,\eta,\zeta)$. Consider the indicator function on X, $g(\xi) = \xi(x) \prod_{y \in V} \xi(y)$. The function g is in $\mathcal{M}(X)$. It is in $\mathcal{D}$ since it depends only on a finite number of coordinates. For $i = 1, 2$ the generators of the processes satisfy for all f in $\mathcal{D}$ and all η in X,

$$\Omega^i f(\eta) \quad = \quad \lim_{t \searrow 0} (S^i(t) f(\eta) - f(\eta))/t \ .$$

When $\eta(x) = \zeta(x) = 0$, then $g(\eta) = g(\zeta) = 0$. Since $S^1(t)g(\eta) \leq S^2(t)g(\zeta)$, it comes $\Omega^1 g(\eta) \leq \Omega^2 g(\zeta)$. Then, it is easy to check that

$$\begin{aligned} \Omega^1 g(\eta) &= c^1(x,\eta) + \sum_{y \in \mathcal{V}_1(x,\eta)\setminus V} c^1(xy,\eta) \ , \\ \Omega^2 g(\zeta) &= c^2(x,\zeta) + \sum_{y \in \mathcal{V}_1(x,\zeta)\setminus V} c^2(xy,\zeta) \ , \end{aligned}$$

and (12) is satisfied. The proof of condition (13) is analogous.

The proof for the other direction is in two stages. Massey's theorem is used to show that conditions (12) and (13) are sufficient when S is finite. The result is then extended to infinite S using a technique of finite approximations. Let S be finite. We will use Liggett's notations by writting η^x for the configuration η changed at x into $1 - \eta(x)$ and η^{xy} when the spins at sites x and y are exchanged. We set for any subset Γ of X, for $i = 0, 1$

$$\begin{aligned} \Gamma_i(\eta,\zeta) &= \{x \in S,\ \eta(x) = \zeta(x) = i,\ \eta^x \in \Gamma\} \\ \Gamma_i(x,\eta) &= \{y \in \mathcal{S}(x),\ \eta(y) = i,\ \eta^{xy} \in \Gamma\} \ . \end{aligned}$$

Let us prove that conditions (12) and (13) imply respectively the following conditions (14) and (15).
For all η, ζ in X with $\eta \leq \zeta$,

for all increasing subsets Γ of X such that $\eta \notin \Gamma$ and $\zeta \notin \Gamma$,

$$\sum_{x \in \Gamma_0(\eta,\zeta)} \left(c^1(x,\eta) + \sum_{y \in \Gamma_1(x,\eta)} c^1(xy,\eta) \right) \quad \leq \quad \sum_{x \in \Gamma_0(\zeta,\eta)} \left(c^2(x,\zeta) + \sum_{y \in \Gamma_1(x,\zeta)} c^2(xy,\zeta) \right) \tag{14}$$

for all decreasing subsets Γ of X such that $\eta \notin \Gamma$ and $\zeta \notin \Gamma$,

$$\sum_{x\in\Gamma_1(\eta,\zeta)} \left(c^1(x,\eta) + \sum_{y\in\Gamma_0(x,\eta)} c^1(xy,\eta) \right) \quad \geq \quad \sum_{x\in\Gamma_1(\zeta,\eta)} \left(c^2(x,\zeta) + \sum_{y\in\Gamma_0(x,\zeta)} c^2(xy,\zeta) \right) \tag{15}$$

Let $\eta \leq \zeta$ and Γ be an increasing subset such that $\eta \notin \Gamma$ and $\zeta \notin \Gamma$. Condition (12) can be applied with x in $\Gamma_0(\eta,\zeta)$ and $V = \{y \in \mathcal{S}(x),\ \eta(y) = \zeta(y) = 1 \text{ and } \eta^{xy} \notin \Gamma\}$. It comes $\mathcal{V}_1(x,\eta)\backslash V = \{y \in \mathcal{S}(x),\ \eta(y) = \zeta(y) = 1,\ \eta^{xy} \in \Gamma\} = \Gamma_1(x,\eta)$,
and $\mathcal{V}_1(x,\zeta)\backslash V \subset \Gamma_1(x,\zeta)$. To show the inclusion above, let consider y in $\mathcal{V}_1(x,\zeta)\backslash V$. Then $\zeta(y) = 1$ and $\eta(y)$ can be 0 or 1. If $\eta(y) = 0$ then $\eta^x \leq \zeta^{xy}$. Since $x \in \Gamma_0(\eta,\zeta)$ then $\eta^x \in \Gamma$. Since Γ is increasing it follows that $\zeta^{xy} \in \Gamma$. If $\eta(y) = 1$ then $\eta^{xy} \in \Gamma$ since $y \notin V$. It follows from $\eta^{xy} \leq \zeta^{xy}$ that $\zeta^{xy} \in \Gamma$. In both cases $\zeta^{xy} \in \Gamma$ and y is in $\Gamma_1(x,\zeta)$.
Applying condition (12), it comes for all x in $\Gamma_0(\eta,\zeta)$,

$$c^1(x,\eta) + \sum_{y\in\Gamma_1(x,\eta)} c^1(xy,\eta) \quad \leq \quad c^2(x,\zeta) + \sum_{y\in\mathcal{V}_1(x,\zeta)\backslash V} c^2(xy,\zeta)$$

and therefore

$$c^1(x,\eta) + \sum_{y\in\Gamma_1(x,\eta)} c^1(xy,\eta) \quad \leq \quad c^2(x,\zeta) + \sum_{y\in\Gamma_1(x,\zeta)} c^2(xy,\zeta) \ .$$

Summing over x in $\Gamma_0(\eta,\zeta)$ and noticing that $\Gamma_0(\eta,\zeta) \subset \Gamma_0(\zeta,\eta)$ leads to (14).
Analogous developments are used to show that (13) implies (15). The conclusion follows by noticing that the conditions (14) and (15) are the conditions of theorem 3.1 written for spin systems with migration.

This ends the proof of our result for finite S. If S is infinite, consider finite sets T_n which increase to S. For each T_n, since (12) and (13) hold for all x in T_n, the corresponding frozen processes can be compared. It follows, as already mentioned in the proof of theorem 3.2, that $S^1(t) \leq_{st} S^2(t)$. □

The conditions in theorems 3.2 and 3.3 may be difficult to check since they compare sums of rates and not individual rates. Conditions that bear on individual rates are given in the next section. They are stronger in the sense that they ensure the existence of a Markov increasing coupling. A Markov coupling is also given for spin systems with migration.

4 Conditions for increasing Markov couplings.

In this section, we shall focus on the construction of explicit Markov couplings for interacting particle systems. Doing so, more tractable conditions will be obtained.

4.1 Generalized Vasershtein coupling.

The Vasershtein coupling also called basic coupling is described in (Liggett, 1985, chap. III p. 125). It concerns spin systems for which $W_x = \{0,1\}$ for all x in S. Necessary and sufficient conditions under which this coupling is increasing are given in (Liggett, 1985). In this subsection we give a generalization of this coupling to the case of finite partially ordered W_x and some sufficient conditions for the coupling to be increasing.

For $i = 1, 2$ consider interacting particle systems with transition rates $c^i_T(\eta,\gamma)$ and generators Ω^i as given in (1). Define the following generator $\widetilde{\Omega}$ on the set $\widetilde{\mathcal{D}}$ of cylinder functions $f(\eta,\zeta)$ on $X \times X$,

$$\forall f \in \widetilde{\mathcal{D}} \ , \ \forall \eta, \zeta \in X \ , \quad \widetilde{\Omega} f(\eta,\zeta) \quad = \quad \sum_{T \in \mathcal{P}_f(S)} \widetilde{\Omega}_T f(\eta,\zeta)$$

$$\begin{aligned} \text{with} \quad \widetilde{\Omega}_T f(\eta,\zeta) \quad &= \quad \sum_{\gamma \in X_T} \min(c^1_T(\eta,\gamma), c^2_T(\zeta,\gamma)) \, [f(\eta^\gamma_T, \zeta^\gamma_T) - f(\eta,\zeta)] \\ &+ \quad \sum_{\gamma \in X_T} [c^1_T(\eta,\gamma) - \min(c^1_T(\eta,\gamma), c^2_T(\zeta,\gamma))] \, [f(\eta^\gamma_T, \zeta) - f(\eta,\zeta)] \\ &+ \quad \sum_{\gamma \in X_T} [c^2_T(\zeta,\gamma) - \min(c^1_T(\eta,\gamma), c^2_T(\zeta,\gamma))] \, [f(\eta, \zeta^\gamma_T) - f(\eta,\zeta)] \quad . \end{aligned} \tag{16}$$

It is easy to check for $f(\eta,\zeta) = g(\eta)$ and $f(\eta,\zeta) = g(\zeta)$ that $\widetilde{\Omega} f(\eta,\zeta) = \Omega^1 g(\eta)$ and $\widetilde{\Omega} f(\eta,\zeta) = \Omega^2 g(\zeta)$ respectively. Therefore $\widetilde{\Omega}$ defines a Markov coupling of the two systems. Theorem 4.1 gives sufficient conditions under which the coupling is increasing.

Theorem 4.1 *For $i = 1, 2$ consider interacting particle systems with transition rates $c^i_T(\eta,\gamma)$ and generators Ω^i defined in (1). If their rates satisfy for T in $\mathcal{P}_f(S)$ and all η, ζ in X such that $\eta \leq \zeta$,*

$$\textit{if } \gamma > \zeta(T) \textit{ or } \gamma \not\sim \zeta(T) \quad \textit{then} \quad c^1_T(\eta,\gamma) \leq c^2_T(\zeta,\gamma) \tag{17}$$

$$\textit{if } \gamma < \eta(T) \textit{ or } \gamma \not\sim \eta(T) \quad \textit{then} \quad c^1_T(\eta,\gamma) \geq c^2_T(\zeta,\gamma) \tag{18}$$

then $\widetilde{\Omega}$ defined in (16) realizes an increasing Markov coupling of the systems.

Proof. Let h be any positive function in $\widetilde{\mathcal{D}}$ and (η,ζ) a point in $K = \{(\eta,\zeta) \in X \times X \ , \ \eta \leq \zeta\}$ where h achieves its maximum on K. To prove that the coupling is increasing it is enough to check (see Liggett, 1985, chap. III p. 127) that $\widetilde{\Omega} h(\eta,\zeta)$ is non positive. Let T be a fixed subset of S. If $\eta \leq \zeta$ then $\eta^\gamma_T \leq \zeta^\gamma_T$ for all $\gamma \in X_T$ and the first sum in (16) is non positive. Regarding the second sum, two cases must be distinguished,

$$\begin{array}{lll} (i) & \gamma \leq \zeta(T) & \Rightarrow \ (\eta^\gamma_T, \zeta) \in K \\ (ii) & \gamma > \zeta(T) \text{ or } \gamma \not\sim \zeta(T) & \Rightarrow \ (\eta^\gamma_T, \zeta) \notin K \ . \end{array}$$

In the same manner, for the third sum in (16),

$$\begin{array}{lll} (i)' & \eta(T) \leq \gamma & \Rightarrow \ (\eta, \zeta^\gamma_T) \in K \\ (ii)' & \gamma < \eta(T) \text{ or } \gamma \not\sim \eta(T) & \Rightarrow \ (\eta, \zeta^\gamma_T) \notin K \ . \end{array}$$

Terms in cases $(i), (i)'$ are always non positive but this is not necessarily true for cases $(ii), (ii)'$. The conditions in theorem 4.1 ensure that these terms are zero.

□

Comment. It is easy to check that the conditions in theorem 4.1 imply conditions (6) and (7) (see the proof of theorem 3.4 in (Forbes and al., 1996)), so that the existence of an increasing coupling actually follows from theorem 3.2 . The above construction of a Markov coupling is interesting since it generalizes the basic coupling for spin systems. In theorem 4.1, some of the conditions concern configurations γ in X_T that cannot be compared with $\eta(T)$ or $\zeta(T)$. Such conditions may not appear very natural having in mind the spin systems case. However, they cannot be removed even for systems in which only transitions between comparable configurations are allowed. The need for additional conditions comes from the occurence of transitions involving spaces which are not totally ordered. When one tries to suppress them, the conditions in Massey's theorem may not be satisfied. Consider for instance, the following example where $S = \{x, y\}$ and $W_x = W_y = \{0, 1\}$. Denote by $(\eta(x), \eta(y))$ a configuration in $X = W_x \times W_y$. The rates $c_T(\eta, \gamma)$ are

$$\begin{aligned} c_T(\eta,\gamma) &= \lambda \quad \text{if } T = S \text{ and } \eta \sim \gamma \\ &\ 0 \quad \text{in any other case.} \end{aligned}$$

The generator can be represented by the following matrix Ω in which indexes 1,2,3,4 correspond respectively to states $(0,0), (0,1), (1,0), (1,1)$.

$$\Omega = \begin{bmatrix} -3\lambda & \lambda & \lambda & \lambda \\ \lambda & -2\lambda & 0 & \lambda \\ \lambda & 0 & -2\lambda & \lambda \\ \lambda & \lambda & \lambda & -3\lambda \end{bmatrix}$$

Two copies of this system ($c^1_T(\eta, \gamma) = c^2_T(\eta, \gamma)$) clearly satisfy condition (17) for all $\gamma > \zeta(T)$ and condition (18) for all $\gamma < \eta(T)$. However the necessary and sufficient conditions of theorem 3.1 are not satisfied.

4.2 Markov couplings for spin systems with migration.

In this part a Markov coupling is given for a couple of spin systems with migration. The conditions under which the coupling is increasing are given in theorem 4.2. They are sufficient conditions. It is easy to check that they imply the necessary and sufficient conditions of theorem 3.4. Recall that $\mathcal{V}_i(x, \eta, \zeta) = \{y \in \mathcal{S}(x) \text{ with } \eta(y) = \zeta(y) = i\}$ $(i = 0, 1)$ and introduce $\mathcal{T}_0(\eta, \zeta) = \{(x, y) \in S \times S \text{ with } x \in \mathcal{V}_0(y, \eta, \zeta) \text{ and } y \in \mathcal{V}_1(x, \eta, \zeta)\}$.

Theorem 4.2 *For $i = 1, 2$ consider spin systems with migration, with transition rates $\{c^i(x, \eta), c^i(xy, \eta)\}$. If for all η, ζ in X with $\eta \le \zeta$,*

$$\begin{aligned} \forall x,\ \eta(x) = \zeta(x) = 0, \quad c^1(x,\eta) + \sum_{y \in \mathcal{V}_1(x,\eta,\zeta)} c^1(xy,\eta) &\le c^2(x,\zeta) + \sum_{y \in \mathcal{V}_1(x,\eta,\zeta)} c^2(xy,\zeta) && (19) \\ \forall x,\ \eta(x) = \zeta(x) = 1, \quad c^1(x,\eta) &\ge c^2(x,\zeta) && (20) \\ \forall (x,y) \in \mathcal{T}_0(\eta,\zeta), \quad c^1(xy,\eta) &\ge c^2(xy,\zeta) && (21) \end{aligned}$$

then there exists a Markov increasing coupling of the systems.

Proof. The coupling is given in terms of its generator $\widetilde{\Omega}$ on $\widetilde{\mathcal{D}}$ by

$$\begin{aligned} \widetilde{\Omega} f(\eta,\zeta) &= \sum_{x:\eta(x)\neq\zeta(x)} c^1(x,\eta)(f(\eta^x,\zeta) - f(\eta,\zeta)) + c^2(x,\zeta)(f(\eta,\zeta^x) - f(\eta,\zeta)) \\ &+ \sum_{x:\eta(x)=\zeta(x)=0} c^1(x,\eta)(f(\eta^x,\zeta^x) - f(\eta,\zeta)) \end{aligned}$$

$$
\begin{aligned}
&+\left[c^2(x,\zeta)+\sum_{y\in\mathcal{V}_1(x,\eta,\zeta)}c^2(xy,\zeta)-c^1(x,\eta)-\sum_{y\in\mathcal{V}_1(x,\eta,\zeta)}c^1(xy,\eta)\right](f(\eta,\zeta^x)-f(\eta,\zeta))\\
+&\sum_{y:\eta(y)=\zeta(y)=1}(c^1(x,\eta)-c^2(x,\zeta))(f(\eta^y,\zeta)-f(\eta,\zeta))+c^2(x,\zeta)(f(\eta^y,\zeta^y)-f(\eta,\zeta))\\
+&\sum_{(x,y)\in\mathcal{T}_0(\eta,\zeta)}c^2(xy,\zeta)(f(\eta^{xy},\zeta^{xy})-f(\eta,\zeta))+\left(c^1(xy,\eta)-c^2(xy,\zeta)\right)(f(\eta^{xy},\zeta^x)-f(\eta,\zeta))\\
+&\sum_{\text{other }\{x,y\}}c^1(xy,\eta)(f(\eta^{xy},\zeta)-f(\eta,\zeta))+c^2(xy,\zeta)(f(\eta,\zeta^{xy})-f(\eta,\zeta)).
\end{aligned}
$$

This expression actually defines a coupling. Conditions (19), (20) and (21) ensure that all expressions for the transition rates in the expression of $\widetilde{\Omega}$ are non negative as required. In addition, for $f(\eta,\zeta)=g(\eta)$ and $f(\eta,\zeta)=g(\zeta)$, it is easy to check that $\widetilde{\Omega}f(\eta,\zeta)=\Omega^1g(\eta)$ and $\widetilde{\Omega}f(\eta,\zeta)=\Omega^2g(\zeta)$ respectively. The coupling is increasing since the set $K=\{(\eta,\zeta)\in X\times X\ ,\ \eta\leq\zeta\}$ is preserved in each allowed transition.

□

In the previous sections we successively investigated two ways of obtaining comparison conditions. In section 3, sufficient and sometimes also necessary conditions were given to compare Markov processes on $X=\bigotimes_{x\in S}W_x$. They ensured the existence of increasing couplings on $X\times X$ without giving them explicitly. In section 4, other comparison conditions were obtained by constructing explicit Markov couplings via their generators. It is not clear whether the existence of an increasing coupling is always equivalent to that of a Markov increasing coupling. This is so for uniformizable Markov processes on countable state space as proved in (Doisy, 1992, theorem 1.4.14 p. 46). The proof uses the decomposition of such Markov processes into a Markov chain and a Poisson process. This theorem does not apply to interacting particle systems when S is infinite. However, for spin systems on a finite or infinite S, the necessary and sufficient conditions in theorem 3.3 are exactly the necessary and sufficient conditions obtained by Liggett (1985) with the basic coupling. Therefore, in this particular case the existence of an increasing coupling is equivalent to that of a Markov increasing coupling.

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