A HILL TYPE ESTIMATE OF THE WEIBULL TAIL-COEFFICIENT

STÉPHANE GIRARD

Abstract. We present a new estimate of the Weibull tail-coefficient. The Weibull tail-coefficient is defined as the regular variation coefficient of the inverse failure rate function. Our estimate is based on the log-spacings of the upper order statistics. Therefore, it is very similar to the Hill estimate for the extreme value index. We prove the weak consistency and the asymptotic normality of our estimate. Its asymptotic as well as its finite sample performances are compared to classical ones.

1. Introduction

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed random variables with cumulative distribution function F. We address the problem of estimating the Weibull tail-coefficient $\theta > 0$ defined when the distribution tail satisfies

(A.1): $1-F(x) = \exp(-H(x))$, $x \geq x_0 \geq 0$, $H^{\leftarrow}(t) = \inf\{x,\ H(x) \geq t\} = t^{\theta}\ell(t)$,

where ℓ is a slowly varying function *i.e.*

$$\ell(\lambda x)/\ell(x) \to 1 \text{ as } x \to \infty \text{ for all } \lambda > 0.$$

The inverse failure rate function $H^{\leftarrow}$ is said to be regularly varying at infinity with index θ and this property is denoted by $H^{\leftarrow} \in \mathcal{R}_{\theta}$.
When ℓ is a constant function, this problems reduces to estimating the shape parameter of a Weibull distribution. In this context, simple and efficient methods exist, see for instance [8] for a moment based estimator. Distributions with non-constant slowly varying functions include for instance normal, gamma and extended Weibull distributions (see Section 3 for their definitions). Such distributions are of great use to model large claims in non-life insurance [4]. Dedicated estimation methods have been proposed since the relevant information on the Weibull tail-coefficient is only contained in the extreme upper part of the sample. A first direction was investigated in [5] where an estimate based on the record values is proposed. An other approach consists of using the k_n upper order statistics $X_{n-k_n+1,n} \leq \cdots \leq X_{n,n}$ where (k_n) is a sequence of integers such that $1 \leq k_n < n$. Our estimate belongs to this family. It is defined by

$$(1)\quad \hat{\theta}_n = \sum_{i=1}^{k_n-1} (\log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n})) \Bigg/ \sum_{i=1}^{k_n-1} (\log_2(n/i) - \log_2(n/k_n)),$$

where $\log_2(t) = \log(\log(t))$, $t > 1$. Some related estimates [3, 7, 16] will be presented below and a precise comparison will be done from both theoretical and practical points of view. Formally, (1) is similar to the expression of the Hill estimate [14]:

$$\hat{\gamma}_n = \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} (\log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n})) \tag{2}$$

of the extreme value index γ for Pareto type distributions, *i.e.* such that

$$(1-F)(\lambda x)/(1-F)(x) \to \lambda^{-1/\gamma} \text{ as } x \to \infty \text{ for all } \lambda > 0.$$

We refer to [9] for a review on this topic. Although estimates (1) and (2) do not address the same problem, it will appear that they share similar properties. In Section 2 we state the main asymptotic properties of our estimate. These results are illustrated in Section 3 on some examples of distributions and are compared in Section 4 with the results obtained by other estimates. Finally the behavior of our estimate on finite sample situations is presented in Section 5.

2. Asymptotic results

In this section, we state our main results on the limiting behavior of the estimate $\hat{\theta}_n$. Its weak consistency is established in Theorem 1 and its asymptotic normality in Theorem 2. The proofs are postponed to Section 6. The weak consistency of $\hat{\theta}_n$ is proved under the usual conditions on the (k_n) sequence (which are also necessary for the weak consistency of the Hill estimate, see [17], Theorem 2):

Theorem 1. *Suppose* **(A.1)** *holds. If* $k_n \to \infty$ *and* $k_n/n \to 0$ *then* $\hat{\theta}_n \xrightarrow{P} \theta$.

The study of the limit distribution of $\hat{\theta}_n$ requires a second order condition on ℓ: There exist $\rho \le 0$ and $b(x) \to 0$ such that uniformly locally on $\lambda \ge 1$ when $x \to \infty$,

$$\textbf{(A.2):} \ \log\left(\frac{\ell(\lambda x)}{\ell(x)}\right) \sim b(x) K_\rho(\lambda),$$

with $K_\rho(\lambda) = \int_1^\lambda u^{\rho-1} du$. It can be shown [11] that necessarily $|b| \in \mathcal{R}_\rho$. The second order parameter $\rho \le 0$ tunes the rate of convergence of $\ell(\lambda x)/\ell(x)$ to 1. The closer ρ is to 0, the slower is the convergence. Condition **(A.2)** is the cornerstone in all proofs of asymptotic normality for extreme value estimates. It is used in [13] to prove the asymptotic normality of the Hill estimate and in [2] for one of its refinements.

Theorem 2. *Suppose* **(A.1)** *and* **(A.2)** *hold. Then*

$$k_n^{1/2}(\hat{\theta}_n - \theta) \xrightarrow{d} \mathcal{N}(0, \theta^2),$$

for any sequence (k_n) *satisfying*

$$k_n \to \infty, \ k_n/n \to 0, \ k_n^{1/2} b(\log(n/k_n)) \to 0 \text{ and } k_n^{1/2}/\log(n/k_n) \to 0.$$

If $\rho < -1$, condition $k_n^{1/2}/\log(n/k_n) \to 0$ implies $k_n^{1/2} b(\log(n/k_n)) \to 0$ and therefore all the corresponding distributions require the same conditions on the sequence (k_n). Conversely, if $-1 < \rho \le 0$, the condition $k_n^{1/2} b(\log(n/k_n)) \to 0$ is the stronger and the convergence is slower. If $\rho = -1$, the condition $k_n^{1/2} b(\log(n/k_n)) \to 0$ may reveal necessary depending on the function b. Some examples are now provided.

3. Examples

In this section, we give some examples of distributions satisfying assumptions **(A.1)** and **(A.2)**.

- Gaussian distribution $\mathcal{N}(\mu, \sigma^2)$, $\sigma > 0$. From [10], Table 3.4.4, we have $H^{\leftarrow}(x) = x^{1/2}\ell(x)$ and an asymptotic expansion of the slowly varying function is given by:
$$\ell(x) = 2^{1/2}\sigma - \frac{\sigma}{2^{3/2}}\frac{\log x}{x} + O(1/x).$$
Therefore $\theta = 1/2$, $\rho = -1$ and $b(x) = \log(x)/(4x)$.
- Gamma distribution $\Gamma(\beta, \alpha)$, $\alpha, \beta > 0$. We use the following parameterization of the density
$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x).$$
From [10], Table 3.4.4, we obtain $H^{\leftarrow}(x) = x\ell(x)$ with
$$\ell(x) = \frac{1}{\beta} + \frac{\alpha - 1}{\beta}\frac{\log x}{x} + O(1/x).$$
We thus have $\theta = 1$, $\rho = -1$ and $b(x) = (1 - \alpha)\log(x)/x$.
- Weibull distribution $\mathcal{W}(\alpha, \lambda)$, $\alpha, \lambda > 0$. The inverse failure rate function is $H^{\leftarrow}(x) = \lambda x^{1/\alpha}$, and then $\ell(x) = \lambda$ for all $x > 0$. Therefore $b(x) = 0$ and we use the usual convention $\rho = -\infty$. A simplified version of Theorem 2 dedicated to this particular case is given in Corollary 1 below.
- Extended Weibull distribution $\mathcal{E}W(\alpha, \beta)$, $\alpha \in (0, 1)$, $\beta \in \mathbb{R}$. This distribution is introduced in [16]. Its distribution tail is given by:
$$1 - F(x) = r(x)\exp(-x^\alpha), \tag{3}$$
where $r \in \mathcal{R}_\beta$. It follows that $H^{\leftarrow}(x) = x^{1/\alpha}\ell(x)$ and the following asymptotic expansion holds:
$$\ell(x) = 1 + \frac{\beta}{\alpha^2}\frac{\log x}{x} + O(1/x).$$
It is easily seen that $\theta = 1/\alpha$, $\rho = -1$ and $b(x) = -\beta\log(x)/(\alpha^2 x)$. In [16], it is remarked that this model encompasses the normal and gamma distributions. Then, an estimate of α dedicated to the model (3) is proposed. Its properties are briefly reviewed and compared to those of $\hat{\theta}_n$ in the next section. In particular, we give in Corollary 2 a specific version of Theorem 2 for this family of distributions.
- Modified Weibull distribution $\mathcal{M}W(\alpha)$, $\alpha > 0$. Let $Y \sim \mathcal{W}(\alpha, 1)$, $\alpha > 0$, and introduce $X = Y \log Y$. Then, the inverse failure rate function of X can be written $H^{\leftarrow}(x) = x^{1/\alpha}\ell(x)$ with $\ell(x) = \alpha\log(x)$. Therefore, we have $\theta = 1/\alpha$, $\rho = 0$ and $b(x) = 1/\log(x)$. Let us note that this distribution does not belong to the Extended Weibull family.

The parameters θ and ρ as well as the auxiliary function $b(x)$ associated to these distributions are summarized in Table 1. We compare theoretically in the next section our estimate to other ones proposed in the literature on some of these examples.

4. Discussion

Our estimate is now compared to other propositions by studying their asymptotic normality properties obtained in some simple situations.

4.1. **Comparison with Broniatowski estimate.**

In [7], another estimate of the Weibull tail coefficient is proposed. It is defined by

$$\tilde{\theta}_n = \frac{1}{k_n} \sum_{i=1}^{k_n - 1} \frac{\log(X_{n-i+1,n})}{\log_2(n/i)}. \tag{4}$$

Both estimates $\hat{\theta}_n$ and $\tilde{\theta}_n$ are based on a similar principle: Introduce $q(t)$, the quantile function defined by

$$q(t) = (1-F)^{\leftarrow}(t) = H^{\leftarrow}(\log(1/t)) = (\log(1/t))^{\theta}\ell(\log(1/t)),$$

under **(A.1)**. The estimate $\tilde{\theta}_n$ is motivated by the approximation

$$\frac{\log(q(t))}{\log_2(1/t)} = \theta + \frac{\log(\ell(\log(1/t)))}{\log_2(1/t)} \simeq \theta. \tag{5}$$

The estimate $\hat{\theta}_n$, as for itself, is motivated by the approximation

$$\begin{aligned} \log(q(t)) - \log(q(s)) &= \theta(\log_2(1/t) - \log_2(1/s)) + \log\left(\frac{\ell(\log(1/t))}{\ell(\log(1/s))}\right) \\ &\simeq \theta(\log_2(1/t) - \log_2(1/s)), \end{aligned} \tag{6}$$

since ℓ is slowly varying. This approximation is similar to the one used in the Hill estimator [14] for estimating the extreme value index. It is important to note that, when ℓ is a constant function, the approximation (6) is exact but not the approximation (5). This remark has an important consequence on the asymptotic behavior of the estimate. From Theorem 2, we easily obtain that, when ℓ is a constant function, the asymptotic normality of $(\hat{\theta}_n - \theta)$ holds under weaker assumptions on k_n:

Corollary 1. *Suppose* **(A.1)** *holds and* ℓ *is a constant function. Then,*

$$k_n^{1/2}(\hat{\theta}_n - \theta) \xrightarrow{d} \mathcal{N}(0, \theta^2).$$

for any sequence (k_n) *satisfying*

$$k_n \to \infty,\ k_n/n \to 0 \text{ and } k_n^{1/2}/\log(n/k_n) \to 0. \tag{7}$$

Obtaining a similar result for $(\tilde{\theta}_n - \theta)$ is not straightforward. In particular, Theorem 2 in [7] requires to verify the condition

$$k_n^{1/2}\log(n/k_n)\log_2(n/k_n)\,\beta_n \to 0, \tag{8}$$

where β_n is a bias term defined by

$$\beta_n = \frac{n}{k_n}\int_0^{k_n/n} \frac{du}{\log_2(1/u)}. \tag{9}$$

It can be shown (see Lemma 2) that there exist no sequence (k_n) verifying condition (8), and thus, it is not possible to make the bias term vanish. This comparison is completed on Section 5 by some simulations.

4.2. **Comparison with Beirlant** *et al* **estimate.** Beirlant *et al* [3] propose to use the mean residual life function $e(x) = E(X - x|X > x)$ to estimate the parameter θ. They remark that, under assumption **(A.1)**, one have

$$(\log x)\frac{e(H^{\leftarrow}(\log x))}{H^{\leftarrow}(\log x)} \sim \theta \tag{10}$$

when $x \to \infty$. They thus introduce the following estimate

$$\breve{\theta}_n = \frac{\log(n/k_n)}{X_{n-k_n+1,n}} \frac{1}{k_n - 1} \sum_{i=1}^{k_n-1} X_{n-i+1,n} - X_{n-k_n+1,n}.$$

A central limit theorem for $\breve{\theta}_n$ is also established (see [3], Theorem 3.2(i)), with the same hypotheses and conclusions as Theorem 2. Consequently, $\hat{\theta}_n$ and $\breve{\theta}_n$ share the same asymptotic properties. Nevertheless, we can again remark that the approximation (10) is not exact when ℓ is a constant function. The performance of $\hat{\theta}_n$ and $\breve{\theta}_n$ on finite sample situations will be compared in Section 5.

4.3. **Estimation within the Extended Weibull family.** In [16], an asymptotic maximum likelihood estimate (AMLE) for the parameter α of the $\mathcal{E}W(\alpha,\beta)$ distribution is proposed. This estimate, which will be denoted by $\hat{\alpha}_n$, is defined by an implicit equation that we do not reproduce here. The authors proved that ([16], Theorem 4.5)

$$k_n^{1/2} \log_2 (n/k_n) \left(\hat{\alpha}_n^{-1} - \alpha^{-1} - \frac{\log(\hat{\alpha}_n/\alpha)}{\log_2 (n/k_n)} \right) \xrightarrow{d} \mathcal{N}(0,1), \tag{11}$$

under conditions

$$k_n \to \infty, \; k_n/n \to 0 \text{ and } k_n^{1/2} \frac{\log_2 (n/k_n)}{\log(n/k_n)} \to 0. \tag{12}$$

As a comparison, the application of Theorem 2 to the particular case of $\mathcal{E}W(\alpha,\beta)$ distributions lead to the following

Corollary 2. *Suppose F is a $\mathcal{E}W(\alpha,\beta)$ distribution. Then,*

$$k_n^{1/2}(\hat{\theta}_n - \theta) \xrightarrow{d} \mathcal{N}(0,\theta^2), \tag{13}$$

for any sequence (k_n) satisfying (7).

We can note that, for a fixed value of k_n, the asymptotic convergence rate in (11) is slightly better than in (13). At the opposite, condition (12) is more severe than condition (7) and then the optimal convergence rates are the same.

5. Numerical experiments

The finite sample performance of our estimate is investigated on 7 different distributions: $\Gamma(0.5, 1)$, $\Gamma(1.5, 1)$, $\mathcal{N}(1.2, 1)$, $\mathcal{W}(2.5, 2.5)$, $\mathcal{W}(0.4, 0.4)$, $\mathcal{M}W(2.5)$, and $\mathcal{M}W(0.4)$. In each case, $N = 100$ samples $(\mathcal{X}_{n,i})_{i=1,\ldots,N}$ of size $n = 500$ were simulated. On each sample $(\mathcal{X}_{n,i})$, the estimate $\hat{\theta}_{n,i}(k_n)$ is computed for $k_n = 2, \ldots, 120$. Finally, the Hill-type plots are built by drawing the points

$$\left(k_n, \frac{1}{N} \sum_{i=1}^{N} \hat{\theta}_{n,i}(k_n) \right)$$

as well as the associated empirical 90% confidence interval. We also present the associated MSE (mean square error) plots obtained by plotting the points

$$\left(k_n, \sum_{i=1}^{N}\left(\hat{\theta}_{n,i}(k_n)-\theta\right)^2\right).$$

The same work is achieved with the estimates $\tilde{\theta}_n$ and $\breve{\theta}_n$ for comparison. The estimate $\hat{\alpha}_n$ is not implemented here since it does not address the same problem as $\hat{\theta}_n$, $\tilde{\theta}_n$ and $\breve{\theta}_n$. For instance, it is not suited for the modified Weibull distribution. Nevertheless, the reader interested in its behavior on simulated and real data can refer to [15].

Results are presented on Figures 2–8. It appears that, in all cases, $\hat{\theta}_n$ gives better results than $\breve{\theta}_n$. It is also seen that $\hat{\theta}_n$ has a smaller bias and a larger variance than $\tilde{\theta}_n$. For this reason, $\hat{\theta}_n$ should be preferred to $\tilde{\theta}_n$ even if the MSE associated to $\hat{\theta}_n$ is larger than $\tilde{\theta}_n$'s one. Indeed, in many cases, the true value θ belongs to the empirical confidence interval associated to $\hat{\theta}_n$ and is outside the interval associated to $\tilde{\theta}_n$.

Let us note that the sign of the bias of $\hat{\theta}_n$ and $\breve{\theta}_n$ is given by the function b. It is positive for the normale, modified Weibull and gamma distributions (with shape parameter less than one) and negative for the gamma distribution with shape parameter more than one. The bias term vanishes for the Weibull distribution. The worst results are obtained for the modified Weibull distribution. In this situation $\rho = 0$ and thus the bias term which is driven by b decreases very slowly.

In practice, the choice of the parameter k_n is the key problem to obtain a correct estimation of θ. If k_n is too small, the variance of $\hat{\theta}_n$ is huge and conversely, if k_n is too large, the bias is very important. A first solution to select the optimal k_n can be based on the drawing of a quantile plot (QQ-plot) adapted to our situation. It consists of drawing the pairs $(\log_2(n/i), \log(X_{n-i+1,n}))$ for $i = 1, \dots, n-1$. From (6), the resulting graph should be approximatively linear (with slope θ), at least for the large values of i. On Figure 1 two cases are presented: The QQ-plot associated to a Weibull distribution which is perfectly linear, and the QQ-plot associated to a normal distribution which is only ultimately linear. A graphical method to select k_n could consist of choosing the point where the graph becomes linear. Of course, this method is not very satisfying and an automatic selection method should be developed. A first step could be to adapt the procedure proposed in [12] for the Hill estimate to our framework.

6. Proofs

Consider $\{E_{1,n}, \dots, E_{n,n}\}$ the order statistics generated by n independent standard exponential random variables. We collect in the next lemma classical results on the asymptotic behavior of these order statistics.

Lemma 1. *Suppose $k_n \to \infty$ and $k_n/n \to 0$. Then,*

- **(i):** $E_{n-i+1,n}/\log(n/i) \xrightarrow{P} 1$, *uniformly on* $i = 1, \dots, k_n$, *and*
- **(ii):** $k_n^{1/2}\left(E_{n-k_n+1,n} - \log(n/k_n)\right) \xrightarrow{d} \mathcal{N}(0,1)$.

Proof. Both results are proved using the representation of the Exp(1) ordered statistics (see [1], p. 72):

$$\{E_{j,n}\}_{j=1,\dots,n} \stackrel{d}{=} \left\{ \sum_{r=1}^{j} \frac{Y_r}{n-r+1} \right\}_{j=1,\dots,n},$$

where $\{Y_1,\dots,Y_n\}$ are independent standard exponential variables.

(i): Introducing for $i=1,\dots,k_n$ the random variables

$$Z_{n,i} = \frac{E_{n-i+1,n}}{\log(n/i)} - 1,$$

straightforward calculations lead to the uniform bounds $0 \le E(Z_{n,i}) \le 1/\log(n/k_n)$ and $\mathrm{var}(Z_{n,i}) \le 2/\log^2(n/k_n)$.

(ii): This result is also found in [3], p. 32. The Lyapounov theorem for triangular array yields

$$k_n^{1/2}\left(E_{n-k_n+1,n} - \sum_{s=k_n}^{n} \frac{1}{s} \right) \xrightarrow{d} \mathcal{N}(0,1).$$

The remaining term is controlled with the well-known equality

$$\sum_{s=1}^{n} \frac{1}{s} = \gamma_E + \log n + O(1/n),$$

where γ_E is the Euler's constant. ∎

The next lemma is used to prove that there exist no sequence (k_n) such that $k_n \to \infty$, $k_n/n \to 0$ and $k_n^{1/2}\log(n/k_n)\log_2(n/k_n)\,\beta_n \to 0$.

Lemma 2. *If $k_n \to \infty$, $k_n/n \to 0$, then, for n large enough, $\beta_n \log_2(n/k_n) > 1/2$.*

Proof. Introducing $t = k_n/(nu)$ and $\mu_n = \log(n/k_n)$ in (9) yields

$$\beta_n = \int_1^{+\infty} \frac{dt}{t^2 \log(\log(t)+\mu_n)}.$$

Since $\mu_n \to \infty$, for n large enough $\log(\log(t)+\mu_n) > 0$ and therefore

$$\beta_n > \int_1^{e^{\mu_n}} \frac{dt}{t^2\log(\log(t)+\mu_n)} > \frac{1}{\log(2\mu_n)} \int_1^{e^{\mu_n}} \frac{dt}{t^2} \sim \frac{1}{\log_2(n/k_n)}.$$

As a conclusion, for n large enough, $\beta_n \log_2(n/k_n) > 1/2$. ∎

Proof of Theorem 1. For the sake of simplicity, in this proof, we note k for k_n. We also write $\hat{\theta}_n = \tau_n^{(2)}/\tau_n^{(1)}$ with

$$\tau_n^{(1)} = \frac{1}{k-1}\sum_{i=1}^{k-1}(\log_2(n/i) - \log_2(n/k)) = \frac{1}{k-1}\sum_{i=1}^{k-1}\log\left(1+\frac{\log(k/i)}{\log(n/k)}\right).$$

The well-known inequality $-x^2/2 \le \log(1+x) - x \le 0$, $x>0$ yields

$$-\frac{1}{2}\frac{1}{\log(n/k)}\frac{1}{k-1}\sum_{i=1}^{k-1}\log^2(k/i) \le \log(n/k)\tau_n^{(1)} - \frac{1}{k-1}\sum_{i=1}^{k-1}\log(k/i) \le 0. \tag{14}$$

Now, since when $k \to \infty$,

$$\frac{1}{k-1}\sum_{i=1}^{k-1}\log^2(k/i) \to \int_0^1 \log^2(x)dx = 2 \text{ and } \frac{1}{k-1}\sum_{i=1}^{k-1}\log(k/i) \to -\int_0^1 \log(x)dx = 1,$$

it follows that

$$\tau_n^{(1)} = \frac{1}{\log(n/k)} + O\left(\frac{1}{\log^2(n/k)}\right). \tag{15}$$

Let us consider now

$$\tau_n^{(2)} = \frac{1}{k-1}\sum_{i=1}^{k-1}(\log(X_{n-i+1,n}) - \log(X_{n-k_n+1,n})).$$

Let $E_{1,n},\dots,E_{n,n}$ be ordered statistics generated by n independent standard exponential random variables. Under **(A.1)**, we have

$$\begin{aligned}
\tau_n^{(2)} &\stackrel{d}{=} \frac{1}{k-1}\sum_{i=1}^{k-1}(\log H^{\leftarrow}(E_{n-i+1,n}) - \log H^{\leftarrow}(E_{n-k+1,n}))\\
&\stackrel{d}{=} \theta\frac{1}{k-1}\sum_{i=1}^{k-1}\log\left(\frac{E_{n-i+1,n}}{E_{n-k+1,n}}\right) + \frac{1}{k-1}\sum_{i=1}^{k-1}\log\left(\frac{\ell(E_{n-i+1,n})}{\ell(E_{n-k+1,n})}\right)\\
&:= \tau_n^{(3)}\theta + \tau_n^{(4)}.
\end{aligned}$$

In order to rewrite $\tau_n^{(3)}$ into a simpler form, we use once again the representation of the Exp(1) ordered statistics ([1], p. 72):

$$\{E_{j,n}\}_{j=1,\dots,n} \stackrel{d}{=} \left\{\sum_{r=1}^{j}\frac{Y_r}{n-r+1}\right\}_{j=1,\dots,n},$$

where $\{Y_1,\dots,Y_n\}$ are independent standard exponential variables. Thus,

$$\begin{aligned}
\left\{\frac{E_{n-i+1,n}}{E_{n-k+1,n}}\right\}_{i=1,\dots,k-1} &\stackrel{d}{=} \left\{1+\sum_{r=n-k+2}^{n-i+1}\frac{Y_r}{n-r+1}\Big/\sum_{r=1}^{n-k+1}\frac{Y_r}{n-r+1}\right\}_{i=1,\dots,k-1}\\
&= \left\{1+\sum_{s=1}^{k-i}\frac{Y_{s-k+n+1}}{k-s}\Big/\sum_{r=1}^{n-k+1}\frac{Y_r}{n-r+1}\right\}_{i=1,\dots,k-1}\\
&\stackrel{d}{=} \left\{1+\frac{F_{k-i,k-1}}{E_{n-k+1,n}}\right\}_{i=1,\dots,k-1}, \qquad (16)
\end{aligned}$$

where $\{F_{1,k-1},\dots,F_{k-1,k-1}\}$ are ordered statistics independent from $E_{n-k+1,n}$ and generated by $k-1$ independent standard exponential variables $\{F_1,\dots,F_{k-1}\}$. Therefore,

$$\tau_n^{(3)} \stackrel{d}{=} \frac{1}{k-1}\sum_{i=1}^{k-1}\log\left(1+\frac{F_{k-i,k-1}}{E_{n-k+1,n}}\right) \stackrel{d}{=} \frac{1}{k-1}\sum_{i=1}^{k-1}\log\left(1+\frac{F_i}{E_{n-k+1,n}}\right),$$

and, similarly to (14),

$$-\frac{1}{2}\frac{1}{E_{n-k+1,n}}\frac{1}{k-1}\sum_{i=1}^{k-1}F_i^2 \le \tau_n^{(3)}E_{n-k+1,n} - \frac{1}{k-1}\sum_{i=1}^{k-1}F_i \le 0.$$

Now, since when $k \to \infty$,

$$\frac{1}{k-1}\sum_{i=1}^{k-1} F_i^2 \xrightarrow{a.s.} 2 \text{ and } \frac{1}{k-1}\sum_{i=1}^{k-1} F_i \xrightarrow{a.s.} 1,$$

by the strong law of large numbers and Lemma 1(i), it yields

$$\tau_n^{(3)} \stackrel{d}{=} \frac{1}{E_{n-k+1,n}} \frac{1}{k-1}\sum_{i=1}^{k-1} F_i + O_P\left(\frac{1}{\log^2(n/k)}\right), \tag{17}$$

and, in view of (15),

$$\frac{\tau_n^{(3)}}{\tau_n^{(1)}} \stackrel{d}{=} \frac{\log(n/k)}{E_{n-k+1,n}} \frac{1}{k-1}\sum_{i=1}^{k-1} F_i + O_P\left(\frac{1}{\log(n/k)}\right) \xrightarrow{P} 1, \tag{18}$$

with Lemma 1(i) and the strong law of large numbers.
Finally, $\tau_n^{(4)}$ can be rewritten using the Karamata representation of ℓ (see e.g. [6]):

$$\ell(x) = c(x) \exp\left(\int_{x_0}^{x} \frac{\delta(u)}{u} du\right), \; x \geq x_0,$$

where $c(x) \to c_0 > 0$ and $\delta(x) \to 0$ as $x \to \infty$. This entails

$$\tau_n^{(4)} = \frac{1}{k-1}\sum_{i=1}^{k-1} \log\left(\frac{c(E_{n-i+1,n})}{c(E_{n-k+1,n})}\right) + \frac{1}{k-1}\sum_{i=1}^{k-1} \int_{E_{n-k+1,n}}^{E_{n-i+1,n}} \frac{\delta(u)}{u} du := \tau_n^{(5)} + \tau_n^{(6)},$$

and the end of the proof is similar to [17], p. 757: $\tau_n^{(5)}$ converges to 0 in probability in view of Lemma 1(i). Besides, $\tau_n^{(6)}$ can be controlled by introducing the random variable

$$\xi_n = \sup\{|\delta(u)|, \; u \geq E_{n-k+1,n}\},$$

and remarking that

$$|\tau_n^{(6)}| \leq \xi_n \frac{1}{k-1}\sum_{i=1}^{k-1} \int_{E_{n-k+1,n}}^{E_{n-i+1,n}} \frac{1}{u} du = \xi_n \tau_n^{(3)} = o_P(\tau_n^{(3)}),$$

with Lemma 1(i). From (15) and (18), we obtain that $\tau_n^{(4)}/\tau_n^{(1)}$ converges to 0 in probability and the conclusion follows with (18): $\hat{\theta}_n = \tau_n^{(2)}/\tau_n^{(1)} \xrightarrow{P} \theta$. ∎

Proof of Theorem 2. Keeping in mind the notations introduced in the previous proof, we have

$$k^{1/2}(\hat{\theta}_n - \theta) = \theta k^{1/2}\left(\frac{\tau_n^{(3)}}{\tau_n^{(1)}} - 1\right) + k^{1/2}\frac{\tau_n^{(4)}}{\tau_n^{(1)}}.$$

In view of (15), (17) and Lemma 1(i),

$$\begin{aligned}
k^{1/2}\left(\frac{\tau_n^{(3)}}{\tau_n^{(1)}}-1\right) &\stackrel{d}{=} \left[k^{1/2}\left(\frac{1}{k-1}\sum_{i=1}^{k-1}F_i-1\right)-k^{1/2}\left(\frac{E_{n-k+1,n}}{\log(n/k)}-1\right)\right]\frac{\log(n/k)}{E_{n-k+1,n}}\\
&+ O_P\left(\frac{k^{1/2}}{\log(n/k)}\right)\\
&\stackrel{d}{=} \left[k^{1/2}\left(\frac{1}{k-1}\sum_{i=1}^{k-1}F_i-1\right)-k^{1/2}\left(\frac{E_{n-k+1,n}}{\log(n/k)}-1\right)\right](1+o_P(1))\\
&+ O_P\left(\frac{k^{1/2}}{\log(n/k)}\right)\\
&\stackrel{d}{=} k^{1/2}\left(\frac{1}{k-1}\sum_{i=1}^{k-1}F_i-1\right)(1+o_P(1))\\
&+ O_P\left(\frac{1}{\log(n/k)}\right)+O_P\left(\frac{k^{1/2}}{\log(n/k)}\right),
\end{aligned}$$

with Lemma 1(ii). The central limit theorem yields

$$\theta k^{1/2}\left(\frac{\tau_n^{(3)}}{\tau_n^{(1)}}-1\right)\stackrel{d}{\to}\mathcal{N}(0,\theta^2).$$

Define $x_n = E_{n-k+1,n}$ and $\lambda_{i,n} = E_{n-i+1,n}/E_{n-k+1,n}$. It is clear, in view of Lemma 1(i) that $x_n \stackrel{P}{\to} \infty$ and $\lambda_{i,n} \stackrel{P}{\to} 1$. Thus, **(A.2)** yields that uniformly in $i = 1,\dots,k-1$:

$$\log\left(\frac{\ell(E_{n-i+1,n})}{\ell(E_{n-k+1,n})}\right)\sim b(E_{n-k+1,n})K_\rho\left(\frac{E_{n-i+1,n}}{E_{n-k+1,n}}\right)(1+o_P(1)).$$

Using the representation (16), we get

$$\begin{aligned}
\tau_n^{(4)} &\stackrel{d}{=} (1+o_P(1))b(E_{n-k+1,n})\frac{1}{k-1}\sum_{i=1}^{k-1}K_\rho\left(1+\frac{F_{k-i,k-1}}{E_{n-k+1,n}}\right)\\
&\stackrel{d}{=} (1+o_P(1))b(E_{n-k+1,n})\frac{1}{k-1}\sum_{i=1}^{k-1}K_\rho\left(1+\frac{F_i}{E_{n-k+1,n}}\right)\\
&\stackrel{d}{=} (1+o_P(1))\frac{b(E_{n-k+1,n})}{E_{n-k+1,n}}\frac{1}{k-1}\sum_{i=1}^{k-1}F_i,
\end{aligned}$$

similarly to (17), since $(\rho-1)x^2/2 \le K_\rho(1+x)-x \le 0$ when $x \ge 0$. Now, (15) and Lemma 1(i) entail

$$k^{1/2}\frac{\tau_n^{(4)}}{\tau_n^{(1)}}\stackrel{d}{=}(1+o_P(1))k^{1/2}b(E_{n-k+1,n})\frac{1}{k-1}\sum_{i=1}^{k-1}F_i\stackrel{d}{=}(1+o_P(1))k^{1/2}b(E_{n-k+1,n}),$$

by the strong law of large numbers. Finally, $b(E_{n-k+1,n})/b(\log(n/k))$ converges to 1 in probability, since $E_{n-k+1,n}/\log(n/k)$ converges to 1 in probability and $|b| \in \mathcal{R}_\rho$. As a conclusion,

$$k^{1/2}\frac{\tau_n^{(4)}}{\tau_n^{(1)}}\stackrel{d}{=}(1+o_P(1))k^{1/2}b(\log(n/k)) = o_P(1),$$

and the result is proved. ■

References

[1] Arnold, B.C, Balakrishnan, N., Nagaraja H.N., (1992), *A First course in order statistics*, Wiley and sons.

[2] Beirlant, J., Dierckx, G., Goegebeur, Y., Matthys, G., (1999), Tail index estimation and an exponential regression model, *Extremes*, **2**, 177–200.

[3] Beirlant J., Broniatowski, M., Teugels, J.L., Vynckier, P. (1995), The mean residual life function at great age: Applications to tail estimation, *Journal of Statistical Planning and Inference*, **45**, 21–48.

[4] Beirlant, J., Teugels, J.L., (1992), Modeling large claims in non-life insurance. *Insurance: Math. Econom.*, **11**, 17–29.

[5] Berred, M., (1991), Record values and the estimation of the Weibull tail-coefficient. *Comptes-Rendus de l'Académie des Sciences* **t. 312**, Série I, 943–946.

[6] Bingham, N.H., Goldie, C.M., Teugels, J.L., (1987), *Regular variation*, Encyclopedia of Mathematics and its Applications, **27**, Cambridge University Press.

[7] Broniatowski, M., (1993), On the estimation of the Weibull tail coefficient, *Journal of Statistical Planning and Inference*, **35**, 349–366.

[8] Cran, G.W., (1988), Moment estimators for the 3-parameter Weibull distribution, *IEEE Transactions on Reliability*, **37**(4), 360–363.

[9] Csörgö, S., Viharos, L. (1998), Estimating the tail index, *Asymptotic methods in probability and statistics*, North-Holland, Amsterdam, 833–881.

[10] Embrechts, P., Klüppelberg, C., Mikosch, T., (1997), *Modelling extremal events*, Springer.

[11] Geluk, J.L., de Haan, L., (1987), Regular Variation, Extensions and Tauberian Theorems. *Math Centre Tracts*, **40**, Centre for Mathematics and Computer Science, Amsterdam.

[12] Guillou, A., Hall, P. (2001), A diagnostic for selecting the threshold in extreme-value analysis, *J. Roy. Statist. Soc. Ser. B*, **63**(2), 293–305.

[13] Häusler, E., Teugels, J.L. (1985), On asymptotic normality of Hill's estimator for the exponent of regular variation, *The Annals of Statistics*, **13**, 743–756.

[14] Hill, B.M., (1975), A simple general approach to inference about the tail of a distribution, *The Annals of Statistics*, **3**, 1163–1174.

[15] Keller, B., Klüppelberg, C., (1991), Statistical estimation of large claims distributions. *Mitteilungen der Schweizerischen Vereinigung der Versicherungsmathematiker*, 203–216.

[16] Klüppelberg, C., Villaseñor, J.A., (1993), Estimation of distribution tails - a semiparametric approach, *Deutschen Gesellschaft für Versicherungsmathematik*, **XXI(2)**, 213–235.

[17] Mason, D.M., (1982), Laws of large numbers for sums of extreme values, *Annals of Probability*, **10**, 756–764.

Stéphane Girard, SMS/LMC, BP 53, 38041 Grenoble cedex 9, France.

stephane.girard@imag.fr

	θ	$b(x)$	ρ
$\mathcal{N}(\mu,\sigma^2)$	$1/2$	$\dfrac{1}{4}\dfrac{\log x}{x}$	-1
$\Gamma(\beta,\alpha\neq 1)$	1	$(1-\alpha)\dfrac{\log x}{x}$	-1
$\mathcal{W}(\alpha,\lambda)$	$1/\alpha$	0	$-\infty$
$\mathcal{E}W(\alpha,\beta\neq 0)$	$1/\alpha$	$-\dfrac{\beta}{\alpha^2}\dfrac{\log x}{x}$	-1
$\mathcal{M}W(\alpha)$	$1/\alpha$	$\dfrac{1}{\log x}$	0

TABLE 1. Parameters θ, ρ and the function $b(x)$ associated to some usual distributions

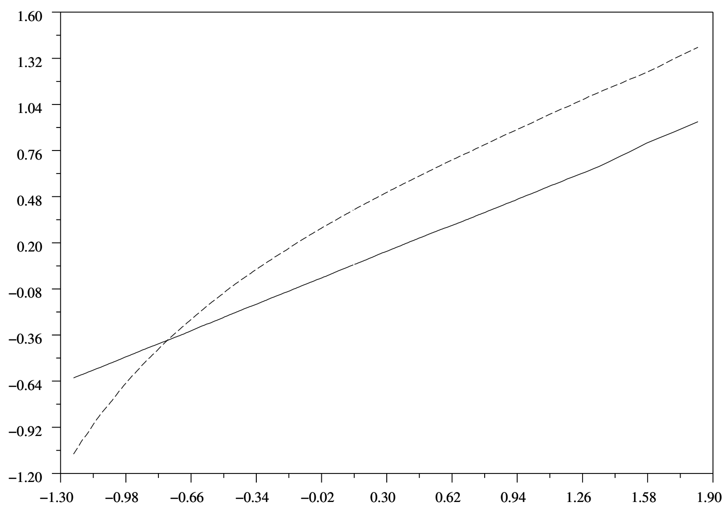
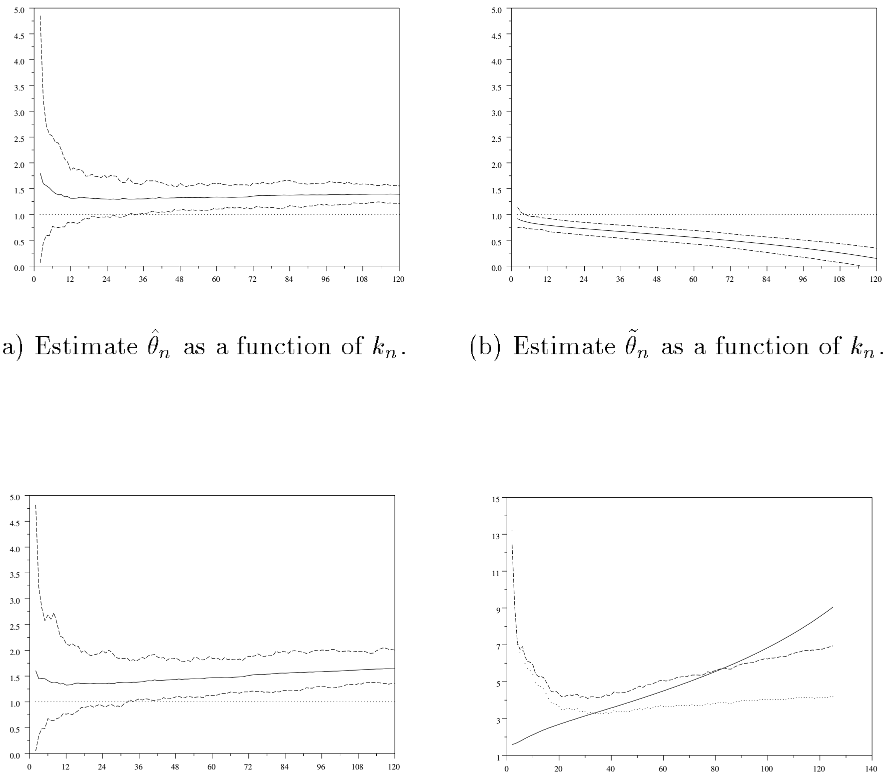


FIGURE 1. Two examples of QQ-plots, $\log(X_{n-i+1,n})$ is plotted versus $\log_2(n/i)$. Solid line: $\mathcal{W}(2,1)$ distribution, dashed line: $\mathcal{N}(1,1)$ distribution



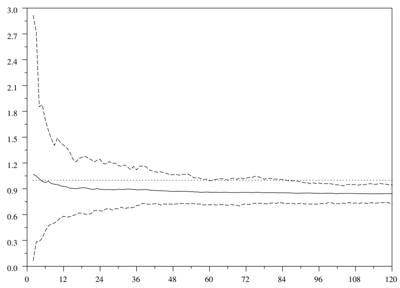
(a) Estimate $\hat{\theta}_n$ as a function of k_n.

(b) Estimate $\tilde{\theta}_n$ as a function of k_n.

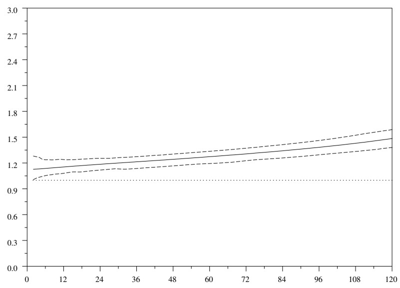
(c) Estimate $\breve{\theta}_n$ as a function of k_n.

(d) Mean square error as a function of k_n.

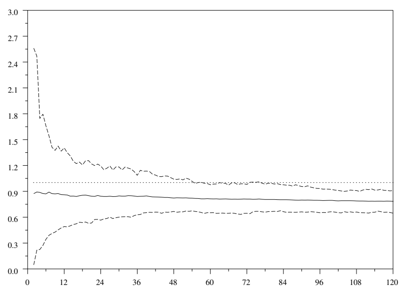
FIGURE 2. Comparison of estimates $\hat{\theta}_n$, $\tilde{\theta}_n$ and $\breve{\theta}_n$ for the $\Gamma(0.5, 1)$ distribution. (a)-(c) Dotted line: true value, solid line: mean over 100 samples, dashed lines: empirical 90% confidence interval. (d) Dotted line: $\hat{\theta}_n$, solid line: $\tilde{\theta}_n$, dashed line: $\breve{\theta}_n$.



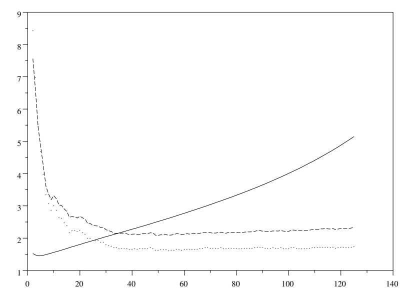
(a) Estimate $\hat{\theta}_n$ as a function of k_n.



(b) Estimate $\tilde{\theta}_n$ as a function of k_n.

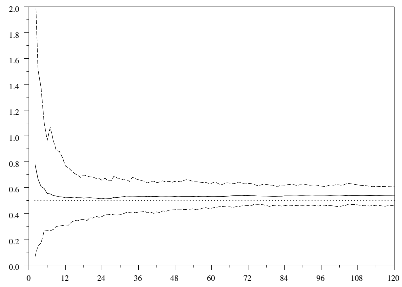


(c) Estimate $\breve{\theta}_n$ as a function of k_n.

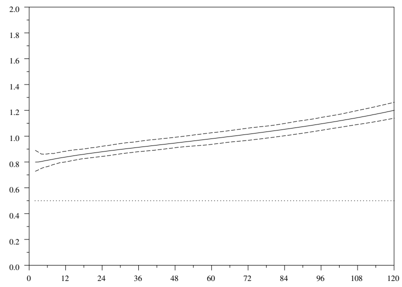


(d) Mean square error as a function of k_n.

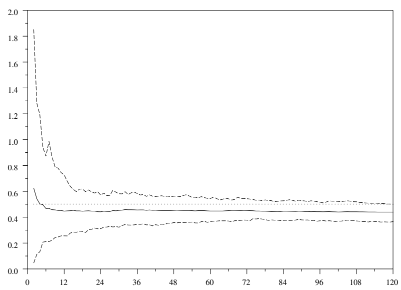
FIGURE 3. Comparison of estimates $\hat{\theta}_n$, $\tilde{\theta}_n$ and $\breve{\theta}_n$ for the $\Gamma(1.5, 1)$ distribution. (a)-(c) Dotted line: true value, solid line: mean over 100 samples, dashed lines: empirical 90% confidence interval. (d) Dotted line: $\hat{\theta}_n$, solid line: $\tilde{\theta}_n$, dashed line: $\breve{\theta}_n$.



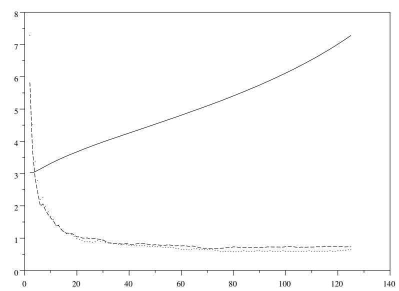
(a) Estimate $\hat{\theta}_n$ as a function of k_n.



(b) Estimate $\tilde{\theta}_n$ as a function of k_n.

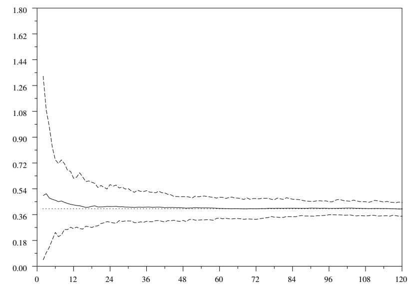


(c) Estimate $\breve{\theta}_n$ as a function of k_n.

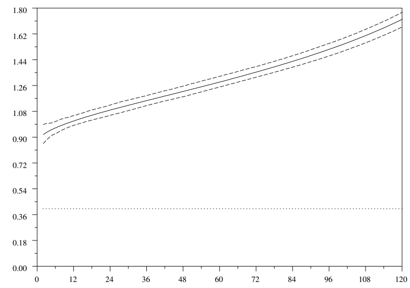


(d) Mean square error as a function of k_n.

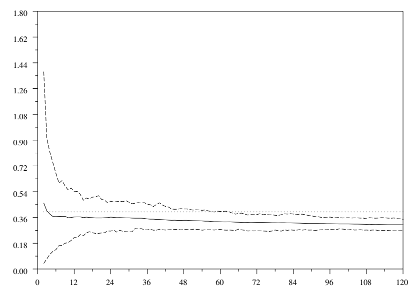
FIGURE 4. Comparison of estimates $\hat{\theta}_n$, $\tilde{\theta}_n$ and $\breve{\theta}_n$ for the $\mathcal{N}(1.2, 1)$ distribution. (a)-(c) Dotted line: true value, solid line: mean over 100 samples, dashed lines: empirical 90% confidence interval. (d) Dotted line: $\hat{\theta}_n$, solid line: $\tilde{\theta}_n$, dashed line: $\breve{\theta}_n$.



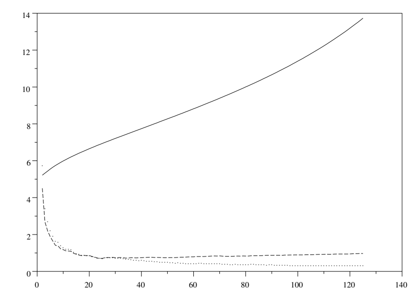
(a) Estimate $\hat{\theta}_n$ as a function of k_n.



(b) Estimate $\tilde{\theta}_n$ as a function of k_n.

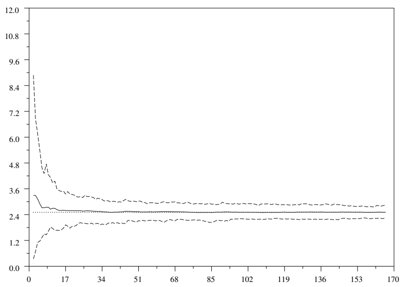


(c) Estimate $\breve{\theta}_n$ as a function of k_n.

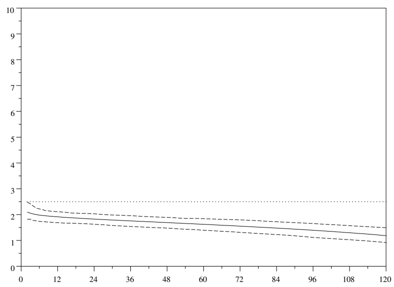


(d) Mean square error as a function of k_n.

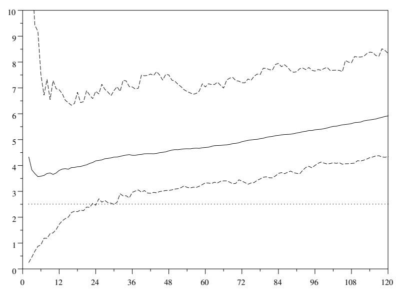
FIGURE 5. Comparison of estimates $\hat{\theta}_n$, $\tilde{\theta}_n$ and $\breve{\theta}_n$ for the $\mathcal{W}(2.5, 2.5)$ distribution. (a)-(c) Dotted line: true value, solid line: mean over 100 samples, dashed lines: empirical 90% confidence interval. (d) Dotted line: $\hat{\theta}_n$, solid line: $\tilde{\theta}_n$, dashed line: $\breve{\theta}_n$.



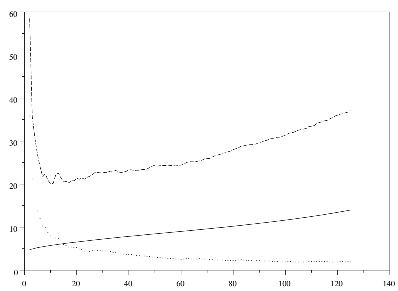
(a) Estimate $\hat{\theta}_n$ as a function of k_n.



(b) Estimate $\tilde{\theta}_n$ as a function of k_n.

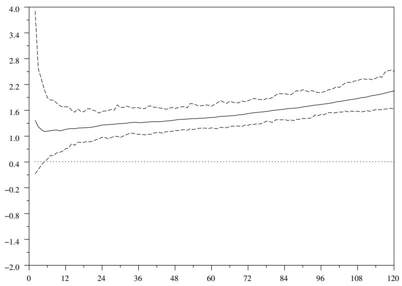


(c) Estimate $\breve{\theta}_n$ as a function of k_n.

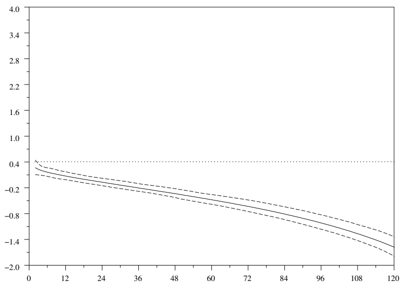


(d) Mean square error as a function of k_n.

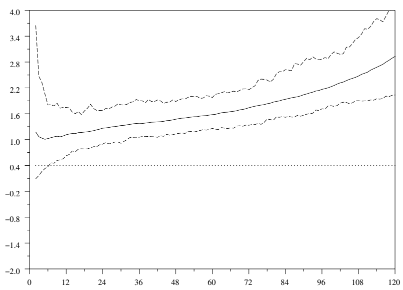
FIGURE 6. Comparison of estimates $\hat{\theta}_n$, $\tilde{\theta}_n$ and $\breve{\theta}_n$ for the $\mathcal{W}(0.4, 0.4)$ distribution. (a)-(c) Dotted line: true value, solid line: mean over 100 samples, dashed lines: empirical 90% confidence interval. (d) Dotted line: $\hat{\theta}_n$, solid line: $\tilde{\theta}_n$, dashed line: $\breve{\theta}_n$.



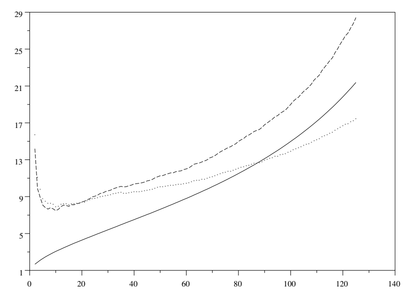
(a) Estimate $\hat{\theta}_n$ as a function of k_n.



(b) Estimate $\tilde{\theta}_n$ as a function of k_n.

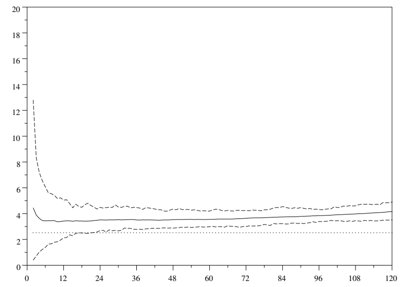


(c) Estimate $\check{\theta}_n$ as a function of k_n.

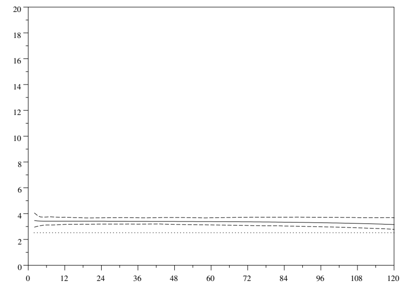


(d) Mean square error as a function of k_n.

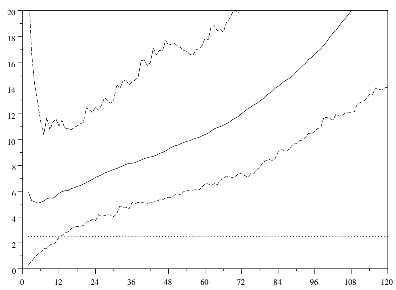
FIGURE 7. Comparison of estimates $\hat{\theta}_n$, $\tilde{\theta}_n$ and $\check{\theta}_n$ for the $\mathcal{M}W(2.5)$ distribution. (a)-(c) Dotted line: true value, solid line: mean over 100 samples, dashed lines: empirical 90% confidence interval. (d) Dotted line: $\hat{\theta}_n$, solid line: $\tilde{\theta}_n$, dashed line: $\check{\theta}_n$.



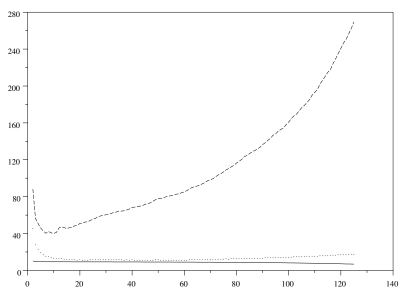
(a) Estimate $\hat{\theta}_n$ as a function of k_n.



(b) Estimate $\tilde{\theta}_n$ as a function of k_n.



(c) Estimate $\breve{\theta}_n$ as a function of k_n.



(d) Mean square error as a function of k_n.

FIGURE 8. Comparison of estimates $\hat{\theta}_n$, $\tilde{\theta}_n$ and $\breve{\theta}_n$ for the $\mathcal{M}W(0.4)$ distribution. (a)-(c) Dotted line: true value, solid line: mean over 100 samples, dashed lines: empirical 90% confidence interval. (d) Dotted line: $\hat{\theta}_n$, solid line: $\tilde{\theta}_n$, dashed line: $\breve{\theta}_n$.