Counting stable sets on Cartesian products of graphs

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Abstract

We study the generating functions for the number of stable sets of all cardinalities, in the case of graphs which are Cartesian products by paths, cycles, or trees. Explicit results are given for products by cliques. Algorithms based on matrix products are derived for grids, cylinders, toruses and hypercubes.

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1 Introduction

The stable sets of an undirected graph (also called independent sets in many references) are subsets of vertices no two of which are connected (see [1] for a general reference). A stable set which is not a strict subset of another stable set is called maximal, and a maximal stable set of highest cardinality is called maximum, its cardinality being the stable number, or independence number of the graph. The two problems of determining maximal and maximum stable sets have received considerable attention, particularly since the computation of the stable number has been proved to be an $\mathcal{NP}$-complete problem by Karp [4] (see section 4 of Zito [12] and references therein). Since any stable set is a subset of a maximal one, the problem of counting all stable sets may comparatively seem of lesser interest, though it appeared long ago in the litterature. In 1943 Kaplanski [3] published his solution to the "Problème des Ménages", where one of the key lemmas consisted in counting the number of stable sets with exactly k vertices on a cycle with m vertices (*cf.* also Riordan [6] p. 198). This number is

$$\frac{m}{m-k}\binom{m-k}{k} \quad .$$

Using hypergeometric functions, one can check directly that the corresponding generating function is

$$\sum_{k=0}^{[m/2]} \frac{m}{m-k}\binom{m-k}{k}\lambda^k = \frac{(1+\sqrt{4\lambda+1})^m + (1-\sqrt{4\lambda+1})^m}{2^m} \quad , \tag{1}$$

where $[m/2]$ is the integer part of $m/2$.

Our main objective in this article is to show that the generating function of the numbers of stable sets can be computed, at least algorithmically, for any graph which is the Cartesian product of a certain "small" graph by a path, a cycle or a tree. Let $G = (V, E)$ be a finite undirected graph without loops or multiple edges. If $x \in V$ is a vertex we denote by $N(x)$ the set of his neighbors on the graph.

$$N(x) = \{y \in S \ s.t. \ \{x, y\} \in E\}.$$

We shall identify stable sets with the corresponding $(0,1)$-incidence vectors, indexed by V. These vectors will be referred to as *stable vectors* of the graph, and their set will be denoted by S_G. They are the vertices of the stable-set polytope ([7] p. 1667).

$$S_G = \left\{ \eta \in \{0,1\}^V \ s.t. \ \forall x \in V \quad \eta(x) = 1 \Longrightarrow \eta(y) = 0 \ \forall y \in N(x) \right\} .$$

It will be convenient to consider the empty set as a particular stable set, corresponding to the null vector, denoted by 0_G. The generating function of all stable sets of G will be denoted by Z_G.

$$Z_G(\lambda) = \sum_{\eta \in S_G} \lambda^{\sum_{x \in V} \eta(x)} \quad . \tag{2}$$

The notation Z_G refers to the interpretation of the generating function as the partition function of a spin system. That spin system may be understood in the sense of interacting particle systems in probability ([11]), in the physical sense of a hard-core lattice model as in [8], or in the sense of graph theory (*cf.* [2]). The function Z_G is a polynomial in λ, the coefficient of order k being the number of stable sets with k vertices exactly. The degree of Z_G as a polynomial in λ is the stable number of G, usually denoted by $\alpha(G)$ (see chapter 31 of [10] for some illustrations). The value of Z_G for $\lambda = 1$ is the cardinality of S_G. For instance, for the cycle with 100 vertices this number is

$$792,070,839,848,372,253,127.$$

The size of the state space S_G prohibits in general direct enumeration, even by computer. However, when the graph G has a particular structure, one can try to take advantage of that structure to find a reasonable algorithm. For instance, maximal stable sets of perfect graphs can be found in polynomial time [7]. Wilf [9] gives a fairly complete description of maximal stable sets of trees. The graphs we consider here are Cartesian products of some subgraph g by paths, cycles, or trees. Provided the set S_g is explicitly known, we show that it is possible to compute Z_G using m products of matrices indexed by S_g, where m is the size of the path, cycle, or tree under consideration. In the case where g is a clique, formulae will be given in propositions 3.1, 3.2 and 3.3. Explicit algorithms will also be given for the cases where G is a grid, a torus, a cylinder, or a hypercube. In section 2 we define ladder graphs and describe our general method of computation for Z_G. Section 3 is devoted to explicit results obtained applying the method of the previous section. Products by cliques are treated first. Then algorithms are described for grids, toruses and hypercubes.

2 Ladder graphs

We define a class of finite graphs for which the computation of the generating function of stable sets will reduce to matrix products. The main result of this section is proposition 2.2, that describes the computation of Z_G for a graph G which is a Cartesian product by a tree. The definition of the Cartesian product (also called Cartesian sum in some references) is classical (see for instance Chartrand and Lesniak [1] p. 11).

Definition 2.1 *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two undirected graphs. The Cartesian product, denoted by $G_1 \times G_2$ is the graph $G = (V, E)$ defined by*

$$\begin{aligned} V &= V_1 \times V_2 \\ E &= \{\ \{(x,y),(x',y')\}\ ;\ (x,y) \textit{ and } (x',y') \in V_1 \times V_2 \qquad \textit{s.t.} \\ &\quad\ x = x' \quad \textit{and} \quad \{y,y'\} \in E_2 \\ &\quad\ \textit{or} \\ &\quad\ y = y' \quad \textit{and} \quad \{x,x'\} \in E_1\ \}\quad . \end{aligned}$$

Let us first explain what is the interest of considering Cartesian products for counting stable states. Assume that G is the Cartesian product of some "small" graph g (of which stable sets have been enumerated) by some other graph $H = (V_H, E_H)$. We shall view G as being formed by copies of g indexed by the vertices of H, and for $x \in V_H$ call g_x the copy on vertex x. Let η be a stable vector of G. For all $x \in V_H$, we denote by η_x its restriction to g_x, which is again a stable vector of g_x. Any stable vector of G can be written

$$\eta \;=\; (\eta_x)_{x \in V_H} \quad .$$

However S_G is far from being the direct product of the S_{g_x}'s. Two stable vectors η_x and η_y corresponding to neighbors x and y have to be compatible in the sense that none of their coordinates are simultaneously equal to 1. The notion of compatibility is of course the key to our study and the compatibility matrix, to be defined below, will be an important tool.

Definition 2.2 *Let $g = (v, e)$ be a finite graph. Two stable vectors ζ and ξ of g are said to be compatible if*

$$\forall x \in v \qquad \zeta(x)\,\xi(x) = 0 \quad .$$

The compatibility matrix P_g of a finite graph g is the square matrix indexed by the elements of S_g defined by

$$\begin{aligned} \forall \zeta, \xi \in S_g\,,\;\; P_g(\zeta, \xi) \;&=\; \lambda^{\sum_{x \in v} \zeta(x)} \quad \textit{if } \zeta \textit{ and } \xi \textit{ are compatible} \\ &=\; 0 \qquad\qquad\quad\;\; \textit{otherwise.} \end{aligned}$$

There is no natural ordering of the elements of S_g. However we shall agree that the first line of P_g corresponds to the empty stable vector 0_g. Hence all its elements are equal to 1.

To count stable vectors on a Cartesian product $g \times H$, the idea is to isolate one vertex x of H, then sum over $\zeta \in S_{g_x}$ the numbers of those stable vectors η which are compatible with the fact that $\eta_x = \zeta$. In the particular case where g is reduced to one point, this idea leads immediately to the following relation.

Proposition 2.1 *Let $G = (V, E)$ be a finite undirected graph. Let x be an element of V We denote by $G - x$ (respectively $G - \overline{x}$) the graph obtained by removing x (respectively $\{x\} \cup N(x)$) and all adjacent edges from G. Then*

$$Z_G \;=\; Z_{G-x} + \lambda\, Z_{G-\overline{x}} \quad . \tag{3}$$

Notice that lemma 1 in Liu [5] is an immediate consequence of equation (3). In the case of Cartesian products by trees, one obtains a recursive way of computing generating functions.

Proposition 2.2 *Let H be a tree and x a vertex of degree d of H. Denote by $y_1, \dots, y_d$ the neighbors of x, and by $H_1, \dots, H_d$ the d connected components (trees) of $H - x$. Let $G = g \times H$, and for $i = 1, \dots, d$, $G_i = g \times H_i$. For all $\eta \in S_g$, denote by Z_G^η the generating function of those stable vectors of G whose restriction to g_x is η.*

$$Z_G^\eta(\lambda) = \sum_{\substack{\xi \in S_G \\ \xi_x = \eta}} \lambda^{\sum_{z \in V} \xi(z)} \quad .$$

Similarly, denote by $Z_{G_i}^\zeta$ the generating function of those stable vectors of G_i whose restriction to g_{y_i} is ζ

$$Z_{G_i}^\zeta(\lambda) = \sum_{\substack{\xi \in S_{G_i} \\ \xi_{y_i} = \zeta}} \lambda^{\sum_{z \in V} \xi(z)} \quad .$$

One has

$$Z_G = \sum_{\eta \in S_g} Z_G^\eta \quad ,$$

and

$$Z_G^\eta = \lambda^{\sum_z \eta(z)} \sum \prod_{i=1}^{d} Z_{G_i}^{\zeta_i}(\lambda) \quad , \tag{4}$$

where the sum on the righthand side of the last identity extends over those d-tuples $(\zeta_1, \dots, \zeta_d)$ of elements of S_g, all compatible with η.

Proof. It suffices to notice that a stable vector coinciding with η on g_x must take compatible values $(\zeta_1, \dots, \zeta_d)$ on the neighbors of x. This is the only restriction. Since H is a tree, $H_1, \dots, H_d$ are disconnected trees, and $G_1, \dots, G_d$ are also disconnected. Once the values of some stable vector on $g_x, g_{y_1}, \dots, g_{y_d}$ have been fixed, any set of stable vectors on $G_1, \dots, G_d$, form a stable vector of $G - x$. Thus the corresponding generating functions multiply. ■

The problem considered here fits in several ways in the general framework of spin models, as described in Jaeger [2]. Here is one. One can look at the stable vectors $(\eta_x)_{x \in V_H}$, as a particular coloring of the vertices of H. Suppose that an orientation of the edges of the tree H has been chosen, and an extra vertex, colored with 0_g, has been added, all leaves being connected to that extra vertex. To the edge joining x to x' can be associated the value of the compatibility matrix $P_g(\eta_x, \eta_{x'})$. The weight of the coloring is the product of all these values, *i.e.* λ elevated to the sum of all of $\eta(x)$'s. The sum of weights Z_G of all colorings is the partition function of that spin model.

Throughout the rest of this section, we shall particularize proposition 2.2 in order to get more explicit results for Cartesian products by paths, cycles and balanced trees. We denote by L_m the path with m vertices and by C_m the cycle with m vertices.

Definition 2.3 *Let $g = (v, e)$ be a finite undirected graph. We call path ladder (respectively cycle ladder) with rung g and length m the Cartesian product $g \times L_m$ (respectively $g \times C_m$).*

The main results concerning path and cycle ladders are given in propositions 2.3 and 2.4. They express the generating function in terms of powers of the matrix P_g. They can be deduced from proposition 2.2, but they are also consequences of the following lemma that gives the interpretation of the coefficients of P_g^m in terms of generating functions.

Lemma 2.1 *For all $\eta, \zeta \in S_g$, the coefficient of order (η, ζ) of P_g^m is the generating function of those stable sets on the path ladder $G = g \times L_{m+1}$ that coincide with η at one end of L_{m+1} and with ζ at the other, divided by $\lambda^{\sum_{z\in v}\zeta(z)}$.*

$$P_g^m(\eta,\zeta) = \lambda^{-\sum_{z\in v}\zeta(z)} \sum_{\substack{\xi\in S_G\\ \xi(1)=\eta\\ \xi(m+1)=\zeta}} \lambda^{\sum_{z\in V}\xi(z)} \quad .$$

Proof. For $m = 1$ this is simply definition 2.2. Assuming the result is true for $m-1$, it is easy to check it for m, by summing over $\zeta' \in S_g$ those stable vectors that coincide with η at the first end, with ζ and the last end, and with ζ' at the next to last vertex. ■

Proposition 2.3 *Let $G = g \times L_m$ be a path ladder. The generating function of the stable sets of G is*

$$Z_G = P_g^{m+1}(0_g, 0_g) \quad . \tag{5}$$

Proof. There is a one-to-one correspondance between stable vectors on $g \times L_m$ and those stable vectors on $g \times L_{m+2}$ which coincide with 0_g at both ends. ■

Proposition 2.4 *Let $G = g \times C_m$ be a cycle ladder with $m > 1$. The generating function of the stable sets of G is*

$$Z_G = trace[P_g^m] \quad . \tag{6}$$

Proof. Here one uses the one-to-one correspondance between stable vectors of $g \times C_m$ and those stable vectors on $g \times L_{m+1}$ whose restrictions at both ends of L_{m+1} coincide. ■

Explicit formulae can also be given for balanced d-ary trees. Let T_m^d be a finite tree consisting of a distinguished vertex O called root with d edges emanating from it. Each neighbor of O also has d more edges emanating from it and so on. Index m refers here to the height of the tree, which is the maximal length of a path from O to the leaves of T_m^d. Such trees can be constructed recursively: T_m^d is obtained by considering d versions of T_{m-1}^d and connecting their roots to a new vertex, the root of T_m^d. Notice that T_m^d is different from a regular (Cayley) tree, since each vertex has degree $d+1$ except for the root, which has only degree d. The generating function for a Cayley tree can easily be deduced from propositions 2.2 and 2.5 below. The recursive construction of T_m^d makes calculations easier. Of course T_m^1 is the path L_m and proposition 2.3 can be deduced from proposition 2.5 below. Since the result is much more simple for a path ladder, we preferred to treat that particular case first.

Definition 2.4 *Let $g = (v, e)$ be a finite undirected graph. We shall refer to the Cartesian product $g \times T_m^d$ as tree ladder with rung g, degree d and height m.*

The generating function of stable sets of a tree ladder can be expressed in terms of the compatibility matrix P_g.

Proposition 2.5 *Let $G = g \times T_m^d$ be a tree ladder. The generating function of the stable sets of G is*

$$Z_G = \sum_{\eta \in S_g} Z_m^\eta \quad ,$$

where the Z_m^η are the components of a vector Z_m indexed on S_g and defined recursively by $Z_0 = (1, 0, \ldots, 0)^t$ and

$$\forall m \geq 1 \ , \qquad Z_m = [\ [\ P_g\ ]^{(1/d)}.Z_{m-1}\ ]^{(d)} \quad , \tag{7}$$

where for any vector or matrix v, $[\,v\,]^{(d)}$ denotes the vector or matrix with coefficients equal to those of v elevated to the power d.

Proof. With the notations of proposition 2.2, one can choose the root O as x, so that the connected components $H_1, \ldots, H_d$ of $T_m^d - O$, are identical copies of T_{m-1}^d. Then (7) is an easy consequence of (4). ∎

3 Applications

Obviously propositions 2.3, 2.4 and 2.5 are of most interest when the compatibility matrix P_g is not too large and can be computed easily. Then taking its m-th power for large m's will be much less expensive than a direct enumeration of the stable sets on G. In this section we apply the results of section 2 to some particular rungs g, namely cliques, paths and cycles. The case of hypercubes will be treated at the end of the section.

Let K_n denote the clique with n vertices. For $g = K_n$, there are only $n+1$ stable vectors in S_g namely 0_g and the n stable vectors with exactly one non null coordinate. All non empty stable sets are equivalent, and the powers of the compatibility matrix P_g can be explicitly computed. We first state the results for the path and cycle cases.

Proposition 3.1 *Let $G = K_n \times L_m$. The generating function of the stable sets of G is*

$$
\begin{aligned}
Z_G \;&=\; \frac{1}{\sqrt{\Delta}}\,\Big[\;\frac{(1+n\lambda)}{2^m}\,\big[\,(1+(n-1)\lambda+\sqrt{\Delta})^m-(1+(n-1)\lambda-\sqrt{\Delta})^m\,\big]\\
&+\;\frac{\lambda}{2^{m-1}}\,\big[\,(1+(n-1)\lambda+\sqrt{\Delta})^{m-1}-(1+(n-1)\lambda-\sqrt{\Delta})^{m-1}\,\big]\;\Big]\\
&\qquad \textit{with}\quad \Delta \;=\; 1+2(n+1)\lambda+(n-1)^2\lambda^2\;. \qquad (8)
\end{aligned}
$$

Proposition 3.2 *Let $G = K_n \times C_m$ with $m > 1$. The generating function of the stable sets of G is*

$$
\begin{aligned}
Z_G \;&=\; \frac{(n-1)(-2)^m\lambda^m+[1+(n-1)\lambda+\sqrt{\Delta}]^m+[1+(n-1)\lambda-\sqrt{\Delta}]^m}{2^m}\\
&\qquad \textit{with}\quad \Delta \;=\; 1+2(n+1)\lambda+(n-1)^2\lambda^2\;. \qquad (9)
\end{aligned}
$$

Formula (1) in the introduction is a particular case of (9) for $n = 1$. Notice that for $n \geq 2$ the stable number $\alpha(G)$ of $G = K_n \times L_m$ or $G = K_n \times C_m$ is m. In both cases Z_G is indeed a polynomial in λ with degree m since odd powers of $\sqrt{\Delta}$ vanish in expressions (8) and (9). Also replacing n or m by 2 in (8) and (9) leads to the same expression as expected, since $L_2 = C_2 = K_2$.

Proof. The matrix P_g and its powers have a very particular structure. Recall that the first index corresponds to 0_g.

$$
P_g^k = \begin{pmatrix}
\alpha_k & \beta_k & \dots & \dots & \beta_k\\
\gamma_k & \delta_k & \epsilon_k & \dots & \epsilon_k\\
\vdots & \epsilon_k & \ddots & \ddots & \vdots\\
\vdots & \vdots & \ddots & \ddots & \epsilon_k\\
\gamma_k & \epsilon_k & \dots & \epsilon_k & \delta_k
\end{pmatrix}
$$

with $\alpha_1 = \beta_1 = 1$, $\delta_1 = 0$, $\gamma_1 = \epsilon_1 = \lambda$.
All we need to compute is $P_g^{k+1}(0_g, 0_g) = \alpha_{k+1}$ for the path case and $trace[P_g^k] = \alpha_k + n\delta_k$ for the cycle case. This can be done recursively.

$$
\begin{aligned}
\begin{bmatrix} P_g^{k+1}(0_g,0_g)\\ P_g^{k+1}(0_g,\zeta)\end{bmatrix} \;&=\; \begin{bmatrix} 1 & n\lambda\\ 1 & (n-1)\lambda\end{bmatrix} \quad \begin{bmatrix} P_g^{k}(0_g,0_g)\\ P_g^{k}(0_g,\zeta)\end{bmatrix}\\
&=\; \begin{bmatrix} 1 & n\lambda\\ 1 & (n-1)\lambda\end{bmatrix}^k \begin{bmatrix} 1\\ 1\end{bmatrix} \qquad (10)
\end{aligned}
$$

and

$$\begin{bmatrix} P_g^{k+1}(0_g,\zeta) \\ P_g^{k+1}(\zeta,\zeta) \\ P_g^{k+1}(\xi,\zeta) \end{bmatrix} = \begin{bmatrix} 1 & 1 & (n-1) \\ \lambda & 0 & (n-1)\lambda \\ \lambda & \lambda & (n-2)\lambda \end{bmatrix} \begin{bmatrix} P_g^{k}(0_g,\zeta) \\ P_g^{k}(\zeta,\zeta) \\ P_g^{k}(\xi,\zeta) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & (n-1) \\ \lambda & 0 & (n-1)\lambda \\ \lambda & \lambda & (n-2)\lambda \end{bmatrix}^k \begin{bmatrix} 1 \\ 0 \\ \lambda \end{bmatrix} \tag{11}$$

where ζ and ξ are any two distinct non null stable vectors.
In the path case solving (10) leads to (8) and in the cycle case (11) leads to (9). ∎

We turn now to the Cartesian product of a clique by a balanced d-ary tree. Proposition 2.5 leads in this case to a recursion on Z_G.

Proposition 3.3 *Let $G_m = K_n \times T_m^d$. For all $m \geq 1$ the generating function of the stable sets of G_m is given by*

$$Z_{G_m} = \left(Z_{G_{m-1}}\right)^d + \frac{\lambda}{n^{d-1}} \left[(n-1) Z_{G_{m-1}} + \left(Z_{G_{m-2}}\right)^d \right]^d \quad , \tag{12}$$

with

$$Z_{G_0} = Z_{K_n} = 1 + n\lambda$$

$$Z_{G_{-1}} = Z_{\emptyset} = 1 \quad .$$

Proof. It follows from the particular form of P_{K_n} that the vector Z_m of proposition 2.5 has only two different coefficients denoted by Z_m^0 and Z_m^1, such that

$$Z_{G_m} = Z_m^0 + n Z_m^1 \quad .$$

One has

$$Z_m^0 = \left(Z_{G_{m-1}}\right)^d \qquad \text{and} \qquad Z_m^1 = \lambda \left(Z_{m-1}^0 + (n-1) Z_{m-1}^1 \right)^d \quad .$$

Combining these identities leads to (12). ∎

We turn now to grids, toruses and cylinders. They can indeed be interpreted as path or cycle ladders by setting $g = L_n$ or $g = C_n$. Applying propositions 2.3 and 2.4, to such a graph, one has to construct the compatibility matrices P_{L_n} and P_{C_n} for paths and cycles. They can be constructed recursively.

Proposition 3.4 *Let k_n be the dimension of P_{L_n} (cardinality of S_{L_n}). One has $k_n = k_{n-1} + k_{n-2}$.*

Let D_n be the $k_{n-2} \times k_{n-1}$ matrix made of the first k_{n-2} rows of $P_{L_{n-1}}$ multiplied by λ.

Let F_n be the $k_{n-1} \times k_{n-2}$ matrix made of the first k_{n-2} columns of $P_{L_{n-1}}$. Then,

$$P_{L_n} = \left(\begin{array}{c|c} P_{L_{n-1}} & F_n \\ \hline D_n & 0 \end{array} \right) .$$

Remark that this construction induces a natural ordering of the elements of S_{L_n} that will be described in the proof. The construction of P_{C_n} is related to that of P_{L_n}.

Proposition 3.5 *Let k_n be the dimension of P_{L_n}. Let v_n be a vector constructed recursively as follows: the recursion starts at $n = 3$ with $v_3 = (1)$. To obtain v_n from v_{n-1}, take coordinates in v_{n-1} which are lower or equal to k_{n-4} and add k_{n-3} to them (by convention $k_{-1} = k_0 = 1$). Then join these values to those in v_{n-1}.*

Let D'_n be the $k_{n-3} \times k_{n-1}$ matrix made of k_{n-3} rows of $P_{L_{n-1}}$ multiplied by λ and indexed by the coordinates of v_n.

Let F'_n be the $k_{n-1} \times k_{n-3}$ matrix made of k_{n-3} columns of $P_{L_{n-1}}$ indexed by the coordinates of v_n.

Then,

$$P_{C_n} = \left(\begin{array}{c|c} P_{L_{n-1}} & F'_n \\ \hline D'_n & 0 \end{array} \right) .$$

Proof. We first examine the respective dimensions of the matrices P_{L_n} and P_{C_n}. They are the values of Z_{L_n} and Z_{C_n} at $\lambda = 1$. Equation (3) yields

$$\begin{aligned} Z_{L_n} &= Z_{L_{n-1}} + \lambda\, Z_{L_{n-2}} \\ Z_{C_n} &= Z_{L_{n-1}} + \lambda\, Z_{L_{n-3}} \quad . \end{aligned} \tag{13}$$

If λ is set to 1, it appears that the number of stable sets of L_n, denoted by k_n, is the n-th term of a Fibonacci sequence starting from $k_0 = 1$ and $k_1 = 2$. It is also the dimension of the matrix P_{L_n}. In addition, the number k'_n of stable sets of C_n (and also the dimension of P_{C_n}) is given by $k'_n = k_{n-1} + k_{n-3}$.

A stable vector of L_n consists of a value for one of the ends, say x_1, together with a stable vector of L_{n-1}. Stable vectors η of L_n are ordered recursively as described now. Assume an ordering is given for stable vectors on L_{n-1}. Stable vectors of L_n which give value 0 to x_1 are put in a first block of size k_{n-1}. Those with $\eta(x_1)=1$ in a second block of size k_{n-2}. Inside the blocks the ordering of L_{n-1} is respected. So that P_{L_n} consists of four blocks:

$$P_{L_n} = \left(\begin{array}{c|c} P_{L_{n-1}} & F_n \\ \hline D_n & 0 \end{array}\right) .$$

The structure of $P_{L_{n-1}}$ is identical,

$$P_{L_{n-1}} = \left(\begin{array}{c|c} P_{L_{n-2}} & F_{n-1} \\ \hline D_{n-1} & 0 \end{array}\right)$$

and it is easy to see that

$$F_n = \left(\begin{array}{c} P_{L_{n-2}} \\ \hline D_{n-1} \end{array}\right) \quad \text{and } D_n = \lambda \left(\begin{array}{c|c} P_{L_{n-2}} & F_{n-1} \end{array}\right) .$$

Thus P_{L_n} can be deduced from $P_{L_{n-1}}$.

The same idea holds for P_{C_n}. A site x_0 of C_n is chosen. Stable vectors of C_n with $\eta(x_0)=0$ are ordered before those with $\eta(x_0)=1$. Then the first block in matrix P_{C_n} is $P_{L_{n-1}}$. The block F'_n is made from k_{n-3} columns of $P_{L_{n-1}}$ and D'_n from k_{n-3} rows of $P_{L_{n-1}}$ after multiplication by λ. Let us determine first the indices of those columns of $P_{L_{n-1}}$ which must be used to construct F'_n. Let v_n be a vector containing these indices. It can be constructed recursively from v_{n-1}. Indices of columns in F'_n (*i.e.* numbers in v_n) correspond to stable vectors of C_n with $\eta(x_0)=1$. Therefore they correspond to stable vectors of L_{n-1} with extremal coordinates $\eta(x_1)=\eta(x_{n-1})=0$. Stable vectors with $\eta(x_1)=0$ are in the first block of the matrix $P_{L_{n-1}}$ (equal to $P_{L_{n-2}}$). Among them we must find those also ending with $\eta(x_{n-1})=0$. But each stable vector of L_{n-1} with $\eta(x_1)=0$ corresponds to a stable vector of L_{n-2}. Thus we are looking for stable vectors of L_{n-2} ending with $\eta(x_{n-2})=0$. We already have numbers in v_{n-1} which are related to stable vectors of L_{n-2} starting and ending with 0. The numbers in v_{n-1} are the indices of the first columns of F'_n. The last ones are the indices of stable vectors of L_{n-2} starting with $\eta(x_1)=1$ and ending with $\eta(x_{n-2})=0$. They are located in the second block of the matrix $P_{L_{n-2}}$ and satisfy $\eta(x_2)=0$. The required indices are determined by taking numbers in v_{n-1} which are lower or equal to k_{n-4} and add k_{n-3} to them. It suffices to join these values to those in v_{n-1} to obtain v_n. The recursion starts at $n=3$. The matrices $P_{C_1}=P_{L_1}$, $P_{C_2}=P_{L_2}$ and $P_{C_3}=P_{K_3}$ are easily calculated and v_3 is set to $(\ 1\ )$. The first values of v_n are given below.

$$\begin{array}{llll} v_3 = (\ 1\ ) & & k_0 = 1 & k_1 = 2 \\ v_4 = (\ 1, 1+2\ ) & = (\ 1, 3\ ) & k_1 = 2 & k_2 = 3 \\ v_5 = (\ 1, 3, 1+3\ ) & = (\ 1, 3, 4\ ) & k_2 = 3 & k_3 = 5 \\ v_6 = (\ 1, 3, 4, 1+5, 3+5\ ) & = (\ 1, 3, 4, 6, 8\ ) & k_3 = 5 & k_4 = 8 \end{array} \quad . \tag{14}$$

■

As an illustration we calculated the number of stable sets for various graphs. This number is given by the value of Z_G at $\lambda=1$. For some ladders with 100 vertices, the results are given in table 3.

Grids and toruses in three dimensions can also be seen as ladders. It suffices to set g to a 2-dimensional grid or torus. To calculate the generating function of stable sets, the compatibility matrix for 2-dimensional grids and toruses is required. As for L_n and C_n (1-dimensional grids and toruses), it is possible to construct this matrix recursively. Equations are analogous to those for P_{L_n} and P_{C_n} above but more complex. Details will not be given.

We turn now to n-dimensional cubes ([1] p. 12). We propose a recursive construction of their compatibility matrix in order to calculate the generating function of their stable sets. Our algorithm is faster than a direct counting of stable sets. Its complexity is obviously still exponential in n. We observe first that cubes Q_n can be constructed recursively.

$$Q_n = Q_{n-1} \times C_2 \tag{15}$$

$$= Q_{n-2} \times C_2 \times C_2 = Q_{n-2} \times C_4 \quad . \tag{16}$$

Thus Q_n is a particular case of path ladder with $g=Q_{n-1}$ and also a cycle ladder with $g=Q_{n-2}$. Let P_n be the compatibility matrix of Q_n (definition 2.2). The generating function Z_{Q_n} of stable sets of Q_n can be calculated using either (15) or (16). Let 0_n be the null stable vector on Q_n. ¿From propositions 2.3 and 2.4, one deduces

$$\begin{aligned} Z_{Q_n} &= P^3_{n-1}(0_{n-1}, 0_{n-1}) \\ &= trace[\ P^4_{n-2}\ ] \quad . \end{aligned}$$

graph	edges	$\alpha(G)$	number of stable sets
$L100$	99	50	$927,372,692,193,078,999,176$
$C100$	100	50	$792,070,839,848,372,253,127$
$L2 \times L50$	148	50	$16,616,132,878,186,749,607$
$L2 \times C50$	150	50	$13,765,255,184,676,885,127$
$L4 \times L25$	171	50	$3,706,188,077,591,198,432$
$L4 \times C25$	175	50	$2,662,270,770,083,033,841$
$C4 \times L25$	196	50	$849,258,745,421,311,327$
$C4 \times C25$	200	50	$643,439,082,960,504,561$
$L5 \times L20$	175	50	$2,840,321,953,949,094,447$
$L5 \times C20$	180	50	$1,904,270,890,157,965,061$
$C5 \times L20$	195	40	$635,511,992,964,231,701$
$C5 \times C20$	200	40	$456,085,875,187,977,011$
$L10 \times L10$	180	50	$2,030,049,051,145,980,050$
$L10 \times C10$	190	50	$974,602,314,570,939,359$
$C10 \times C10$	200	50	$498,819,827,260,367,617$
$K4 \times L25$	246	25	$5,527,939,700,884,757$
$K4 \times C25$	250	25	$4,721,424,167,835,361$
$K5 \times L20$	295	20	$233,528,957,822,051$
$K5 \times C20$	300	20	$203,080,369,893,131$
$K10 \times L10$	540	10	$12,010,471,551$
$K10 \times C10$	550	10	$11,035,502,511$
$L5 \times K20$	1030	5	$3,393,261$
$C5 \times K20$	1050	5	$3,240,081$
$K5 \times K20$	1150	5	$2,514,181$
$L4 \times K25$	1275	4	$408,176$
$C4 \times K25$	1300	4	$393,151$
$K4 \times K25$	1350	4	$362,501$
$L2 \times K50$	2500	2	$2,551$
$K100$	4950	1	101

Table 1: Various ladder graphs with 100 vertices.

Our main problem is to construct the matrix P_n. Its size increases very quickly with n. For instance, P_3 is of dimension 35, P_4 is of dimension 743 and P_5 of dimension 254475. Here is the recursive calculation of P_n.

Proposition 3.6 *Let e_1, e_2 be stable vectors of Q_{n-1}. Notation $e_1 \times e_2$ represents the stable vector of $Q_n = Q_{n-1} \times L_2$ which coincides with e_1 on the first version of Q_{n-1} and with e_2 on the second ($e_1 \times e_2$ is a stable vector iff e_1 and e_2 are compatible, i.e. $P_{n-1}(e_1, e_2) \neq 0$).*
For two stable vectors $e_1 \times e_2$ and $e_3 \times e_4$,

$$P_n(e_1 \times e_2, e_3 \times e_4) = P_{n-1}(e_1, e_3)P_{n-1}(e_2, e_4) \quad .$$

Proof. The relationship between P_n and P_{n-1} is based on the equality $g \times C_4 = g \times C_2 \times L_2$ that holds for all g. The graph $g \times C_2$ is constructed by joining two identical versions of g. If e_1, e_2 are stable vectors of g, $e_1 \times e_2$ represents the stable vector of $g \times C_2$ which coincides with e_1 on the first version of g and with e_2 on the second. For instance, $0_n = 0_{n-1} \times 0_{n-1}$. Taking $g = Q_{n-1}$, we have $g \times C_2 = Q_n$. If f_1 and f_2 are stable vectors of $g \times C_2$, they can be identified to $f_1 = e_1 \times e_2$ and $f_2 = e_3 \times e_4$. Therefore, to construct P_n we first need to list stable vectors $e_1 \times e_2$ on Q_n. They consist of products of compatible stable vectors of Q_{n-1} and thus satisfy $P_{n-1}(e_1, e_2) \neq 0$. Then we shall look at incompatible vectors on Q_n. The two vectors $e_1 \times e_2$ and $e_3 \times e_4$ are not compatible on Q_n iff e_1 and e_3 or e_2 and e_4 are not compatible on Q_{n-1}. Proposition 3.6 follows. ∎

The generating functions for Q_3, Q_4 and Q_5 are given in table 3. For Q_6 we only computed the number of stable sets, which is

$$19,768,832,143 \ .$$

Z_{Q_3}	1	$+8\lambda$	$+16\lambda^2$	$+8\lambda^3$	$+2\lambda^4$	
Z_{Q_4}	1	$+16\lambda$	$+88\lambda^2$	$+208\lambda^3$	$+228\lambda^4$	$+128\lambda^5$
	$+56\lambda^6$	$+16\lambda^7$	$+2\lambda^8$			
Z_{Q_5}	1	$+32\lambda$	$+416\lambda^2$	$+2880\lambda^3$	$+11760\lambda^4$	$+29856\lambda^5$
	$+48960\lambda^6$	$+54304\lambda^7$	$+44240\lambda^8$	$+29920\lambda^9$	$+17952\lambda^{10}$	$+9088\lambda^{11}$
	$+3672\lambda^{12}$	$+1120\lambda^{13}$	$+240\lambda^{14}$	$+32\lambda^{15}$	$+2\lambda^{16}$	

Table 2: Generating functions for Q_3, Q_4 and Q_5.

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