Stochastic comparison for resource sharing models

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Abstract

New models for resource sharing situations are defined. These models are Markovian versions of the drinking philosophers problem introduced by Chandy and Misra. They can be viewed as interacting particle systems, for which the classical stochastic comparison results for spin systems are generalized. Several applications to resource sharing models are proposed.

Key Words : Resource sharing, Philosophers process, Interacting particle system, Stochastic comparison

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1 Introduction

Depending on the application context, many different ways of modelling resource sharing situations have been proposed. Dijkstra's dining philosophers problem [3] is one of the first systematic attempts in the area. It describes a situation where the users (philosophers) are the vertices of a graph the edges of which (chopsticks) represent resources. When one of the users starts working, he needs all available resources (adjacent edges) and therefore none of his neighbors can work. One major drawback of this model is that it does not take into account the fact that in a real system, a given user may need only part of the available resources. Two users sharing common resources may be working at the same time provided they do not need the same resources. In 1984, Chandy and Misra [1, 2] proposed the drinking philosophers problem, where the users are allowed to block varying subsets of their resources provided that a given resource should be used at each time by at most one user.

The dining as well as the drinking philosophers problems were deterministic situations and the main question in both cases was to find an algorithm that would ensure fairness, *i.e.* give to each user a correct amount of working time. One way of solving the problem of fairness is to define a stochastic evolution of the situation, namely a Markov process with values in the set of configurations of busy and idle users. The first to propose such a Markovian version (Philosophers Process or PP) was Zenie [15] (*cf.* [14] and references therein).

But in the mean time quite a lot of work had been done on the modelling of communication networks [7]. A communication network has also a set of resources (the lines or channels) which the users share by placing calls. Each call follows a route, thus blocking a certain subset of the lines. A Loss Network (LN, *cf.* Kelly [8] is a stochastic process that describes the evolution of the occupation status of the communication lines, under a certain random dynamics of the calls. One could roughly summarize by saying that a PP takes the point of view of the users whereas a LN describes the evolution of resources. The two models are closely related and particular versions of the PP were considered as early as 84 (Suhov [13]) and 85 (Kelly [7]) in the context of communication networks.

We propose a new family of models for resource sharing which can be viewed as Markovian versions of the drinking philosophers problem, thus being a natural generalization of the PP. We shall distinguish the points of view of users and resources, by calling User Process (UP) a Markov process with values in a set of mappings that associate to each user the set of resources he his currently working with. A Resource Process (RP) will be a Markov process with values in the set of subsets of resources, the state of the process being the set of resources currently blocked (*cf.* section 2 for exact definitions).

Defining new models is of interest only if efficient mathematical techniques of treatment can be proposed for these models. Reversibility is one such technique, and it has already proved its value for PPs [14, 5] as well as for LNs [8]. We shall refer to it here only marginally (*cf.* sections 2 and 4). Our main stress is on stochastic comparison, which is another classical tool for stochastic networks (*cf.* [12] for a general reference).

Since the PP is a particular case of spin system (*cf.* Liggett [9] for a general reference), classical results applied in that case. The situation here is more complicated. A User Process can be seen as a spin system where the state of each site (user) is a subset of resources instead of 0 or 1. A Resource Process is also an interacting particle system for which the sites are resources, in state 0 if they are available, 1 if not. But a given configuration may change at several sites at a time and classical results for spin systems do not apply either. Hence we have to develop a generalization of the standard theory. The main theoretical results in this paper are conditions of comparability for general interacting particle systems with states in a partially ordered set (theorems 3.2 and 3.3). The proof uses Massey's results [10]. Several applications of these conditions to resource sharing models are proposed. In particular sufficient conditions will be given for a reversible, translation invariant User Process on $\mathbb{Z}$ to be ergodic (theorem 4.1), thus generalizing theorem 2.1 p. 359 of [14]. However comparison results for User Processes are somewhat limited in their extent due to the constraints induced by sharing. The application of stochastic comparison to Resource Processes yields different monotonicity results for the set of occupied resources in terms of transition rates and of the initial configuration (theorems 5.1 and 5.2). These results justify theoretically the performance evaluation of a resource sharing system through Monte-Carlo simulation.

The article is organized as follows. Section 2 contains the description of the proposed models together with their relations with previous models, and several examples. Section 3 is devoted to the statement and proof of our main results on the stochastic comparison of general interacting particle systems. Applications to User Processes and Resource Processes are developed respectively in sections 4 and 5.

2 Resource sharing models

Let us denote by R a finite set of resources and by U a finite set of users. The set of subsets of R will be denoted by $\mathcal{P}(R)$. In this section, we shall define UPs as Markov processes taking values in $X = \mathcal{P}(R)^U$ and RPs as Markov processes taking values in $X = \mathcal{P}(R)$. Both are stochastic versions of the deterministic drinking philosophers problem of Chandy and Misra [1]. Furthermore we shall consider natural ways of defining UPs and RPs from the knowledge of the behavior of each user when isolated from the others. For UPs this will be done by truncation from independent behaviors. For RPs it will correspond to superposition. At the end of the section, aggregability conditions under which the global subset of resources being blocked in a UP evolves as a RP (theorem 2.1) are given.

We shall consider configurations in $\mathcal{P}(R)^U$, *i.e.* mappings η defined on U, with values in $\mathcal{P}(R)$. For each user u, $\eta(u)$ will be interpreted as the subset of resources u is currently working with. In Chandy and Misra's [1] colorful setting, $\eta(u)$ is the current cocktail that philosopher u is drinking, with $\eta(u) = \emptyset$ if u is sober. The UPs we want to define are Markov processes with values in a set of configurations. We shall see them as interacting particle systems, and use Liggett [9] as a general reference, trying to stay as close as possible to that reference regarding notations. Our processes will be allowed to modify a given configuration by changing the status of only one user at a time. If η is a configuration and γ a subset of R, η_u^γ will denote the configuration deduced from η by changing the status of u into γ.

$$\forall u \in U,\ \forall \gamma \in \mathcal{P}(R), \qquad \begin{array}{rcll} \eta_u^\gamma(v) &=& \eta(v) & \text{if } v \neq u \\ \eta_u^\gamma(u) &=& \gamma & . \end{array}$$

We want to define non negative transition rates between configurations η and η_u^γ. They will be denoted by $c(u, \eta, \gamma)$ and interpreted as the rate at which the philosopher u changes his current cocktail into γ when the general configuration is η. By definition, we call User Process (UP) with rates $c(u, \eta, \gamma)$ a Markov process $\{\eta_t\ ;\ t \geq 0\}$ on the set of configurations for which the transition rate between any two configurations η and ζ is

$$\begin{array}{rl} c(u, \eta, \gamma) & \quad \text{if } \zeta = \eta_u^\gamma \\ 0 & \quad \text{in any other case.} \end{array}$$

The resource sharing interpretation imposes an obvious limitation on possible configurations, and also on transition rates, since each resource can be used by at most one user. We shall denote by $\mathcal{A}$ the set of all admissible configurations.

$$\mathcal{A} = \{\eta \in \mathcal{P}(R)^U \text{ s.t. } \forall u, v \in U\ ,\ \eta(u) \cap \eta(v) = \emptyset\} \quad .$$

Any UP is bound to stay in $\mathcal{A}$ by the following requirement.

$$\forall u \in U,\ \forall \eta \in \mathcal{A},\ \forall \gamma \in \mathcal{P}(R), \quad c(u, \eta, \gamma) = 0 \quad \text{if} \quad \eta_u^\gamma \notin \mathcal{A} \quad . \tag{1}$$

Many natural ways of defining the transition rates $c(u, \eta, \gamma)$ can be proposed.

As a first example, we show that UPs generalize PPs. Assume that to each user u is associated a particular subset R_u of resources, so that u has only two possibilities, either use all R_u or stay idle. Define the rates $c(u, \eta, \gamma)$ by

$$c(u, \eta, \gamma) = \begin{cases} \lambda & \text{if} \quad \eta(u) = \emptyset\ ,\ \gamma = R_u \text{ and } \eta(v) \cap R_u = \emptyset\ ,\ \forall v \neq u \\ 1 & \text{if} \quad \eta(u) = R_u \\ 0 & \text{in any other case.} \end{cases}$$

Define on U a graph structure G by putting an edge between any two users u, v such that $R_u \cap R_v \neq \emptyset$. Since only two states are possible, one may as well write $\eta(u) = 1, 0$ instead of $\eta(u) = R_u, \emptyset$. The process so defined is the PP on the graph G (*cf.* [14, 5]).

We shall call Resource Process (RP) a Markov process with values in $\mathcal{P}(R)$. From the point of view of applications, a very natural way to define a UP is by truncation of individual RPs. To each user u in U, associate the rates $\lambda_u(\alpha, \beta)$ of a RP. These rates are interpreted as the rates at which the user u would exchange subsets of resources if he was alone. The corresponding RP could be called the *individual RP* of u. One can define a global UP as the direct product of these individual RPs, truncated to the set $\mathcal{A}$ by removing those transitions which are not compatible with (1). This interpretation corresponds to the following definition for the rates $c(u,\eta,\gamma)$.

$$\forall u \in U,\ \forall \eta \in \mathcal{A},\ \forall \gamma \in \mathcal{P}(R), \quad c(u,\eta,\gamma) = \begin{cases} \lambda_u(\eta(u),\gamma) & \text{if } \ \eta_u^\gamma \in \mathcal{A} \\ 0 & \text{in any other case.} \end{cases} \tag{2}$$

The operation of truncation is particularly adapted to reversible processes. Indeed, assume that for each $u \in U$, the rates $\lambda_u(\alpha,\beta)$ are those of a reversible process on $\mathcal{P}(R)$ having a unique irreducibility class, with reversible measure π_u. Let the rates $c(u,\eta,\gamma)$ be defined as in (2) and assume the resulting UP is irreducible on $\mathcal{A}$. Then it is reversible with respect to the measure ν defined by

$$\forall \eta \in \mathcal{A}, \quad \nu(\eta) = Z^{-1} \prod_{u\in U} \pi_u(\eta(u)) \quad ,$$

with

$$Z = \sum_{\eta\in\mathcal{A}} \prod_{u\in U} \pi_u(\eta(u)) \quad .$$

This is an easy consequence of corollary 1.10 p. 26 of Kelly [6]. The PP mentioned above is a particular case of a reversible UP. There are many natural ways to define reversible individual RPs. Here are three examples.

Example 2.1 *Independent resources.*

Assume that each user takes or gives back one resource at a time, with a rate depending only on the user and the resource. The rates of his individual RP will be of the following type

$$\forall \alpha,\beta \in \mathcal{P}(R), \quad \lambda_u(\alpha,\beta) = \begin{cases} a_u(r) & \text{if } \ r \notin \beta \ ,\ \alpha = \beta \cup \{r\} \\ b_u(r) & \text{if } \ r \notin \alpha \ ,\ \beta = \alpha \cup \{r\} \\ 0 & \text{in any other case.} \end{cases}$$

This RP is obviously reversible with respect to the following measure

$$\begin{aligned} \forall \alpha \neq \emptyset, \qquad \pi_u(\alpha) &= \pi_u(\emptyset) \prod_{r\in\alpha} \frac{b_u(r)}{a_u(r)} \ , \\ \pi_u(\emptyset) &= \left(1 + \sum_{\alpha\neq\emptyset} \prod_{r\in\alpha} \frac{b_u(r)}{a_u(r)}\right)^{-1} \quad . \end{aligned}$$

Example 2.2 *Back to empty set.*

Assume that each user is obliged to give back the set of resources he is using before taking another. The rates of his individual RP will be of the following type

$$\forall \alpha,\beta \in \mathcal{P}(R), \quad \lambda_u(\alpha,\beta) = \begin{cases} a_u(\alpha) & \text{if } \ \beta = \emptyset \\ b_u(\beta) & \text{if } \ \alpha = \emptyset \\ 0 & \text{in any other case.} \end{cases}$$

This RP is obviously reversible with respect to the following measure

$$\begin{aligned}\forall \alpha \neq \emptyset, \qquad \pi_u(\alpha) &= \pi_u(\emptyset)\frac{b_u(\alpha)}{a_u(\alpha)}\,, \\ \pi_u(\emptyset) &= \left(1+\sum_{\alpha\neq\emptyset}\frac{b_u(\alpha)}{a_u(\alpha)}\right)^{-1} \quad .\end{aligned}$$

Example 2.3 *Target dependent rates.*

Assume that the rate at which a given subset of resources is taken depends only on that set, and that the rates at which resources are dropped are constant. Then, each user u attributes a weight $\mu_u(\beta)$ to each subset β of R and the rates are

$$\forall \alpha, \beta \in \mathcal{P}(R), \qquad \lambda_u(\alpha,\beta) = \mu_u(\beta) \quad .$$

This RP is reversible with respect to the probability measure which is proportional to μ_u.

In a resource sharing network the incompatibilities are usually symbolized by a graph or hypergraph structure, where the users are vertices and the resources are edges (*cf.* [2]). In our setting this graph structure is implicit in the more general definition of the rates $c(u,\eta,\gamma)$. The geometric structure can be recovered in the following way. For each user u call *individual resource set* R_u the set of all resources in R he may possibly use

$$R_u \;=\; \cup\{\gamma \in \mathcal{P}(R) \; s.t. \; \exists \eta \in \mathcal{A} \quad c(u,\eta,\gamma) > 0\} \quad . \tag{3}$$

Then to each pair $\{u,v\}$ of users, one can associate the intersection of their individual resource sets. The elements of this intersection can be viewed as the edges of an undirected hypergraph (*cf.* Chandy and Misra [1]). One could also define *resource sites* in which certain users but not all, could block resources. Instead of presenting geometrical settings in full generality, we shall describe only the example of a cyclic network in which each user shares resources with two neighbors.

Example 2.4 *Cyclic networks.*

Let $G \subset \{\{u,v\}\,,\; u,v \in U\}$ be a cyclic graph structure on the set of users U, that will be identified to $\mathbb{Z}/n\mathbb{Z}$. To each user u in U corresponds a set of resources R_u which is partitioned into three subsets R_u^l, R_u^r and R_u^i. The subsets R_u^l and R_u^r are shared with the users on the left and on the right respectively

$$\forall u \in \mathbb{Z}/n\mathbb{Z}, \quad R_u^l = R_{u-1}^r\,,$$

whereas resources in R_u^i are not shared with anyone. Define now the rates $\lambda_u(\alpha,\beta)$ for each individual RP and then the rates $c(u,\eta,\gamma)$ of the global UP by truncation as in (2). The individual resource processes may be copies of the same RP in the following sense. Let R_0 be a reference resource set, partitioned into three subsets R_0^l, R_0^r and

R_0^i. Let $\lambda(\alpha,\beta)$ be the rates of a RP on R_0. Assume that for each u there exists a one to one mapping σ_u from R_0 onto R_u that preserves partitions and rates.

$$\sigma_u(R_0^l) = R_u^l \ , \ \sigma_u(R_0^r) = R_u^r \ , \ \sigma_u(R_0^i) = R_u^i$$

and

$$\lambda_u(\sigma_u(\alpha), \sigma_u(\beta)) = \lambda(\alpha,\beta) \quad .$$

Then the resulting rates $c(u,\eta,\gamma)$ will be invariant through translations of $\mathbb{Z}/n\mathbb{Z}$,

$$c(u,\eta(.),\gamma) = c(u+1,\eta(.-1),\gamma) \quad . \tag{4}$$

A UP on $\mathbb{Z}/n\mathbb{Z}$ satisfying (4) will be called *translation invariant.* As a particular case, assume each edge of G corresponds to only one resource. Each user has two resources, one on his right and one on his left, exactly as in the original dining philosophers problem [3]. Define each individual RP by its rates $\lambda_u(\alpha,\beta)$ between the 4 subsets of the two individual resources l and r. If these rates are the same for all users, the resulting UP is translation invariant. The PP on a round table of [14] is a particular case.

Even if in reality the set of users is always finite, it may be useful to use infinite particle systems as large scale approximations. The mathematical definition of a UP with an infinite set of users raises problems that are out of the scope of this paper (*cf.* chapter I of [9] and references therein). However, since ergodicity proofs are one of the most important applications of stochastic comparison techniques, we found interesting to give in section 4 an ergodicity result for translation invariant UPs defined on the infinite graph $\mathbb{Z}$ exactly the same way as discussed above for the cyclic graph $\mathbb{Z}/n\mathbb{Z}$. This UP on $\mathbb{Z}$ will be an interacting particle system with a finite phase space and finite range interactions, and its existence follows from classical results (*cf.* [9] p.27).

Let us turn now to the point of view of resources. In the following developments, it will be convenient to consider a RP as a Markov process on $\{0,1\}^R$. A subset of R will be identified to a configuration η in $\{0,1\}^R$ by setting $\eta(r)=1$ if r belongs to the subset, $\eta(r)=0$ otherwise. Then we shall often write $\eta \in \mathcal{P}(R)$ for $\eta \in \{0,1\}^R$. Besides, $\eta\Delta\zeta$ will be the subset $(\eta\backslash\zeta)\cup(\zeta\backslash\eta)$ of the sites r of R where configurations η and ζ differ. For U a finite set of users, let $(R_u)_{u\in U}$ be a collection of distinct finite subsets indexed by U. Each R_u can be viewed as the individual resource set of user u in analogy with equation (3). The whole set of resources is

$$R = \bigcup_{u\in U} R_u \, .$$

For each u in U, we consider a local RP with rates

$$\{c_u(\eta,\zeta)\,,\ \eta,\zeta\in\mathcal{P}(R)\}\,.$$

Only transitions involving resources in R_u are allowed and therefore $c_u(\eta,\zeta)$ can be positive only if η and ζ differ on coordinates in R_u ($\eta\Delta\zeta \in \mathcal{P}(R_u)$). A *superposition* of such processes is a RP with generator Ω defined, for all function $f(\eta)$ on $\mathcal{P}(R)$ by

$$\begin{aligned} \Omega f(\eta) &= \sum_{u\in U}\sum_{\zeta\in\mathcal{P}(R)} c_u(\eta,\zeta)\,(f(\zeta)-f(\eta)) \\ &= \sum_{u\in U}\sum_{\gamma\in\mathcal{P}(R_u)} c_u(\eta,\eta^\gamma_{R_u})\left(f(\eta^\gamma_{R_u})-f(\eta)\right) \quad . \end{aligned} \tag{5}$$

Resource sharing models are built in practice upon a set U of users, a set R of resources and the knowledge of the behavior of each user considered alone. The behavior of each user is described by an individual RP. From these data, a UP may be defined by truncation as in equation (2). Particular cases will be studied in section 4. However, general results for UPs are very hard to obtain, mainly because of the mathematical difficulty to handle the set of admissible configurations. An alternative is to construct from the same data a RP by superposition. However, for a resource sharing interpretation, we shall adopt a *bayesian* point of view, to be described precisely in section 5 together with a practical application. Stochastic comparison and monotonicity are tools much more adapted to such models than to UPs. Results from section 3 will apply straightforwardly.

There exists many other ways to define RPs from UPs. For a UP any coordinate as well as any union of coordinates is a stochastic process with values in $\mathcal{P}(R)$. However it is not Markovian in general. We give below the aggregability conditions under which the union of all coordinates of a UP is a RP. They are straightforward applications of the classical conditions of Rosenblatt [11].

Theorem 2.1 *Let $\{\eta_t\ ;\ t\geq 0\}$ be a UP with rates $c(u,\eta,\gamma)$. Define for all $t\geq 0$*

$$\rho_t = \bigcup_{u\in U}\eta_t(u) \quad .$$

Then $\{\rho_t\ ;\ t\geq 0\}$ is a Markov process (RP) with rates $c(\alpha,\beta)$ iff

$$\forall\eta,\ \bigcup_{u\in U}\eta(u)=\alpha\,,\quad \sum c(u,\eta,\gamma) = c(\alpha,\beta) \quad , \tag{6}$$

where the sum extends over all pairs (u,γ) such that $\bigcup_{v\in U}\eta^\gamma_u(v)=\beta$.

The condition (6) is obviously a very strong one, hardly likely to be satisfied in practice, except for very symmetric networks. To give just a few examples, we shall see how they could hold in the particular case of the reversible UP defined by truncation from the individual RPs of examples 2.1, 2.2 and 2.3. For the first case, all resources are taken (if available) and dropped independently with rates depending on the resource and the user. Condition (6) holds if all the rates depend only on the resources, and not on the user. In that case each resource is taken and dropped according to a binary Markov process, and all these binary processes are independent. The process $\{\rho_t\ ;\ t \geq 0\}$ is merely the product of these independent binary processes. For a UP defined by truncation from the individual RP of example 2.2, condition (6) is even more restrictive. For the sake of simplicity all individual processes are supposed to be copies of the same one, *i.e.* the rates at which subsets of resources are requested and dropped do not depend on the user. This is far from being sufficient. Assume the rates at which a subset of several resources is requested and dropped by some given user are non null. Then if condition (6) holds, no two different users can simultaneously block resources in that same subset. In that case the rate at which the whole subset is dropped has to be null since only one user can change status at a time. So the rates $a_u(\alpha)$ and $b_u(\beta)$ have to be very particular. Here are two examples.

$$\begin{array}{rcl} a_u(\alpha) & = & a(r) \quad \text{if} \quad \alpha = \{r\}\ , \text{ and } \quad 0 \quad \text{in any other case} \\ b_u(\beta) & = & b(r) \quad \text{if } a_u(\beta) \neq 0\ , \text{ and } \quad 0 \quad \text{in any other case} \quad . \end{array}$$

or

$$\begin{array}{rcl} a_u(\alpha) & = & a(R_u) \quad \text{if} \quad \alpha = R_u\ , \text{ and } \quad 0 \quad \text{in any other case} \\ b_u(\beta) & = & b(R_u) \quad \text{if } a_u(\beta) \neq 0\ , \text{ and } \quad 0 \quad \text{in any other case} \quad . \end{array}$$

Up to minor changes, the above remarks still hold if example 2.2 is replaced by example 2.3 in the definition of individual RPs.

The UPs and RPs discussed above are Markov processes on finite or infinite product of partially ordered sets. For such processes, both terminology and formalism are borrowed from the field of interacting particle systems (see Liggett [9]. In [9], section 1 of chapter II is devoted to applications of stochastic comparison to spin systems for which configurations, with values in $\{0,1\}$ can change at one site at a time. Our objective in the next section is to extend this to more general Feller processes on a product space.

3 Stochastic comparison

Throughout this section, S will denote a countable set of sites. For all x in S, W_x denotes a finite space. It will play the role of the phase space for site x. We shall consider interacting particle systems which are Feller processes on the state space

$$X = \bigotimes_{x \in S} W_x\ .$$

For all x in S, W_x is endowed with a partial order $\leq_x$. For each subset T of S, this order naturally extends to a partial order $\leq_T$ on $X_T = \bigotimes_{x \in T} W_x$,

$$\forall \alpha, \beta \in X_T, \qquad \alpha \leq_T \beta \ \Leftrightarrow\ \forall x \in T, \quad \alpha(x) \leq_x \beta(x)\ .$$

The local dynamics of interacting particle systems are specified by a collection of transition rates

$$\{c_T(\eta,\gamma)\,,\, \eta \in X,\ T \in \mathcal{P}_f(S),\ \gamma \in X_T\}\ .$$

The notation $\mathcal{P}_f(S)$ refers to the set of finite subsets of S. For all T in $\mathcal{P}_f(S)$ and all η in X, $c_T(\eta,\gamma)$ is a continuous function of η (X is endowed as usual with the product topology). The local dynamics are assumed to determine the evolution of the system as a whole in a unique way. Existence and uniqueness conditions can be found in [9], p. 27. An interacting particle system is associated to a Markov generator Ω. Let $\mathcal{D}$ be the set of functions on X depending on a finite number of coordinates. The restriction of Ω to $\mathcal{D}$ is defined by

$$\forall f \in \mathcal{D},\ \forall \eta \in X, \quad \Omega f(\eta) \;=\; \sum_{T \in \mathcal{P}_f(S)} \Omega_T f(\eta)\ , \tag{7}$$

with

$$\Omega_T f(\eta) \;=\; \sum_{\gamma \in X_T} c_T(\eta,\gamma)\,(f(\eta_T^\gamma) - f(\eta))\ , \tag{8}$$

and

$$\eta_T^\gamma(x) = \begin{cases} \gamma(x) & \text{if } x \in T \\ \eta(x) & \text{otherwise.} \end{cases}$$

The corresponding Feller semigroup $S(t)$ is defined on the set $\mathcal{C}(X)$ of continuous functions on X . For each T in $\mathcal{P}_f(S)$, the local rates $\{c_T(\eta,\gamma)\,,\,\eta \in X\ ,\ \gamma \in X_T\}$ specify the dynamics of a Feller process on X. Such processes are referred to as *T-local processes* since only transitions involving coordinates in T are allowed. Their generators are the Ω_T's of equation (8). Interacting particle systems as described by (7) are superpositions of T-local processes. Their global dynamics are defined as superposition of local dynamics involving only finite parts of the sites. UPs and RPs are particular cases of this setting. For UPs, transitions may involve only one user at a time, and the state space of each user (site) is $\mathcal{P}(R)$, whereas for a RP, sites are resources, with values in $\{0,1\}$ (free-occupied) and several resources can change status at the same time.

Stochastic comparison for Markov processes on countable partially ordered sets has been investigated by Massey [10]. This reference gives necessary and sufficient conditions for two Markov processes to be compared. These conditions involve the transition rates and the increasing subsets of the state space (definition 3.4 below). They are not of great help when an infinite or a large complex state space is considered. They are not well adapted to interacting particle systems since they do not take into account the locality of the rates. The aim of this section is to give conditions for stochastic comparison that are local in the sense that they bear on the local rates and not on the rates of the process as a whole. Section 3.1 is devoted to basic material on stochastic comparison. Our main results (theorems 3.2 and 3.3) are presented in section 3.2. Theorem 3.2 gives local sufficient conditions. For interacting particle systems on a finite set of sites S, they rely on Massey's result for countable state space (theorem 3.1). For infinite S, we shall use finite approximations to prove our result. Theorem 3.3 will apply to UPs. We actually prove that for a class of systems including UPs, the sufficient conditions of theorem 3.2 are also necessary. Although expressed in terms of the local rates, the conditions in theorems 3.2 and 3.3 may remain difficult to check since they compare sums of rates and not individual rates. In theorem 3.4 conditions that only bear on individual rates are given. They are more restrictive but easier to check in practice.

3.1 Basics for stochastic comparison.

Let us recall some basic tools for stochastic comparison (see Stoyan [12]). The notation $\alpha <_T \beta$ will often be used instead of $\alpha \leq_T \beta$ and $\alpha \neq \beta$. The notation $\alpha \sim_T \beta$ will be used when α and β are comparable with $\leq_T$ in the sense that $\alpha \leq_T \beta$ or $\beta \leq_T \alpha$. The opposite case will be denoted $\alpha \not\sim_T \beta$. When no confusion arises, subscript T will be dropped. The partial order on X induces a stochastic order $\leq_{st}$ on the set of probability measures on X. The notation $\mathcal{M}(X)$ denotes the set of continuous functions on X, increasing for the partial order $\leq_X$ on X .

Definition 3.1 *Two measures μ_1 and μ_2 on X satisfy $\mu_1 \leq_{st} \mu_2$ whenever*

$$\forall f \in \mathcal{M}(X) , \qquad \int f \, d\mu_1 \leq \int f \, d\mu_2 .$$

Feller processes on X are compared through their semigroups. If certain inequalities between the distributions of the processes hold initially, then they continue to hold at later times.

Definition 3.2 *Two Feller semigroups $S^1(t)$ and $S^2(t)$ satisfy*

$$S^1(t) \ \leq_{st} \ S^2(t)$$

iff for all probability measures μ_1 and μ_2 on X,

$$\mu_1 \leq_{st} \mu_2 \ \Rightarrow \ \forall t \geq 0 , \quad \mu_1 S^1(t) \leq_{st} \mu_2 S^2(t) . \tag{9}$$

Using Strassen's theorem it is easy to give the equivalent formulation below.

Lemma 3.1 *Consider the Feller semigroups $S^1(t)$ and $S^2(t)$. The following two statements are equivalent:*
- (a) $S^1(t) \ \leq_{st} \ S^2(t)$.
- (b) $\eta \leq \zeta \ \Rightarrow \ \forall t \geq 0 , \ \ \forall f \in \mathcal{M}(X) , \ \ S^1(t)f(\eta) \leq S^2(t)f(\zeta)$.

Assume that the processes to be compared are initially in configurations η and ζ with $\eta \leq \zeta$. Then assertion (b) above says that

$$\forall t \geq 0 , \ \forall f \in \mathcal{M}(X) , \qquad \mathbb{E}[f(\eta_t)] \leq \mathbb{E}[f(\zeta_t)] . \tag{10}$$

Stochastic comparison is closely related to monotonicity properties. Consider two copies of a single Feller process on X ($S^1(t) = S^2(t)$ for all positive t). The previous property can be used to study the evolution of the process starting from comparable initial configurations. Let us recall definition 2.3 chapter I p. 72 in [9] .

Definition 3.3 *A Feller process on X with Feller semigroup $S(t)$ is said to be monotone if for all probability measures μ and ν on X,*

$$\mu \leq_{st} \nu \qquad \Rightarrow \qquad \forall t \geq 0,\ \mu S(t) \leq_{st} \nu S(t) \ . \tag{11}$$

In the following section we shall give conditions (theorems 3.2 and 3.3) to compare semigroups in the sense of definition 3.2 .

3.2 Local conditions for stochastic comparison.

Massey [10] gives necessary and sufficient conditions to compare Markov processes on a countable partially ordered state space X. These conditions are written in terms of the rates of the processes and of increasing subsets of X .

Definition 3.4 *A subset Γ of X is increasing if it satisfies*

$$\eta \in \Gamma \text{ and } \eta \leq \zeta \ \ \Rightarrow \ \ \zeta \in \Gamma \quad .$$

It is decreasing if it satisfies

$$\eta \in \Gamma \text{ and } \eta \geq \zeta \ \ \Rightarrow \ \ \zeta \in \Gamma \quad .$$

Massey's theorem (5.3 p. 359 of [10]) is recalled below.

Theorem 3.1 *Consider two Markov processes on a countable partially ordered state space X, with rates $\{c^i(\eta,\gamma)\ ,\ \eta,\gamma \in X\}$ and semigroups $S^i(t)$ $(i = 1, 2)$. They satisfy*

$$S^1(t) \leq_{st} S^2(t)$$

iff for all η, ζ in X with $\eta \leq \zeta$ and all increasing sets Γ such that $\eta \in \Gamma$ or $\zeta \notin \Gamma$,

$$\sum_{\gamma \in \Gamma} c^1(\eta,\gamma) \ \ \leq \ \ \sum_{\gamma \in \Gamma} c^2(\zeta,\gamma) \quad . \tag{12}$$

For large partially ordered X, increasing subsets may be difficult to enumerate and condition (12) is not easily checked. In particular, it can be used to compare interacting particle systems on a finite S but it is not of great help when S is large. However, one can try to take advantage from the fact that for such systems only particular transitions are allowed. The global dynamics are superposition of local dynamics specified by the rates $c^i_T(\eta,\gamma)$. The subsets T involved are usually much smaller than S . In UPs as in spin systems for instance they are all singletons. In theorem 3.2, sufficient conditions under which interacting particle systems can be compared are given. They take into account the locality of the rates.

Theorem 3.2 *For $i = 1, 2$, consider interacting particle systems on a countable set S, with rates $c_T^i(\eta,\gamma)$ and semigroups $S^i(t)$. If for all T in $\mathcal{P}_f(S)$, for all η,ζ in X with $\eta \leq \zeta$,*

(a) *for all increasing subsets Γ_T of X_T such that $\eta(T) \notin \Gamma_T$ and $\zeta(T) \notin \Gamma_T$,*

$$\sum_{\gamma\in\Gamma_T} c_T^1(\eta,\gamma) \ \leq \ \sum_{\gamma\in\Gamma_T} c_T^2(\zeta,\gamma) \tag{13}$$

(b) *for all decreasing subsets Γ_T of X_T such that $\eta(T) \notin \Gamma_T$ and $\zeta(T) \notin \Gamma_T$,*

$$\sum_{\gamma\in\Gamma_T} c_T^1(\eta,\gamma) \ \geq \ \sum_{\gamma\in\Gamma_T} c_T^2(\zeta,\gamma) \tag{14}$$

then $S^1(t) \ \leq_{st} \ S^2(t)$.

Proof. In order to simplify formulations in the proof, we remark that conditions (13) and (14) can be written under the following more compact form (15).
For all T in $\mathcal{P}_f(S)$, for all η,ζ in X with $\eta \leq \zeta$ and for all increasing subsets Γ_T of X_T such that $\eta(T) \in \Gamma_T$ or $\zeta(T) \notin \Gamma_T$,

$$\sum_{\gamma\in\Gamma_T} c_T^1(\eta,\gamma) \ \leq \ \sum_{\gamma\in\Gamma_T} c_T^2(\zeta,\gamma) \ . \tag{15}$$

Theorem 3.2 is first proved for finite S using theorem 3.1 . Then it is extended to infinite S using a finite approximations technique.

Let S be finite. Then X is also finite. Assume (15) holds and let T be a subset of S. For all increasing subsets Γ of X such that $\eta \in \Gamma$ or $\zeta \notin \Gamma$, consider the subsets Γ_T^η and Γ_T^ζ of X_T respectively defined by

$$\Gamma_T^\eta \ = \ \{\gamma \in X_T\,,\, \eta_T^\gamma \in \Gamma\} \quad \text{and} \quad \Gamma_T^\zeta \ = \ \{\gamma \in X_T\,,\, \zeta_T^\gamma \in \Gamma\} \ .$$

They are both increasing subsets of X_T with $\Gamma_T^\eta \subset \Gamma_T^\zeta$. Using (15) with $\Gamma_T = \Gamma_T^\eta$ yields

$$\sum_{\gamma\in\Gamma_T^\eta} c_T^1(\eta,\gamma) \ \leq \ \sum_{\gamma\in\Gamma_T^\eta} c_T^2(\zeta,\gamma) \ .$$

Then, it follows from

$$\sum_{\gamma\in\Gamma_T^\eta} c_T^2(\zeta,\gamma) \ \leq \ \sum_{\gamma\in\Gamma_T^\zeta} c_T^2(\zeta,\gamma)$$

that for all η,ζ in X with $\eta \leq \zeta$ and all increasing subsets Γ of X such that $\eta \in \Gamma$ or $\zeta \notin \Gamma$, we have

$$\sum_{\gamma\in\Gamma_T^\eta} c_T^1(\eta,\gamma) \ \leq \ \sum_{\gamma\in\Gamma_T^\zeta} c_T^2(\zeta,\gamma) \ . \tag{16}$$

Since X is finite, by theorem 3.1 this last statement is equivalent to

$$S^1_T(t) \quad \leq_{st} \quad S^2_T(t)$$

where $S^i_T(t)$, for $i = 1, 2$, are the semigroups of the T-local processes. Since this is true for all subsets T of S, summing each side of (16) over T leads to (12) and thus to

$$S^1(t) \ \leq_{st} \quad S^2(t) \ .$$

Therefore, for a finite S, to compare interacting particle systems on S it is enough to check conditions (15) for all subsets T of S . To show that this remains valid for infinite S, we shall introduce frozen processes (definition 3.5) analogous to finite approximations for spin systems, defined in [9] chapter III p. 138 .

Definition 3.5 *Consider an interacting particle system on S with rates*

$$\{c_T(\eta, \gamma) \, , \, \eta \ \in \ X, \ T \ \in \ \mathcal{P}_f(S), \ \gamma \ \in \ X_T\}$$

and generator Ω . Let V be a finite subset of S and η in X . The process frozen to η on $S\backslash V$ is the interacting particle system on S with rates

$$\{c_T(\eta_V^{\beta(V)}, \gamma) \, , \, \beta \in X, \ T \subset V, \ \gamma \in X_T\} \ .$$

Its generator and its semigroup are denoted by $\Omega_{V,\eta}$ and $S_{V,\eta}(t)$.

Lemma 3.2 below shows that it suffices to compare frozen processes for a family of finite subsets increasing to S. Actually condition (17) is weaker than saying that the frozen processes are comparable.

Lemma 3.2 *For $i = 1, 2$, consider interacting particle systems on a countable set S, with rates $c^i_T(\eta, \gamma)$. Let T_n be finite sets which increase to S . For all n and all $\eta \in X$, denote by $S^i_{n,\eta}(t)$ the semigroup of the process frozen to η on $S\backslash T_n$. If for all $\eta \leq \zeta$ and for all n,*

$$\forall f \in \mathcal{M}(X) \ , \quad S^1_{n,\eta}(t) f(\eta) \ \ \leq \ \ S^2_{n,\zeta}(t) f(\zeta) \tag{17}$$

then $S^1(t) \ \leq_{st} \ S^2(t)$.

Proof. It is easy to check that

$$\forall f \in \mathcal{D}, \, \forall \eta \in X \ , \quad \lim_{n\to\infty} \Omega^i_{n,\eta} f \quad = \quad \Omega^i f \ .$$

By theorem 2.12 in [9], this leads to

$$\forall f \in \mathcal{C}(X), \, \forall \eta \in X \ , \quad \lim_{n\to\infty} S^i_{n,\eta}(t) f \quad = \quad S^i(t) f \ , \tag{18}$$

uniformly for t in compact sets. Let $\eta \leq \zeta$. If for all n,

$$\forall f \in \mathcal{M}(X) \ , \quad S^1_{n,\eta}(t) f(\eta) \ \ \leq \ \ S^2_{n,\zeta}(t) f(\zeta) \ .$$

Taking the limit in n leads to

$$\forall f \in \mathcal{M}(X)\,, \quad S^1(t)f(\eta) \;\leq\; S^2(t)f(\zeta)\,,$$

which is enough (lemma 3.1) to prove $S^1(t) \;\leq_{st}\; S^2(t)$.

Frozen processes are processes on S but they can be identified to processes on finite subsets of S. The process frozen to η on $S\backslash V$ can be associated to the interacting particle system on V with rates

$$\{c_T(\eta_V^\beta, \gamma)\,,\, \beta \in X_V,\ T \subset V,\ \gamma \in X_T\}\,.$$

Moreover, it is easy to show that equation (17) between frozen processes can be obtained by comparing the associated processes on T_n . The proof of theorem 3.2 for infinite S is then as follows. Take $\eta \leq \zeta$ and let T_n be finite sets increasing to S. For all T_n consider the frozen processes to η and ζ on T_n with semigroups $S^1_{n,\eta}(t)$ and $S^2_{n,\zeta}(t)$. Since T_n is finite and (15) is satisfied for all subsets T of T_n, we can use the first part of the proof with $S = T_n$ to show that for all $\eta \leq \zeta$,

$$\forall f \in \mathcal{M}(X)\,, \quad S^1_{n,\eta}(t)f(\eta) \;\leq\; S^2_{n,\zeta}(t)f(\zeta)\,.$$

The conclusion is then given by lemma 3.2 .

Turn now to a particular case. When only one site can change at a time, the rates $c^i_T(\eta, \gamma)$ can be rewritten $c^i(u, \eta, \gamma)$, using the notation of section 2 for UPs though UPs are actually only particular examples of such processes. For these processes, the sufficient conditions of theorem 3.2 are also necessary. More generally, the same holds as soon as the sets T on which the transitions occur do not overlap.

Theorem 3.3 *For $i = 1, 2$, consider processes on a countable set S, with rates*

$$\{c^i(u, \eta, \gamma)\,,\, \eta \in X,\ u \in S,\ \gamma \in W_u\}$$

and semigroups $S^i(t)$. They satisfy $S^1(t) \;\leq_{st}\; S^2(t)$ iff for all u in S, for all η, ζ in X with $\eta \leq \zeta$,

(*a*) *for all increasing subsets Γ_u of W_u such that $\eta(u) \notin \Gamma_u$ and $\zeta(u) \notin \Gamma_u$,*

$$\sum_{\gamma \in \Gamma_u} c^1(u, \eta, \gamma) \;\leq\; \sum_{\gamma \in \Gamma_u} c^2(u, \zeta, \gamma)$$

(*b*) *for all decreasing subsets Γ_u of W_u such that $\eta(u) \notin \Gamma_u$ and $\zeta(u) \notin \Gamma_u$,*

$$\sum_{\gamma \in \Gamma_u} c^1(u, \eta, \gamma) \;\geq\; \sum_{\gamma \in \Gamma_u} c^2(u, \zeta, \gamma)\,.$$

Proof. Assume $S^1(t) \leq_{st} S^2(t)$. Let u be in S and η, ζ in X such that $\eta \leq \zeta$. Let Γ_u be an increasing subset of W_u such that $\eta(u) \notin \Gamma_u$ and $\zeta(u) \notin \Gamma_u$. Consider the indicator function on X,

$$g(\xi) = 1\!\!1_{\{\xi(u) \in \Gamma_u\}} \; .$$

The function g is in $\mathcal{M}(X)$ since Γ_u is increasing. It is in D since it depends only on the configuration at site u . For $i = 1, 2$, the generators of the processes satisfy for all f in D and all η in X,

$$\Omega^i f(\eta) \;=\; \lim_{t \searrow 0} (S^i(t) f(\eta) - f(\eta))/t \; .$$

Since $g(\eta) = g(\zeta) = 0$ and $S^1(t)g(\eta) \leq S^2(t)g(\zeta)$ it comes $\Omega^1 g(\eta) \leq \Omega^2 g(\zeta)$. Then, it is easy to check that

$$\begin{aligned} \Omega^1 g(\eta) &= \sum_{\gamma \in \Gamma_u} c^1(u, \eta, \gamma) \; , \\ \Omega^2 g(\zeta) &= \sum_{\gamma \in \Gamma_u} c^2(u, \zeta, \gamma) \; , \end{aligned}$$

and (a) is satisfied. The proof is analogous for (b) with Γ_u decreasing and

$$g(\xi) = 1\!\!1_{\{\xi(u) \notin \Gamma_u\}} \; .$$

The other direction is a straightforward application of theorem 3.2 .

The conditions in theorems 3.2 and 3.3 may be difficult to check since they compare sums of rates and not individual rates. Conditions that compare individual rates are given in theorem 3.4 below.

Theorem 3.4 *For $i = 1, 2$, consider interacting particle systems on a countable set S, with rates $c^i_T(\eta, \gamma)$ and semigroups $S^i(t)$. If their transition rates satisfy for all finite subsets T of S and all configurations $\eta \leq \zeta$,*

$$\begin{array}{lll} (a) & \text{if } \gamma > \zeta(T) \text{ or } \gamma \not\sim \zeta(T) & \text{then} \quad c^1_T(\eta, \gamma) \leq c^2_T(\zeta, \gamma) \\ (b) & \text{if } \gamma < \eta(T) \text{ or } \gamma \not\sim \eta(T) & \text{then} \quad c^1_T(\eta, \gamma) \geq c^2_T(\zeta, \gamma) \end{array}$$

then $S^1(t) \leq_{st} S^2(t)$.

Proof. It suffices to check that the conditions above imply the conditions of theorem 3.2 . To do so, fix T and let Γ_T be an increasing subset of X_T such that $\eta(T) \notin \Gamma_T$ and $\zeta(T) \notin \Gamma_T$. If γ is in Γ_T, then γ cannot be smaller than $\zeta(T)$ since $\zeta(T)$ is not in Γ_T . Thus

$$\Gamma_T \;\subset\; \{\gamma \in X_T, \gamma > \zeta(T) \text{ or } \gamma \not\sim \zeta(T)\} \; ,$$

so that condition (a) of theorem 3.2 comes clearly from condition (a) above. In the same manner, let Γ_T be a decreasing subset of X_T such that $\eta(T) \notin \Gamma_T$ and $\zeta(T) \notin \Gamma_T$. If γ is in Γ_T, then γ cannot be greater than $\eta(T)$ since $\eta(T)$ is not in Γ_T . Thus

$$\Gamma_T \;\subset\; \{\gamma \in X_T, \gamma < \eta(T) \text{ or } \gamma \not\sim \eta(T)\} \; ,$$

and (b) above implies (b) in theorem 3.2 .

The conditions in theorem 3.4 will be referred to as basic conditions since they can be independently recovered by using a generalization of the so-called basic coupling or Vasershtein coupling. Details can be found in [4]. As already mentioned, the material in this section is motivated by its possible applications to UPs and RPs. We shall develop such applications in the following two sections.

4 Stochastic Comparison for UPs on $\mathbb{Z}$

In this section, stochastic comparison tools are used to prove an ergodicity theorem for a class of UPs on $\mathbb{Z}$. This result generalizes [14] for the PP on $\mathbb{Z}$, along the same line of proof. In particular, the ergodicity result justifies some statistical procedures to evaluate the performance of the system. The computation of the variance of performance estimators will also be possible. This will be discussed at the end of the section.

The processes under consideration are extended versions of the cyclic network defined in example 2.4 of section 2 to the infinite network on $\mathbb{Z}$. They are UPs defined by truncation from irreducible and reversible RPs as in equation (2). Users are located on $\mathbb{Z}$. As in Chandy and Misra's original setting [1], resources are located on the left and on the right of each user. User u can ask for two resources, l_u and r_u. Let $R_u = \{l_u, r_u\}$ denote the set of available resources for user u. Resources are shared in the sense that for each u, $l_u = r_{u-1}$. Assume that all users act individually the same Markovian way. Each individual RP is a copy of a generic Markov process with rates $\lambda(\alpha, \beta)$ on the set of 4 elements $\{\emptyset, l, r, lr\}$ with l_u identified to l and r_u identified to r. We shall assume that resources cannot be instantaneously exchanged, that is

$$\lambda(l, r) = \lambda(r, l) = 0.$$

The UP is an interacting particle system on

$$X = \bigotimes_{u \in \mathbb{Z}} \mathcal{P}(R_u)$$

with rates given by,

$\forall u \in \mathbb{Z}$, $\forall \eta \in \mathcal{A}$, $\forall \gamma \in \mathcal{P}(R_u)$,

$$c(u, \eta, \gamma) = \begin{cases} \lambda_u(\eta(u), \gamma) & \text{if } \gamma \cap \eta(u-1) = \emptyset \text{ and } \gamma \cap \eta(u+1) = \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Reversibility for the UP on $\mathbb{Z}$ is guaranteed by the reversibility of the individual RPs. This follows from an easy extension to the infinite case of corollary 1.10, p.26 in [7] that will not be detailed here. The application of stochastic comparison to UPs is not straightforward. The natural order induced by inclusion in each subset R_u does not lead to statisfying conclusions. Indeed, consider the following situation. Let η be the empty configuration and $\zeta = \eta^{l_u}_{u-1}$. Then the rate at which u can take l_u under

η is $\lambda_u(\emptyset, l_u) > 0$, although it is null under ζ. This makes it impossible to apply the results obtained in section 3 with the natural ordering. Hence, it is necessary to define a different stochastic ordering on X. A natural idea is to reverse the order on odd coordinates as in [14]. This new order is denoted by $\preceq$. Configurations satisfy $\eta \preceq \zeta$ iff $\eta(u) \subset \zeta(u)$ for all even u and $\eta(u) \supset \zeta(u)$ for all odd $u \in \mathbb{Z}$. With respect to this new order on X, the necessary and sufficient conditions for stochastic monotonicity of the whole UP express locally (*i.e.* in terms of the RPs). These conditions are given below for the generic process with rates $\lambda(\alpha, \beta)$. Translation invariance allows to shift them to all users

$$\begin{aligned} \lambda(\emptyset, l) + \lambda(\emptyset, lr) &\leq \lambda(r, lr) \\ \lambda(\emptyset, r) + \lambda(\emptyset, lr) &\leq \lambda(l, lr) \\ \lambda(lr, \emptyset) + \lambda(lr, l) &\leq \lambda(r, \emptyset) \\ \lambda(lr, \emptyset) + \lambda(lr, r) &\leq \lambda(l, \emptyset) \end{aligned} \tag{20}$$

Actually, these conditions are exactly the necessary and sufficient conditions for the individual RP to be a monotone Markov process with respect to inclusion in $\mathcal{P}(R_0)$ (see theorem 3.1).

Recall that a Feller process is ergodic if from any initial distribution, it converges to a unique stationary distribution (see [9]). The ergodicity of a translation invariant reversible UP ensures fairness in the access of users to resources.

Theorem 4.1 *A UP on $\mathbb{Z}$ defined as in equation (19) by truncation from translation invariant reversible individual RPs is ergodic provided that these RPs are monotone (conditions (20)).*

Proof. Once monotonicity conditions with respect to some stochastic ordering are given, the technique of proof is rather classical. It is based upon the finite approximations of the process. These approximations are built up by conditioning the state of the process to extremal configurations outside the set $[-n, +n]$ (see [9]). There exists a unique stationary distribution for all finite approximations of size n. Moreover, the process is reversible and the interactions between users are of nearest neighbor type. Hence, the stationary distribution is that of a homogeneous Markov chain on $[-n, +n]$. The transitions of this chain are defined on the sets of left and right resources for successive users. The corresponding transition matrix on $\{\emptyset, l, r, lr\}$ is the following

$$P = \begin{pmatrix} p_{\emptyset\,\emptyset} & p_{\emptyset\,l} & p_{\emptyset\,r} & p_{\emptyset\,lr} \\ p_{l\,\emptyset} & p_{l\,l} & p_{l\,r} & p_{l\,lr} \\ p_{r\,\emptyset} & 0 & p_{r\,r} & 0 \\ p_{lr\,\emptyset} & 0 & p_{lr\,r} & 0 \end{pmatrix} \tag{21}$$

where the quantities $p_{\alpha\,\beta}$ are positive transition probabilities. Since the Markov chain is ergodic, the UP must be so by stochastic comparison (see [9] and [14] for details).

Example 4.1 .

Here is an example of rates to which theorem 4.1 applies.

$$\lambda(\emptyset, l) = \lambda(\emptyset, r) = a \quad ; \quad \lambda(l, lr) = \lambda(r, lr) = b$$

$$\lambda(lr, l) = \lambda(lr, r) = c \quad ; \quad \lambda(l, \emptyset) = \lambda(r, \emptyset) = d$$

$$\lambda(\emptyset, lr) = ab \quad ; \quad \lambda(lr, \emptyset) = cd$$

where a, b, c, d are positive real numbers. Conditions (20) hold provided that

$$a \leq b/(1+b) \quad \text{and} \quad c \leq d/(1+d)\,.$$

Example 4.2 .

When the generic process is that of example 2.1 (independent resources) with two resources, the monotonicity conditions (20) are obviously satisfied. According to theorem 4.1, the corresponding UP on $\mathbb{Z}$ is then ergodic.

When larger resource sets are considered, as in example 2.4, the same technique applies again. The change of ordering on odd coordinates leads to non empty conditions for the ergodicity of the corresponding UPs on $\mathbb{Z}$. In particular it is easily seen that the individual RPs must be monotone with respect to the natural order (transitions between non comparable configurations are forbidden). However for general RPs, the monotonicity of the individual RPs is not sufficient for the monotonicity of the whole UP since local monotonicity conditions do not imply conditions (20). Generally, additional local conditions must be taken into account. Listing all conditions may be possible but tedious in general. For this reason, each particular case must be handled with individually. For example 2.1 with more than two resources, it is easy to check that the monotonicity conditions of theorem 3.2 are satisfied. Thus, the conclusion holds for the corresponding UP on $\mathbb{Z}$.

Looking further, a useful consequence of ergodicity is the possibility to define a statistical structure for the estimation of some equilibrium quantities. Consider again a line network consisting in a large and unknown number of users. Associate to this system a UP defined as in equation (19). Assume that the equilibrium state of a small number n of neighboring users is observed. For instance, we wish to estimate the proportion π_l of users using only their left resource l. The statistical structure for the observations can be parametrized by the coefficients of the transition matrix P defined in equation (21). An explicit formula can be obtained for the quantity π_l in terms of the transition probabilities $p_{\alpha\,\beta}$. Indeed, π_l is the weight corresponding to resource l in the the invariant probability π of P. Naturally, to estimate π_l, we shall choose

$$\overline{X} = \frac{1}{n} \sum_{u=1}^{n} X_u$$

with $X_u = 1$ when u uses resource l and $X_u = 0$ otherwise. Again, the variance of this estimator expresses in terms of powers of the transition matrix P and the invariant probability π_l

$$\sigma^2(\overline{X}) = \frac{1}{n}\pi_l\,(1-\pi_l) + 2\sum_{k=1}^{n-1} \frac{n-k}{n^2}\,\pi_l\,(P^k(l,l) - \pi_l)\,. \tag{22}$$

As a particular case, we turn back to the Philosophers Process on $\mathbb{Z}$. User u can use l and r together but not separately (see section 2). The state space for user u reduces to $\{\emptyset, lr\}$. The stationary distribution of the PP is that of a Markov chain indexed by integers. The transition matrix for this Markov chain is

$$P = \begin{pmatrix} 1-p & p \\ 1 & 0 \end{pmatrix}$$

with $p = (2\lambda + 1 - (4\lambda + 1)^{1/2})/2\lambda$ (see Ycart [14]). Equation (22) reduces then to

$$\sigma^2(\overline{X}) = \frac{p}{(1+p)^2} \left\{ \frac{1}{n} + 2 \sum_{l=1}^{n-1} (-1)^l p^l \frac{n-l}{n^2} \right\}.$$

5 Stochastic Comparison and Resource Processes

In this section, we focus on RPs as defined in section 2. From a given UP, a logical way to define a process in the set of resources is by taking the union of all resource sets used at each time (*cf.* theorem 2.1). If the order on configurations is the partial order defined by inclusion on each coordinate, then the union of all coordinates is an increasing function. If the UP is monotone then the union of its coordinates is a monotone process. But the UPs under consideration cannot be monotone with respect to the natural ordering, as already pointed out in last section. So that the cases in which the aggregated RP would be monotone as a consequence of the monotonicity in the underlying UP are too particular to be worth developing here. At the end of section 2, we mentioned another way of defining RPs. We shall develop this approach here. The RPs under consideration are defined by superposition from a collection of local RPs indexed by a finite user set U. We shall define a class of RPs to which the comparison results of section 3 apply. To show that these RPs are actually relevant to model resource sharing, a (somewhat caricatural) model for the administration of a math library is given and followed by some natural applications of monotonicity.

Let $(R_u)_{u \in U}$ be distinct finite subsets indexed by U. The whole set of resources is

$$R = \bigcup_{u \in U} R_u .$$

For all u in U, we consider a R_u-local RP with rates

$$\{c_u(\eta, \zeta) \, , \, \eta, \zeta \in \mathcal{P}(R)\} .$$

Recall that only transitions involving resources in R_u are allowed. We consider a RP defined by superposition of such processes as in equation (5). Such a process is a special case of the interacting particle systems discussed in section 3. The results of that section can therefore be applied directly to compare such RPs. The corresponding rates $c_T(\eta, \gamma)$ (with the notation of section 3) are defined by

$$\begin{array}{rrcll} \forall u \in U\,, & c_T(\eta,\gamma) & = & 0 & \text{if } T \neq R_u \text{ and} \\ & c_{R_u}(\eta,\gamma) & = & c_u(\eta,\eta^{\gamma}_{R_u})\,. & \end{array}$$

To simplify we shall write $c_u(\eta,\gamma)$ instead of $c_u(\eta,\eta^{\gamma}_{R_u})$, for all $\eta \in \mathcal{P}(R)$ and $\gamma \in \mathcal{P}(R_u)$. Consider two RPs with rates

$$\{c^i_u(\eta,\gamma)\,,\, \eta \in \mathcal{P}(R),\; \gamma \in \mathcal{P}(R_u),\; u \in U\}$$

and semigroup $S^i(t)$, for $i = 1, 2$. Theorem 3.4 of section 3 applies. If for all u in U and all η, ζ in $\mathcal{P}(R)$ with $\eta \subset \zeta$,

$$\begin{array}{llll} (a) & \text{if } \gamma \supset \zeta(R_u) \text{ or } \gamma \not\sim \zeta(R_u) & \text{then} & c^1_u(\eta,\gamma) \leq c^2_u(\zeta,\gamma) \\ (b) & \text{if } \gamma \subset \eta(R_u) \text{ or } \gamma \not\sim \eta(R_u) & \text{then} & c^1_u(\eta,\gamma) \geq c^2_u(\zeta,\gamma) \end{array} \tag{23}$$

then $S^1(t) \;\leq_{st}\; S^2(t)$.

In other words, if for all u, the R_u-local RPs can be compared then their superpositions can also be compared. Monotonicity conditions are easily deduced by taking $c^1_u = c^2_u$ in the above conditions. A first type of information that is intuitive but hard to reach by other techniques can be deduced from conditions (23). It concerns the monotonicity in the *birth and death* rates of the mean value of any increasing function on $\mathcal{P}(R)$. Let $c^1_u(\eta,\gamma)$ be the rates of a monotone RP. Consider another RP with rates $c^2_u(\eta,\gamma)$ satisfying

$$\begin{array}{rcll} c^2_u(\eta,\gamma) & = & b_u\; c^1_u(\eta,\gamma) & \text{if } \gamma \supset \eta(R_u)\,, \\ c^2_u(\eta,\gamma) & = & d_u\; c^1_u(\eta,\gamma) & \text{if } \gamma \subset \eta(R_u)\,, \\ c^2_u(\eta,\gamma) & = & c^1_u(\eta,\gamma) & \text{in any other cases,} \end{array} \tag{24}$$

with $b_u \geq 1$ and $1 \geq d_u > 0$. The following statement is a direct consequence of theorem 3.4.

Theorem 5.1 *Let $f(\eta)$ be an increasing function and μ an arbitrary probability measure on $\mathcal{P}(R)$. For $i = 1, 2$, let $\{\eta^i_t\;;\; t \geq 0\}$ be the RPs with rates $c^i_u(\eta,\gamma)$ satisfying equations (24). Under the above monotonicity conditions on $c^1_u(\eta,\gamma)$ (conditions (23) with $c^1_u = c^2_u$),*

$$\mathbb{E}_\mu[f(\eta^1_t)] \;\;\leq\;\; \mathbb{E}_\mu[f(\eta^2_t)]\;.$$

Suppose that an increasing function $f(\eta)$ is chosen to evaluate a RP with rates $c_u(\eta,\gamma)$. Then the mean value of this function at time t increases as the birth rates of the process increase or as the death rates decrease.

A second type of information is related to the monotonicity with respect to the initial conditions of the global RP. Note that R is an extremal configuration for the inclusion. Let $\{\eta_t\;;\; t \geq 0\}$ be a monotone RP and $S(t)$ the associated semigroup. For an arbitrary probability measure μ on X, the following relation is satisfied.

$$\forall t \geq 0, \quad \mu S(t) \leq \delta_R S(t)\;,$$

where δ_R is the Dirac mass at R. In the same way, the empty configuration $\emptyset$ is minimal for the inclusion. Then, for an arbitrary probability measure μ on X

$$\forall t \geq 0, \quad \delta_\emptyset S(t) \leq \mu S(t) \; .$$

Finally, the following statement can be viewed as an extension of theorem 2.3 p. 135 chapter III in [9].

Theorem 5.2 *Let $f(\eta)$ be an increasing function. Let $\{\eta_t \; ; \; t \geq 0\}$ be a RP with rates $c_u(\eta, \gamma)$ and μ be an arbitrary probability measure on X. Under the monotonicity conditions obtained from (23) with $c_u^1 = c_u^2 = c_u$,*

$$\mathbb{E}_\emptyset[f(\eta_t)] \leq \mathbb{E}_\mu[f(\eta_t)] \leq \mathbb{E}_R[f(\eta_t)] \, .$$

To show that these models are relevant for resource sharing and to illustrate the previous results, we turn to a (caricatural) model of a math library. Books are resources. A more or less rough classification of the resources exists. Books are classified according to their topics of contents : algebra, analysis, geometry, probability theory and so on. Users from different departments borrow books in their topic of interest for an exponentially distributed duration. For many reasons, the librarian cannot build a UP. He does not even know exactly which user possesses each borrowed book. Sometimes, users forget to fill the borrowing form or make it unreadable. Futhermore, the evolution of the users might not be of spin nature. Indeed, users may exchange books without returning them to the library or a user may borrow some books for other users and for himself. To these cases correspond synchronized or non spin transitions in the state space for the users. To overcome the difficulties due to his lack of information about users' behavior, the librarian can look at the evolution of the set of wandering resources. Doing so, a set of new formal users is implicitely defined. A topic class identifies to a formal user u for the librarian. It is an element of the set U of all topics. Its individual resource set R_u is the subset of books relevant to the topic. A book may cover two or more different topics. Thus, subsets R_u may overlap. For each topic u, the librarian knows the individual rates

$$\{\lambda_u(\alpha, \beta), \; \alpha, \beta \in \mathcal{P}(R_u)\}.$$

From these rates, the librarian can define a RP with $R = \cup_{u \in U} R_u$. When a subset of books disappears from the library, this is due to one user among those that have the books in their resource set. Therefore if $\eta \subset \zeta$ the corresponding rate is actually a sum of the type

$$\sum_{u : \eta \Delta \zeta \subset R_u} \lambda_u(\eta(R_u), \zeta(R_u)),$$

When a return or an exchange of a subset of books occurs, the situation is different. Several users were likely to return or exchange the books but only one of them actually did it. The corresponding rate cannot be a sum of individual rates anymore. To each topic class R_u, the librarian assigns a positive weight for the topic user u to be responsible for the change from η to ζ. Denote this weight by $p_{\eta,\zeta}(u)$ and assume that $\sum_{u\in U} p_{\eta,\zeta}(u) = 1$. User u cannot be responsible for a change in the library involving books which are not in his resource set R_u. Therefore, if $\eta\Delta\zeta \not\subset R_u$ then $p_{\eta,\zeta}(u) = 0$. The librarian will actually consider local RPs as follows

$$\begin{aligned}
\text{if } \eta(R_u) \subset \gamma, \quad c_u(\eta,\gamma) &= \lambda_u(\eta(R_u),\gamma), \\
\text{if } \eta(R_u) \not\subset \gamma, \quad c_u(\eta,\gamma) &= p_{\eta,\eta^\gamma_{R_u}}(u)\, \lambda_u(\eta(R_u),\gamma)
\end{aligned}$$

and then define a RP by superposition as in equation (5).

In what follows, we shall specify additional conditions on the local rates. Under these conditions the RP will be monotone. We shall assume that when some books are returned to the library by a user, this user will more likely replace the returned books by others than leave the library with empty hands. Also, when new books are borrowed by a user, this user will more often give back books previously borrowed rather than to keep all books. In other words, this means that the librarian will observe more exchanges of books than pure returns or pure borrowings of books. Thus, a natural assumption is that for each u, for all η in $\mathcal{P}(R)$ and all γ in $\mathcal{P}(R_u)$ with $\gamma \not\sim \eta(R_u)$,

$$c_u(\eta,\gamma) \;\geq\; \max\{c_u(\eta \cup \eta^\gamma_{R_u},\gamma), c_u(\eta \cap \eta^\gamma_{R_u},\gamma)\}\,. \tag{25}$$

Regarding pure returns or pure borrowings of books, a natural assumption is that users borrow or return more frequently small sets of books than large ones. This leads to the conditions (26) below. For all u and all η,ζ in $\mathcal{P}(R)$ with $\eta \subset \zeta$,

$$\begin{aligned}
\text{if } \gamma \supset \zeta(R_u) \text{ then} \quad c_u(\eta,\gamma) &\leq c_u(\zeta,\gamma) \\
\text{if } \gamma \subset \eta(R_u) \text{ then} \quad c_u(\eta,\gamma) &\geq c_u(\zeta,\gamma)\,.
\end{aligned} \tag{26}$$

We may also have to compare exchanges of books for which none of the above situations can be favored. In this case, the corresponding rates are assumed to be equal. Then, for all u and all η,ζ in $\mathcal{P}(R)$ with $\eta \subset \zeta$,

$$\text{if } \gamma \not\sim \eta(R_u) \text{ and } \gamma \not\sim \zeta(R_u) \text{ then} \quad c_u(\eta,\gamma) \;=\; c_u(\zeta,\gamma)\,. \tag{27}$$

It is easy to check that conditions (25), (26), (27) imply (23) with $c^1_u = c^2_u$. Thus, the corresponding RP is monotone. This allows the librarian to evaluate the state of his library by choosing a score function $f(\eta)$ which is monotone with respect to the inclusion operator. For instance, the number of books published in 1995 is an increasing function of the configuration of the library. Useful monotonicity properties are deduced from theorem 5.1 and theorem 5.2. In practice, a Monte Carlo estimation of the mean value $\mathbb{E}_\mu[f(\eta_t)]$ can be obtained by starting a numerical simulation of the RP from one of the two extremal configurations $\emptyset$ (no book borrowed) and R (no book on the shelves).

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