Contributions to extreme-value analysis

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Abstract: This report summarizes my contributions to extreme-value statistics. I worked on the estimation of the extreme-value index, Weibull-tail coefficient and end-point estimation, together with the design of confidence intervals and tests for distribution tails. I also studied conditional extremes, that is the situation where the extreme-value behavior depends on a covariate. I investigated how extreme-value statistics can benefit from recent neural networks advances. Finally, I developed some applications to risk measure estimation.

Contributions

Extreme value theory is a branch of statistics dealing with the extreme deviations from the bulk of probability distributions. More specifically, it focuses on the limiting distributions for the minimum or the maximum of a large collection of random observations from the same arbitrary (unknown) distribution. Let $x_1 < \cdots < x_n$ denote n ordered observations from a random variable X representing some quantity of interest. A p_n-quantile of X is the value q_{p_n} such that the probability that X is greater than q_{p_n} is p_n, i.e. $\mathbb{P}(X > q_{p_n}) = p_n$. When $p_n < 1/n$, such a quantile is said to be extreme since it is usually greater than the maximum observation x_n. To estimate such extreme quantiles requires therefore specific methods to extrapolate information beyond the observed values of X. Those methods are based on Extreme value theory. This kind of issues appeared in hydrology. One objective was to assess risk for highly unusual events, such as 100-year floods, starting from flows measured over 50 years.

The decay of the survival function $\mathbb{P}(X > x) = 1 - F(x)$, where F denotes the cumulative distribution function associated to X, is driven by a real parameter called the **extreme-value index** γ. I have proposed several estimators for this parameter, see [1, 2, 3, 4, 5, 6] and for the associated extreme quantiles [7, 8]. When this parameter is positive, the survival function is said to be heavy-tailed, when this parameter is negative, the survival function vanishes above its right end point. If this parameter is zero, then the survival function decreases to zero at an exponential rate. An important part of our work is dedicated to the study of such distributions. For instance, in reliability, the distributions of interest are included in a semi-parametric family whose tails are decreasing exponentially fast. These so-called **Weibull tail-distributions** encompass a variety of light-tailed distributions, such as Weibull, Gaussian, gamma and logistic distributions. Let us recall that a cumulative distribution function F has a Weibull tail if it satisfies the following property: There exists $\theta > 0$ such that for all $\lambda > 0$,

$$\lim_{y \to \infty} \frac{\log(1 - F(\lambda y))}{\log(1 - F(y))} = \lambda^{1/\theta}.$$

Dedicated methods have been proposed to estimate the Weibull tail-coefficient θ since the relevant information is only contained in the extreme upper part of the sample. More specifically, the estimators I proposed are based on the log-spacings between the upper order statistics [9, 10, 11, 12, 13, 14]. See also [15, 16, 17, 18, 19] for the estimation of the associated extreme quantiles. These methods can also be seen as an improvement of the ET method [20, 21, 22]. Of course, the choice of a tail model is an important issue, see [23, 24, 25] for the introduction of **goodness-of-fit tests** and [26] for implementation details. The design of **confidence intervals** is addressed in [27, 28, 29].

I also addressed the estimation of extreme level curves. This problem is equivalent to estimating quantiles when covariate information is available and when their order converges to one as the sample size increases. We show that, under some conditions, these so-called **extreme conditional quantiles** can still be estimated through a kernel estimator of the conditional survival function. Sufficient conditions on the rate of convergence of their order to one are provided to obtain asymptotically Gaussian distributed estimators. Making use of this result, some estimators of the extreme-value parameters are introduced and extreme conditional quantiles estimators are deduced [30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43]. Finally, the tail copula is widely used to describe the dependence in the tail of multivariate distributions. In some situations such as risk management, the dependence structure may be linked with some covariate. The tail copula thus depends on this covariate and is referred to as the conditional tail copula. The aim of [44] is to propose a nonparametric estimator of the conditional tail copula and to establish its asymptotic normality.

In the **multivariate context**, we focus on extreme geometric quantiles [45, 46]. Their asymptotics are established, both in direction and magnitude, under suitable integrability conditions, when the norm of the associated index vector tends to one.

I worked on the interplay between **learning methods based on neural networks**, on the one hand, and the statistics of extreme values, on the other. I have shown that Generative Adversarial Networks (GANs) in their original formulation could not reproduce the behavior of heavy-tailed distributions [47, 48]. We proposed a modification to the architecture of these networks to overcome this significant limitation, and established theoretical convergence results as a function of the number of hidden units in a layer [49]. These are the first results on the behavior of GANs at the tail of a distribution. These results are extended to the multivariate setting in [50]. We have also shown how to adapt the regression principle of neural networks to approximate the quantile function of a distribution. This method allows for a dramatic improvement in the quality of estimates of extreme quantities [51, 52].

Applications of extreme-value theory are found in statistical learning [53, 54], ecology [55], hydrology [56, 57, 58, 59, 60] and more generally in risk estimation. In this case, I investigate the extreme properties of several **risk measures** as alternatives to the classical Value-at-Risk: Conditional tail expectation [61, 62], Conditional tail moment [63, 64], proportional hazard premium [65], expectiles [66, 67, 68, 69, 70, 71, 72, 73] and L_p quantiles [74, 75, 76]. See also [77] for a measure of variability in the distribution tail.

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