A moving window approach for nonparametric estimation of extreme level curves

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Joint work with Stéphane Girard and Alexandre Lekina

Let $Y \in \mathbb{R}$ be a real random value associated to a non-random covariate $x \in E$ where E is a metric space indowed by a metric d.

Goal: Estimate a conditional extreme quantile $q(\alpha, t)$ of order $1 - \alpha$ defined by

$$\mathbb{P}(Y \geq q(\alpha, x) | x) = \alpha$$

Difficulties:

- The quantile order $1 - \alpha$ can be very close to 1 (large quantile).
- The quantile is a function of the covariate x.
- The space E can be of infinite dimension.

Main assumption: The conditional distribution of Y given $x \in E$ is a heavy tailed distribution *i.e.* for $\lambda > 0$,

$$\lim_{\alpha \to 0} \frac{q(\lambda\alpha, x)}{q(\alpha, x)} = \lambda^{-\gamma(x)},$$

- $\gamma(.)$ is an unknown positive function of the covariate x called the conditional tail index.
- The conditional extreme quantile $q(., x)$ decreases to infinity at a polynomial rate as $\alpha \to 0$.

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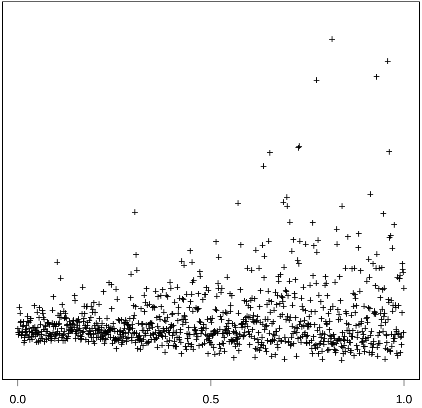

Example: Estimation at the point $t = 0.5$ using $n = 1000$ observations (Y_i, x_i), $i = 1, \ldots, n$ for $E = [0, 1]$.

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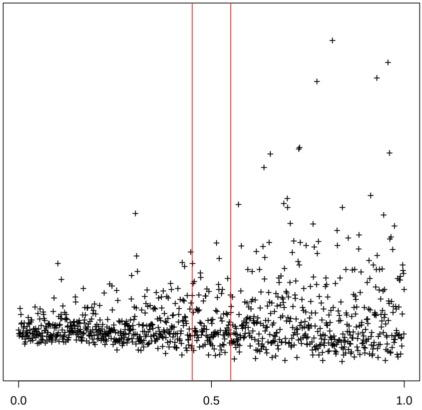
- We thus choose a positive sequence $h_{n,t}$ tending to zero as $n \to \infty$ and we define the slice $S_t = (0, \infty) \times B(t, h_{n,t})$ where $B(t, h_{n,t})$ is the ball of center t and radius $h_{n,t}$.



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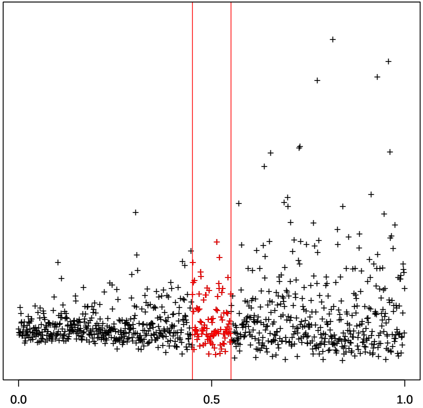

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- We select the observations Y_i's for which $x_i \in B(t, h_{n,t})$.
- These observations are denoted by $\{Z_i(t), i = 1, \ldots, m_{n,t}\}$ where $m_{n,t}$ is the number of x_i's in the ball $B(t, h_{n,t})$.
- The associated order statistics are denoted by

$$Z_{1,m_{n,t}}(t) \leq \ldots \leq Z_{m_{n,t},m_{n,t}}(t).$$

Influence of the rate of convergence of α

We consider three situations for the rate of convergence of α to zero:

- **(S.1)** Slow convergence of α to zero:

$$\alpha \to 0 \text{ and } m_{n,t}\alpha \to \infty.$$

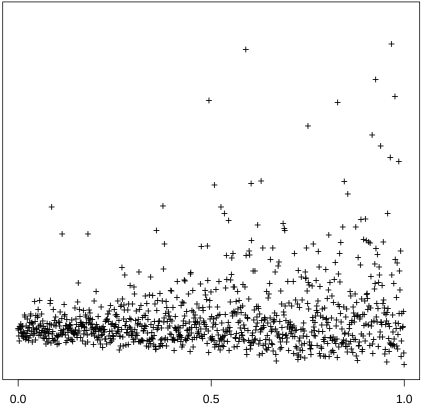
- **(S.2)** Fast convergence of α to zero:

$$\alpha \to 0 \text{ and } m_{n,t}\alpha \to c \in [1, \infty).$$

- **(S.3)** Very fast convergence of α to zero:

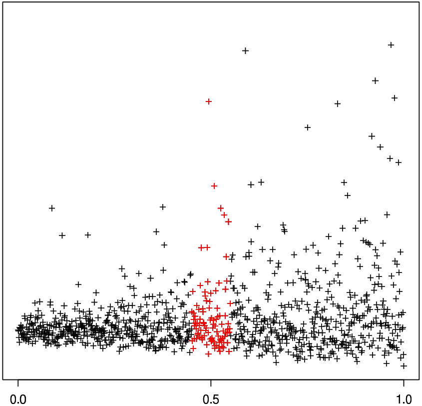
$$\alpha \to 0 \text{ and } m_{n,t}\alpha \to c \in [0, 1).$$

Example



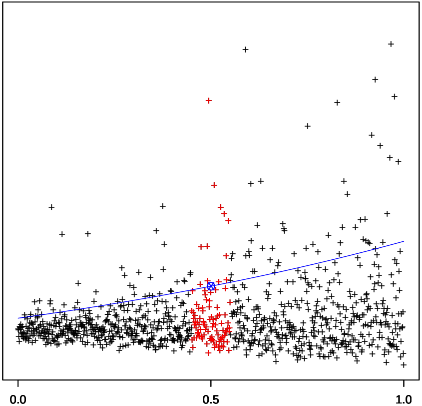
Using $n = 1000$ observations, we are interested in the estimation of the extreme quantile of order α at the point $t = 0.5$ ($E = [0, 1]$).

Example



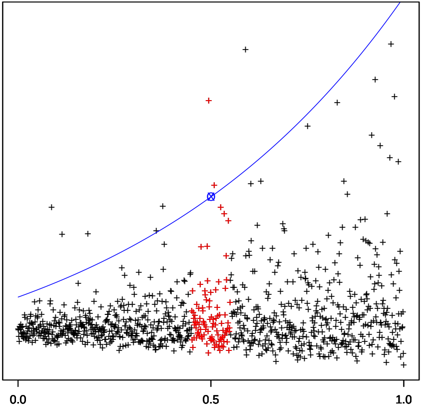
We select the observations in the slice S_t with $h_{n,t} = 0.05$ $(m_{n,t} = 100)$.

Example



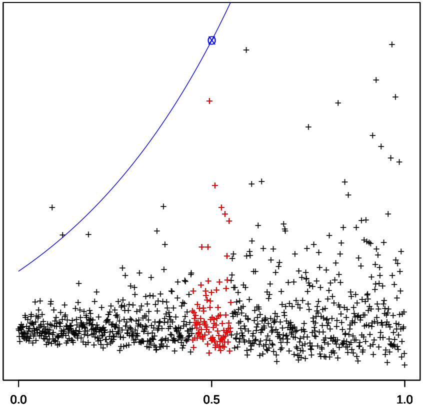
(S.1): Theoretical value of the quantile of order $\alpha = 10/100 = 0.1$

Example



(S.2): Theoretical value of the quantile of order $\alpha = 1/100 = 0.01$

Example



(S.3): Theoretical value of the quantile of order $\alpha = 0.1/100 = 0.001$

Situation **(S.1)**: The conditional quantile is in the range of the observations. We use the estimator:

$$\hat{q}_1(\alpha, t) = Z_{m_{n,t} - \lfloor m_{n,t}\alpha \rfloor + 1, m_{n,t}}(t).$$

Asymptotic distribution

If α satisfies **(S.1)**, under some assumptions on the conditional distribution,

$$(m_{n,t}\alpha)^{1/2} \left(\frac{\hat{q}_1(\alpha, t)}{q(\alpha, t)} - 1 \right) \xrightarrow{d} \mathcal{N}(0, \gamma^2(t))$$

Situation **(S.2)**: The conditional quantile is located near the boundary of the sample but still in the range of the data. We can also use the estimator $\hat{q}_1(\alpha, t)$.

Asymptotic distribution

If α satisfies **(S.2)**, under some assumptions on the conditional distribution,

$$\left(\frac{\hat{q}_1(\alpha, t)}{q(\alpha, t)} - 1\right) \xrightarrow{d} \mathcal{E}(c, \gamma(t)),$$

where $\mathcal{E}(c, \gamma(t))$ is a non-degenerated distribution (but not Gaussian !!)

Comments:

- The asymptotic distribution is not Gaussian and its expression is quite complicated.
- In this situation, estimator $\hat{q}_1$ is not consistent.

Situation **(S.3)**: The conditional quantile is beyond the range of the observations. Thus, we can not use the estimator $\hat{q}_1$. We propose to use the estimator:

$$\hat{q}_2(\alpha, t) = \hat{q}_1(\beta, t) \left(\frac{\beta}{\alpha}\right)^{\hat{\gamma}_n(t)},$$

where β satisfies **(S.1)** and $\hat{\gamma}_n(t)$ is a point wise estimator of the conditional tail index.

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Estimator $\hat{q}_2(\alpha, t)$ can be decomposed in two parts:

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Estimator $\hat{q}_2(\alpha, t)$ can be decomposed in two parts:

- An estimator of a conditional quantile of order β satisfying **(S.1)** (*i.e.* an order statistics)
- An extrapolation term depending on $\hat{\gamma}_(t)$

Asymptotic distribution

If β satisfies **(S.1)**, if there exists a positive sequence $\upsilon_n(t)$ and a distribution $\mathcal{D}$ such that $\upsilon_n(t)(\hat{\gamma}_n(t) - \gamma(t)) \xrightarrow{d} \mathcal{D}$, then, under some assumptions on the conditional distribution, two situations arise:

i) The asymptotic distribution is driven by $\hat{q}_1(\beta, t)$ and then

$$(m_{n,t}\alpha)^{1/2} \left(\frac{\hat{q}_2(\alpha, t)}{q(\alpha, t)} - 1 \right) \xrightarrow{d} \mathcal{N}(0, \gamma^2(t)).$$

ii) The asymptotic distribution is driven by $\hat{\gamma}_n(t)$ and then

$$\frac{\upsilon_n(t)}{\log(\beta/\alpha)} \left(\frac{\hat{q}_2(\alpha, t)}{q(\alpha, t)} - 1 \right) \xrightarrow{d} \mathcal{D}.$$

- Estimator $\hat{q}_2$ can be used in the three situations.
- For the conditional tail index estimator, we can use for instance the Hill type estimator proposed by L. Gardes & S. Girard (2008):

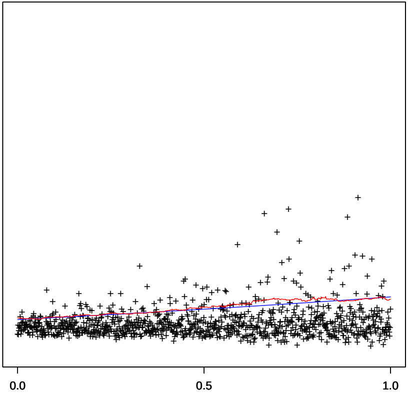
$$\hat{\gamma}_n^H(t) = \frac{1}{k_{n,t}} \sum_{i=1}^{k_{n,t}} i \log \left(\frac{Z_{m_{n,t}-i+1,m_{n,t}}}{Z_{m_{n,t}-i,m_{n,t}}} \right),$$

where $k_{n,t} = m_{n,t}\beta$.

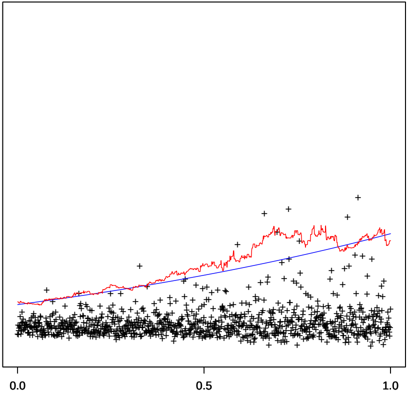
- We generate $n = 1000$ observations $\{(Y_i, x_i), i = 1, \ldots, n\}$ under the model: $x \in E = [0, 1]$, and the conditional extreme quantile of Y given x is defined by:

$$q(\alpha, x) = \left\{ \log \left(\frac{1}{1-\alpha} \right) \right\}^{-\gamma(x)} \quad \text{(Fréchet distribution)}$$

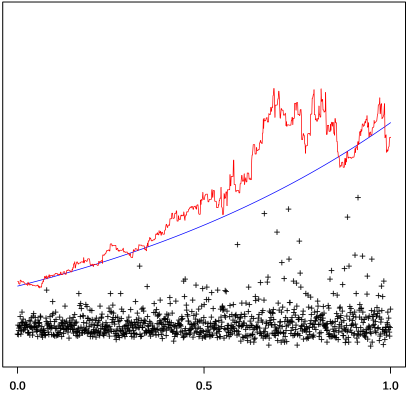
- To estimate $q(\alpha, t)$ we use the estimator $\hat{q}_2(\alpha, t) = \hat{q}_1(\beta, t)(\beta/\alpha)^{\hat{\gamma}_n^H(t)}$ where $\hat{\gamma}_n^H(t)$ is the conditional Hill type estimator, the observations are selected using a ball of radius $h_{n,t} = 0.1$ for all $t \in E$ (*i.e.* $m_{n,t} = 200$) and $\beta = 0.3$.



Estimation of the function $q(\alpha, .)$ with $\alpha = 20/200$ (**(S.1)**).



Estimation of the function $q(\alpha, .)$ with $\alpha = 2/200$ (**(S.2)**).



Estimation of the function $q(\alpha, .)$ with $\alpha = 0.2/200$ ((**S.3**)).

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